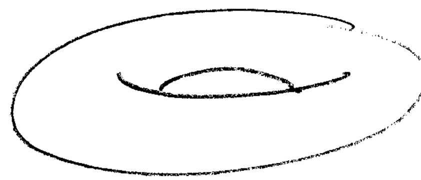
BSMJC Feb 21, 2011 David Curtin

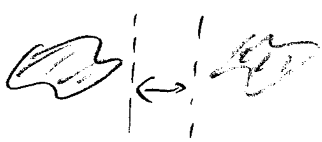
HOMOLOGY & COHOMOLOGY

1. Rough Idea

- Consider the Torus: A 2D-Manifold (can embed in 3D space to display) with two 2D-"Holes"



- Want to classify it's topology. One fruitful avenue of classification: count "holes" (of various dimension)"



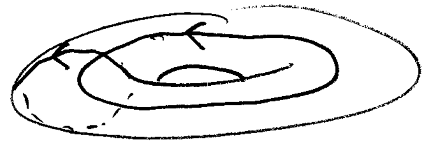
disconnected: "1D-hole"

circle: "2D-hole"

Sphere: "3D-hole" (interior)

etc...

- Powerful method to classify topology & count holes: HOMOTOPY GROUPS $\pi_n(X)$. Classify n-dimensional loops that are deformable into each other eg for torus



two cycles

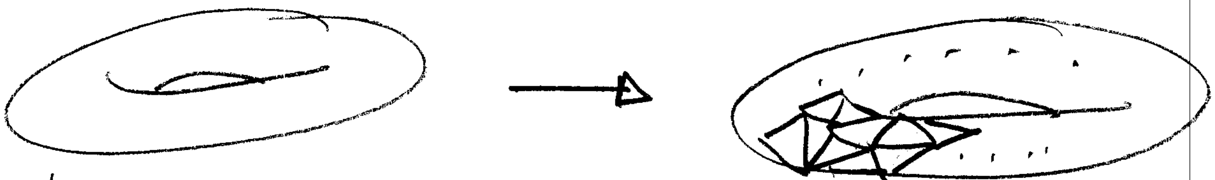
$\pi_1(T) = \mathbb{Z}^2$

→ $\boxed{\pi_n \text{ "detects" } (n+1)\text{-dim holes}}$ (loops around holes can't be contracted to a pt)

→ π_1 is easy to calculate, $\pi_2, \pi_3 \ldots$ are MUCH HARDER. Could we use something simpler?

- HOMOLOGY

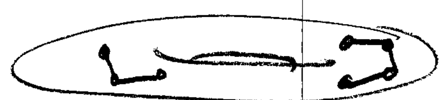
Imagine representing Torus by a polyhedron made up of generalized triangles (simpletes) {  , ... }. ← includes interior



(kinda like a 3D computer graphics polygon)

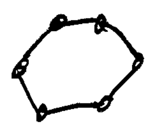
↳ called "Triangulation of Torus". Not unique! (That's OK)

↳ There is a way to algebraically represent all possible p-dim sub-polyhedra of the triangulation. They form the p-chain group C_p



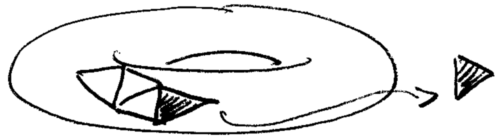
↳ eg 1-D subpolyhedra = "lines connecting vertices of the triangulation"

→ Can then ask: which ones of those chains have NO BOUNDARY?

eg "o—o" has boundary "• •".  has no boundary

↳ they form the group $Z_p \subset C_p$

↳ Can furthermore ask: which of those boundary-less chains are themselves just the boundaries of a higher-dim sub-polyhedron of the torus

↳ eg  has boundary , which itself has no boundary

↳ they form the group $B_p \subset Z_p \subset C_p$

Now, the interesting question is: "WHICH p-CHAINS ARE BOUNDARY-LESS BUT ARE NOT THEMSELVES BOUNDARIES?" → THEY ENCLOSE "HOLES" !!

this cycle encloses one of the "holes" of the torus

Define p^{th} HOMOLOGY GROUP

$$H_p \equiv \frac{Z_p}{B_p}$$

free & finitely generated

cyclic "torsion subgroup"

H_p is abelian, and can show $H_p = G \oplus T$

- rank G counts $(p+1)$-dim holes! → $H_1(T) = \mathbb{Z}^2$
- cyclic torsion subgroup contains info on how space is TWISTED!

