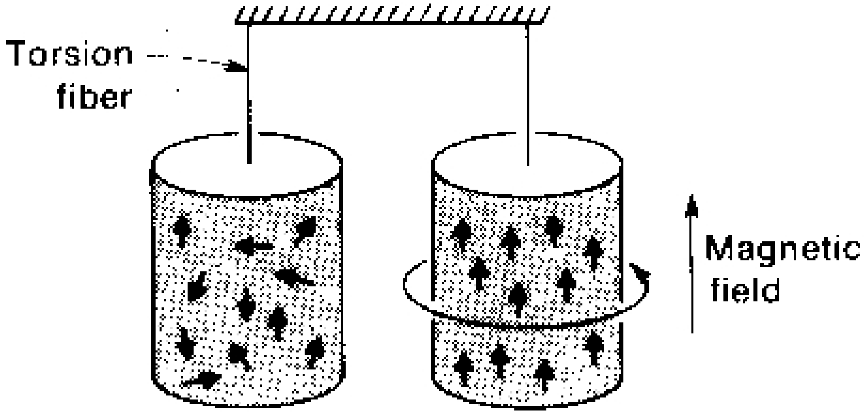
10) Angular momentum and magnetism

The Einstein-de Haas effect (1915) shows that angular momentum and magnetism are related:

When an initially unmagnetized iron rod is suddenly magnetized along its length, it tends to twist about this axis.

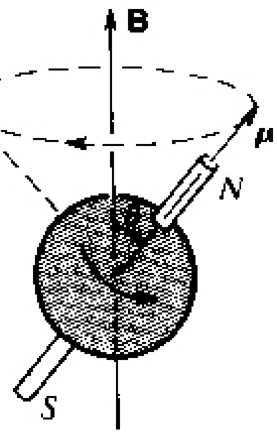


If the components of the angular momentum are quantized, the components of the magnetic moment of atoms should also be quantized.

This was experimentally shown in the Stern Gerlach Experiment (1922).

Force on a magnetic dipole

$\vec{\mu} = I \cdot \vec{A}$



$\vec{\mu} = q_m \cdot \vec{d}$

One can describe a magnetic moment by two oppositely charged magnetic monopoles $\pm q_m$ a distance d apart on which the monopole force acts: $\vec{F} = q_m \vec{B}$

$$F_z = q_m B_z(z + d_z) - q_m B_z(z) = q_m \cdot \tfrac{\partial}{\partial z} B_z d_z$$
$$= \mu_z \tfrac{\partial}{\partial z} B_z$$

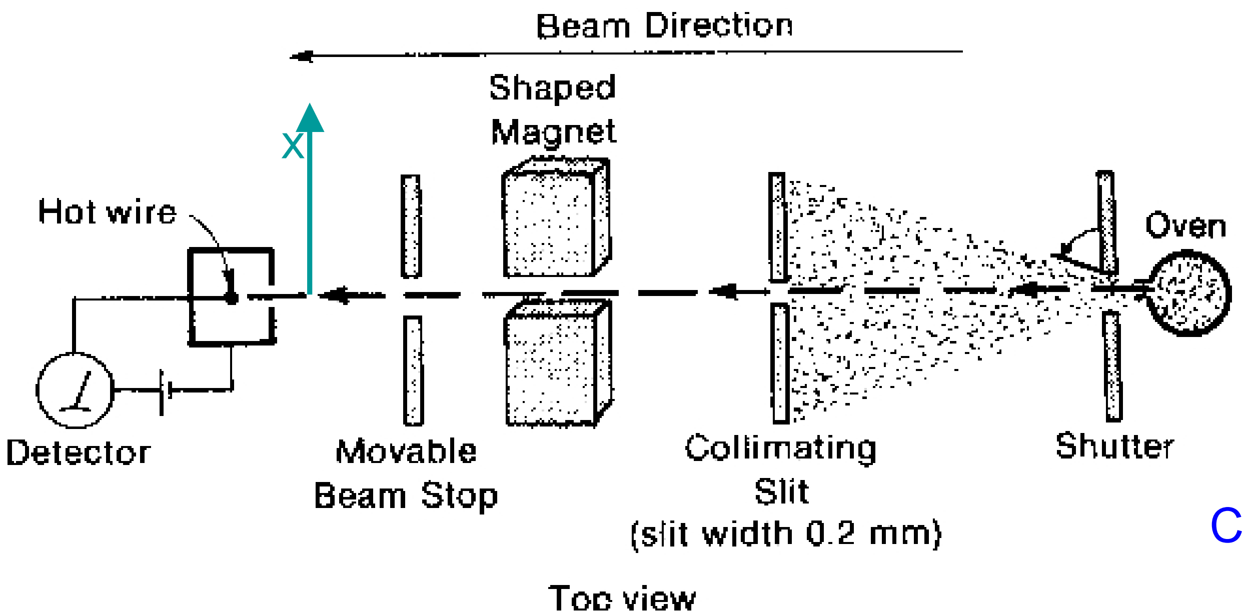
If z component of the angular momentum is quantized then

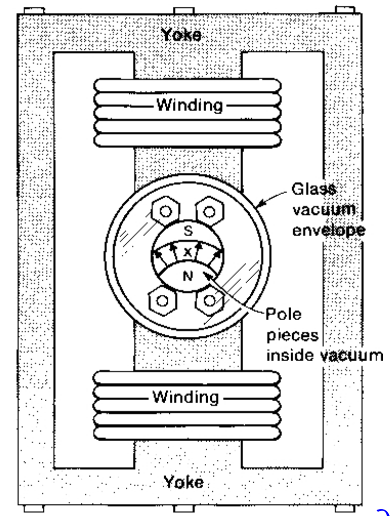
- z component of the magnetic moment is quantized
- z component of the force in an inhomogeneous magnetic filed is quantized

Georg.Hoffstaetter@Cornell.edu

The Stern Gerlach experiment

04/22/2005





Creation of a constant $\frac{\partial}{\partial z} B_z$

- Neutral atoms are used so that forces on the charge do not dominate the motion
- The outgoing current is measured directly by depositing ions on a glass plate or by detection of the ion flux at different x coordinates by an ionizing hot wire.