(Note homology groups have down-index)

- Homology is awesome, but very tedious to calculate.
  Also needs "global" information : boundary operator "∂"

  ↳ COHOMOLOGY is dual to homology and much more powerful because its construction only requires local information ("∂" → exterior derivative), and it also has an additional RING STRUCTURE that allows it to (potentially?) distinguish between two spaces with identical homology groups.

- COHOMOLOGY:
  Construction of dual to C_p, Z_p, B_p, H_p:

  Recall: $\omega = T_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p}$ antisym rank p tensor

  $d\omega = \frac{\partial T_{i_1 \dots}}{\partial x^{i_{p+1}}} dx^{i_{p+1}} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_p}$ is a $p+1$ form

  C_p = group of p-cycles

  ↳ roughly speaking, a p-form on our manifold M $\omega \in \Omega^p(M)$ defines a map $\omega : C_p \to \mathbb{R}$ by $(c \in C_p)$

  $$\langle \omega, c \rangle \equiv \int_c \omega$$

  ⇒ Stokes' Thm : boundary op. & ext. derivative are formal ADJOINTS of each other

  $$\langle d\omega, c \rangle = \langle \omega, \partial c \rangle$$

  ⇒ C^p = set of all p-forms (co-chains) on M

  (all defined with respect to some manifold M)

  Z^p = p-forms satisfying $d\omega = 0$: $\boxed{\text{space of CLOSED } p\text{-forms}}$

  B^p = p-forms that can be written as $\omega = d\eta$: $\boxed{\text{space of EXACT } p\text{-forms}}$

  (up-index!)

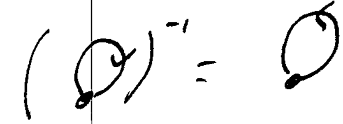
- <u>De Rahm Cohomology group:</u> set of closed p-forms on M that are NOT exact

$$H^p = Z^p / B^p$$

- also counts $(p+1)$-dim holes
- has additional Ring structure

2. Homotopy Groups

• 1D loop: map $\alpha:[0,1] \longrightarrow$ topological space X s.t. $\alpha(0)=\alpha(1)=x_0\in X$

↳ product:  inverse: 

↳ $\pi_1(X,x_0)$ = set of homotopy classes of 1-D loops based on x_0

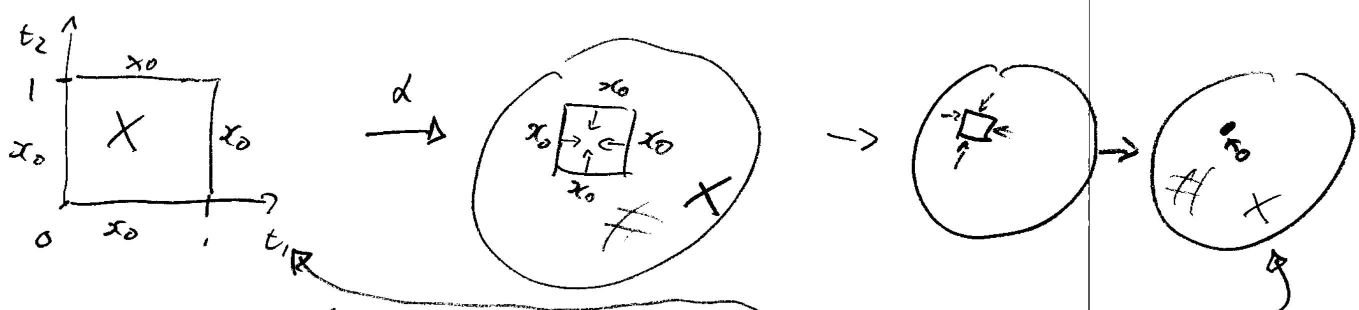
↳ X path connected: $\pi_1(X,x_0)\cong\pi_1(X,x_1)=\pi_1(X)$

↳ There is a "calculating theorem" that gives algorithm for calculating $\pi_1(X)$

• higher-dimensional loops: map $\alpha: I_n \longrightarrow X$ s.t. $\partial I_n \longrightarrow x_0\in X$

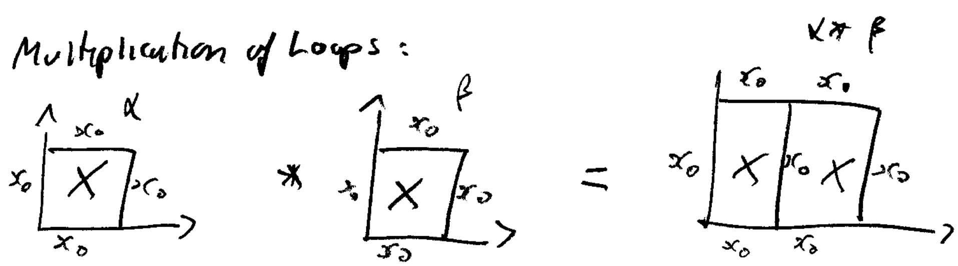
(I_n is closed n-dim cube, of course top. equivalent to S_n but easier to handle)

↳ 2D example good with pictures:



Connection to $\cancel{SU(2)} \to U(1)$ monopole: This is the sphere S^2@∞, and this is the field space of the Higgs field. π_2 classifies maps $S_2 \longrightarrow M_0$ (vacuum manifold)

↳ Multiplication of Loops:

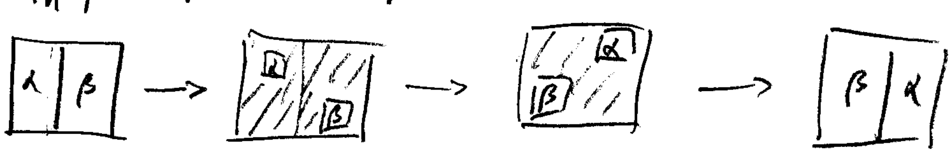


again, particular $x_0 \in X$ does not matter for path-connected X, often omit.

↳ form group of equivalence classes of n-loops: $\pi_n(X,x_0)$

Note: $\pi_n(X\otimes Y) = \pi_n(X)\oplus\pi_n(Y)$

↳ π_n, $n>1$, is always abelian: $\alpha*\beta=\beta*\alpha$



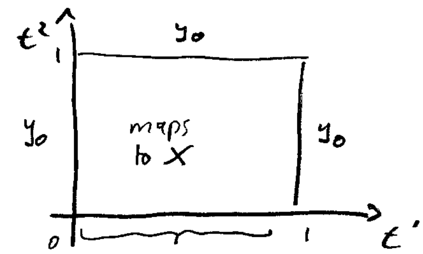
/// = mapped to x_0

$\longrightarrow$ = homotopic to

- Relative homology groups : for closed $Y \subset X$, $\alpha : I_n \to X$

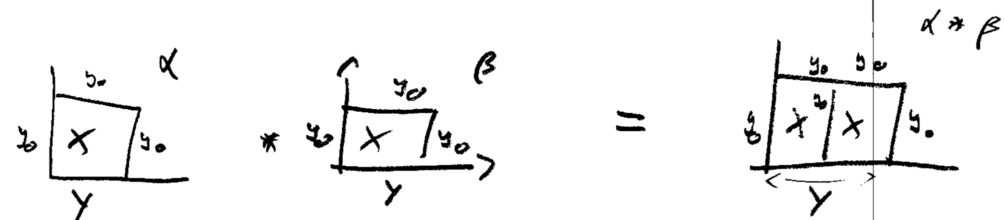
(*) s.t. all faces of $I_n \to y_0 \in Y$ except for one open face $J_{n-1} \to Y$ (ie a path in Y

eg $n = 2$



maps to Y → $(n-1)$-dim path in Y

↳ product :



↳ Define group of equivalent loops that can be deformed into each other while preserving property (*) : $\pi_n(X, Y, y_0)$

→ $\boxed{\text{hence two relative loops with totally inequivalent } (n-1)\text{-paths in } Y \text{ on } J_{n-1} \text{ are equivalent!!!}}$

- Exact Sequence :