Au on Glass

Otto Stern

Germany 1888-USA 1969

- Splitting of the beam shows the quantization of the angular momentum.
- Top and bottom is not split due to energy spread of the beam.

Georg.Hoffstaetter@Cornell.edu

Angular momentum

Classical angular momentum: $\vec{L} = \vec{x} \times \vec{p} = \begin{cases} yp_z - zp_y \\ zp_x - xp_z \\ xp_y - yp_x \end{cases}$

$$x = r\cos\varphi\sin\vartheta \qquad y = r\sin\varphi\sin\vartheta \qquad z = r\cos\vartheta$$

$$\hat{L}_z\Psi = (x\hat{p}_y - y\hat{p}_x)\Psi = -i\hbar\left(x\tfrac{\partial}{\partial y} - y\tfrac{\partial}{\partial x}\right)\Psi = -i\hbar\tfrac{\partial}{\partial\varphi}\Psi$$

$$\tfrac{\partial}{\partial\varphi} = \tfrac{\partial x}{\partial\varphi}\tfrac{\partial}{\partial x} + \tfrac{\partial y}{\partial\varphi}\tfrac{\partial}{\partial y} + \tfrac{\partial z}{\partial\varphi}\tfrac{\partial}{\partial z} = -r\sin\varphi\sin\vartheta\tfrac{\partial}{\partial x} + r\cos\varphi\sin\vartheta\tfrac{\partial}{\partial y} = x\tfrac{\partial}{\partial y} - y\tfrac{\partial}{\partial x}$$

Eigenfunctions of this operator describe a particle with well defined z-component of angular momentum, since its measurement has no spread.

$$-i\hbar\tfrac{\partial}{\partial\varphi}\Psi_{L_z} = \hbar m\Psi_{L_z} \quad \to \quad \Psi_{L_z} \propto e^{im\varphi} \text{ and } L_z = m\hbar$$

Periodic boundary conditions lead to the quantization of the angular momentum.

$$\Psi(\varphi + 2\pi) = \Psi(\varphi) \quad \to \quad m \in \{0, \pm1, \pm2, \ldots\}$$

Georg.Hoffstaetter@Cornell.edu

Angular momentum commutators

$$\vec{L} = \vec{x} \times \vec{p} = \begin{cases} yp_z - zp_y \\ zp_x - xp_z \\ xp_y - yp_x \end{cases}$$

$$[\tfrac{\partial}{\partial z}, z]\Psi = \tfrac{\partial}{\partial z} z\Psi - z\tfrac{\partial}{\partial z}\Psi = \Psi \quad \rightarrow \quad [\tfrac{\partial}{\partial z}, z] = 1$$

$$[\hat{L}_x, \hat{L}_y]\Psi = [y\hat{p}_z - z\hat{p}_y, z\hat{p}_x - x\hat{p}_z]\Psi = \left([y\hat{p}_z, z\hat{p}_x - x\hat{p}_z] - [z\hat{p}_y, z\hat{p}_x - x\hat{p}_z]\right)\Psi$$

$$= \left([y\hat{p}_z, z\hat{p}_x] - [y\hat{p}_z, x\hat{p}_z] - [z\hat{p}_y, z\hat{p}_x] + [z\hat{p}_y, x\hat{p}_z]\right)\Psi$$

$$= -i\hbar\left([y\tfrac{\partial}{\partial z}, z\hat{p}_x] + [z\hat{p}_y, x\tfrac{\partial}{\partial z}]\right)\Psi = -i\hbar\left(y\hat{p}_x[\tfrac{\partial}{\partial z}, z] + x\hat{p}_y[z, \tfrac{\partial}{\partial z}]\right)\Psi$$

$$= i\hbar\left(x\hat{p}_y - y\hat{p}_x\right)\Psi = i\hbar\hat{L}_z\Psi$$

Rotating the coordinate system can lead to a cyclic interchange of the coordinate axes:

$$[\hat{L}_x, \hat{L}_y]\Psi = i\hbar\hat{L}_z\Psi, \quad [\hat{L}_y, \hat{L}_z]\Psi = i\hbar\hat{L}_x\Psi, \quad [\hat{L}_z, \hat{L}_x]\Psi = i\hbar\hat{L}_y\Psi$$

Georg.Hoffstaetter@Cornell.edu

Simultaneous eigenvalues

If two operators do not commute, there can not be a complete set of functions which simultaneously are eigenfunctions of both operators.

Otherwise one could expand any wave function as

$$\Psi = \sum_{\text{all } a,b} C_{a,b}\Phi_{a,b}(x) \quad \text{with} \quad \hat{A}\Phi_{a,b}(x) = a\Phi_{a,b}(x), \quad \hat{B}\Phi_{a,b}(x) = b\Phi_{a,b}(x)$$

And the operators would commute:

$$[\hat{A},\hat{B}]\Psi = [\hat{A},\hat{B}] \sum_{\text{compl.set}} C_{a,b}\Phi_{a,b}(x) = \sum_{\text{compl.set}} C_{a,b}(ab - ba)\Phi_{a,b}(x) = 0$$

When the wave function is in an eigenstate of one component of the angular momentum, then this component is quantized and the other two cannot be specified, i.e. they have a spread when measured.