$i_* : \pi_n(Y, y_0) \to \pi_n(X, y_0)$ is the trivial inclusion map

$j_* : \pi_n(X, y_0) \to \pi_n(X, Y, y_0)$ is another trivial inclusion map (think of $\pi_n(X, y_0)$ as $\pi_n(X, y_0, y_0)$)

$\partial_* : \pi_n(X, Y, y_0) \to \pi_{n-1}(Y, y_0)$ is the boundary homomorphism that takes a relative loop $\alpha : I_n \to X$ and keeps only the $(n-1)$-dim loop in Y that corresponds to the exceptional face J_{n-1}

⇒ EXACT SEQUENCE $\ldots \to \pi_n(Y, y_0) \xrightarrow{i_*} \pi_n(X, y_0) \xrightarrow{j_*} \pi_n(X, Y, y_0) \xrightarrow{\partial_*} \pi_{n-1}(Y, y_0) \xrightarrow{i_*} \ldots$

ie image of one map = kernel of the next

↳ USEFUL, eg "$\{0\} \to A \to B \to \{0\}$" ⇒ "$A \cong B$"

3. Simplexes

$\sigma^0 = \bullet$ $\quad \sigma^1 = \bullet\!\!-\!\!\bullet$ $\quad \sigma^2 =$ $\quad \sigma^3 =$

- **m-simplex** σ^n = generalized triangle in n-D

  ↳ $\sigma^m = \left\{ x = \sum_{i=1}^{m+1} \lambda_i x_i \;\middle|\; \lambda_i \geq 0,\ \sum \lambda_i = 1,\ x_i \text{ are linearly independent} \right\}$

  ↳ $\forall x \in \sigma^m$ has barycentric coords $\{\lambda_i\}$, x is the center of mass of σ^m with masses λ^i placed on points x_i

  ↳ often write $\sigma^m = [x_1, x_2, \ldots, x_{m+1}]$

  ↳ j^{th} face of $\sigma^m = [x_1, \ldots, \hat{x}_j, \ldots x_{m+1}]$ $\quad \wedge =$ "omitted"

  $= \{ \lambda_i x_i \mid \lambda_j = 0 \}$

- A **simplical complex** K is a finite collection of distinct simplexes in some $\mathbb{R}^n$ s.t.

  1) $\forall \sigma^p \in K \to$ all faces $\in K$ $\qquad$ 2) $\sigma^p, \sigma^q \in K \to \sigma^p \cap \sigma^q = \emptyset$ or common face of σ^p, σ^q

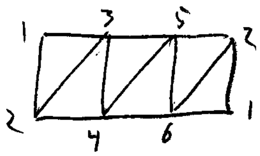
  ↳ ie simplexes that "fit together nicely"

  ↳ $|K| = \bigcup \{K\}$ is the **polyhedron** constructed out of simplex K.

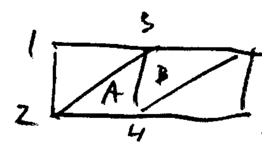
- **Triangulation** of space X = polyhedron $|K|$ that is homeomorphic to X, not unique.

  ↳ eg Moebius strip

=



(why not



?

simplexes A, B have same vertices!)

$= \{123\} + \{234\} + \{345\} + \{456\} + \{256\} + \{126\} +$ faces

- **oriented p-simplex**

  obtained from p-simplex $\sigma^p = [v_0, \ldots, v_p]$ and choosing an ordering of vertices v_i.

  Then $\pm\sigma_p$ = equivalence class of $\substack{\text{odd} \\ \text{even}}$ permutations of given ordering. eg $[v_2, v_1] = -[v_1, v_2]$

- **Boundary Operator** for oriented simplex $\sigma^p = [v_0, \ldots, v_p]$: $\partial\sigma^p = \sum_{j=0}^{p} (-1)^j [v_0, \ldots, \hat{v}_j, \ldots, v_p]$

This satisfies $\partial^2 = 0$.

Examples:

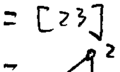
1) $\sigma = [12] = \overset{1}{\bullet}\!\!\longrightarrow\!\!\overset{2}{\bullet} \to \partial\sigma = [2] - [1] = \overset{1}{\bullet} \quad \overset{2}{\bullet}$

2) $\sigma = [12] + [23] = \overset{1}{\bullet}\!\!\to\!\!\overset{2}{\bullet}\!\!\to\!\!\overset{3}{\bullet}$ $\quad \partial\sigma = [2] - [1] + [3] - [2] = [3] - [1]$ $\quad \overset{1}{\bullet} \quad \overset{3}{\bullet}$

3) $\sigma = [123] =$

$\partial\sigma = [23] - [13] + [12]$

$=$



$\partial^2\sigma = [3] - [2] - [3] + [1] + [2] - [1] = 0$ ✓

4. Homology

- (Assume for now top space X is triangulable.) $H_p(X)$ is a topological invariant we will calculate using a triangulation K of X.

- For a given triangulation K of topological space X with dim n.
  - $C_p(K)$ = free abelian group generated by the oriented p-simplexes of K
    $= \{ c_p \mid c_p = \sum_{i=1}^{l_p} f_i \sigma_i ,\ f_i \in \mathbb{Z},\ l_p = \#\ p\text{-simplexes in } K \}$
    = "p-chain group of K"
    - addition: $\sum f_i \sigma_i + \sum g_i \sigma_i = \sum (g_i + f_i) \sigma_i$
    - set $C_p(K) = \{0\}$ for $p > n$
    - boundary operator acts linearly on $c_p \in C_p(K)$
  - $Z_p(K) = \ker \partial C_p \subset C_p$ = p-chains $c_p \in C_p$ satisfying $\partial c_p = 0$
    = "p-dim **cycle** group"
  - $B_p(K)$ = p-cycles for which $\exists\ c_{p+1} \in C_{p+1}$ s.t. $\partial c_{p+1} = b_p \in B_p(K)$
    = "p-dim **boundary** group" $= \operatorname{Im} \partial C_{p+1}$

- $\underset{\text{p-chain}}{C_p(K)} \subset \underset{\text{p-cycles}}{Z_p(K)} \subset \underset{\text{p-boundaries}}{B_p(K)}$. $\Longrightarrow H_p(K) \equiv \dfrac{Z_p(K)}{B_p(K)}$ = p^{th} homology group

  i.e. H_p = group of p-cycles that are not trivially cycles, i.e. are just boundaries.

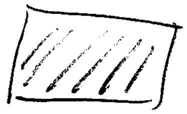
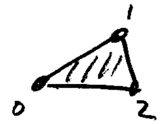
- H_p is independent of triangulation, even though Z_p, B_p, C_p are not.

- **Thm**: if K is contractible to a pt, $H_p(K) = \begin{cases} \{0\} & p \neq 0 \\ \mathbb{Z} & p = 0 \end{cases}$

- $\boxed{H_p \cong \dfrac{Z_p}{B_p} \cong G_p + T_p}$ G_p = abelian, uniquely generated by finite # generators. # generators = p^{th} Betty Number = # (p+1)-dim Holes in K!

  T_p = cyclic torsion subgroup. Contains twist info.

Example 1

Homology groups of  $\longrightarrow K =$  $= \{[012] + \text{faces}\}$

Chain groups:

$C_0(K) = \{a_0[0] + b_0[1] + c_0[2]\} \cong \mathbb{Z}^3$ $\qquad C_2(K) = \{a_3[012]\} \cong \mathbb{Z}$

$C_1(K) = \{a_1[12] + b_1[02] + c_1[01]\} \cong \mathbb{Z}^3$ $\qquad C_{3+}(K) = \{0\}$

Cycle groups:

- $\forall c_0 \in C_0(K)$, $\partial c_0 = 0 \rightarrow Z_0(K) = C_0(K) \cong \mathbb{Z}^3$
- if $z = a_1[12] + b_1[02] + c_1[01] \in C_1(K)$, $\partial z = 0 = (-a_1 - b_1)[0] + (a_1 - c_1)[1] + (b_1 + c_1)[2]$

  $\rightarrow a_1 = -b_1 = c_1 \Rightarrow Z_1(K) = \{a([12] - [02] + [01])\} \cong \mathbb{Z}$ (lin x )

- if $c \in C_2(K)$, $\partial c \neq 0 \Rightarrow Z_2(K) = \{0\}$

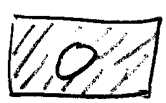
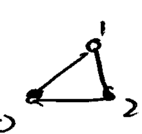
Boundary groups:

- $B_0(K) = \partial C_1(K) = \{(-a_1 - b_1)[0] + (a_1 - c_1)[1] + (b_1 + c_1)[2]\} \cong \mathbb{Z}_2$ since the coeffs add to zero
- $B_1(K) = \partial C_2(K) = \{a_3([12] - [02] + [01])\} = Z_1(K) \cong \mathbb{Z}$
- $B_2(K) = \partial C_3(K) = \{0\}$

Homology groups:

- $H_0(K) = \dfrac{Z_0(K)}{B_0(K)} \cong \dfrac{\mathbb{Z}_3}{\mathbb{Z}_2} = \mathbb{Z}$; $\quad H_1(K) = \dfrac{Z_1}{B_1} = \{0\}$; $\quad H_2(K) = \dfrac{Z_2}{B_2} = \{0\}$

Example 2

Homology group of  $\longrightarrow K' =$  $= \{[01] + [12] + [02] + \text{faces}\}$

$\rightarrow H_0(K') = H_0(K) = \mathbb{Z}$ since 1,2 simplex structure of K, K' the same

$\rightarrow$ $\underline{H_1(K')}$: $C_1(K') = C_1(K) = \mathbb{Z}^3$; $\quad Z_1(K') = Z_1(K) = \mathbb{Z}$

BUT $C_2(K') = 0 \rightarrow B_1(K') = \{0\}$

$$\Rightarrow H_1(K') = \frac{Z_1(K')}{B_1(K')} = \mathbb{Z} \text{ : detects hole!!}$$

Remarks

- Similar to homotopy groups, can define relative homology groups which also define an exact sequence. For $K \supset L$,

$$\cdots \xrightarrow{\partial_*} H_p(L) \xrightarrow{i_*} H_p(K) \xrightarrow{j_*} H_p(K;L) \xrightarrow{\partial_*} H_{p-1}(L) \xrightarrow{i_*} \cdots$$

This is actually MORE POWERFUL for homotopy groups, since there is eg an excision thm whereby $H_p(K,L) = H_p(K-L_0, L-L_0)$ where $L_0 \subset$ interior of L. THIS ALLOWS FOR RELATIVELY EASY CALCULATION OF $H_p(X)$ FOR MANY CASES.

↳ even so, can always calculate H_p by "brute force".

→ H_p contains a little less info than π_p, but we can always (?) calculate it!!!

- $H_p(X \times Y, \mathbb{Q}) = \bigoplus_{p=k+q} \left(H_k(X, \mathbb{Q}) \otimes H_q(Y, \mathbb{Q}) \right)$

$\mathbb{Q}$ means we take C_p coeffs to be rational instead of $\mathbb{Z}$. this removes torsion subgroup of H_p.

- H_p gives us Euler characteristic: $\chi(K) = \sum_{p=0}^{n} (-1)^p \underbrace{l_p}_{\#p\text{-simplexes}} = \sum_{p=0}^{n} (-1)^p \operatorname{rank}(G_p)$

Generalizing this to arbitrary topological space X

- Consider top. space X (not necessarily triangulable). (This will also be more suited to discussion of COHOMOLOGY!)

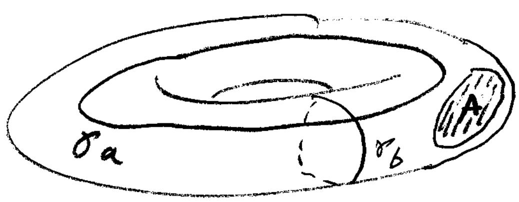
- Define "standard simplex" Δ_p : literally just representative abstract examples of p-simplexes, used to define the topology of a p-dim path

$\Delta_0 = \bullet \quad \Delta_1 =$  $\quad \Delta_2 =$  includes interior! etc

← regard these (?) as actual manifolds in their own right!

- Define singular p-chains λ_i^p in X as maps $\lambda_i^p : \Delta_p \to X$

eg $X =$ Torus



$\lambda^1_{a,b} : \Delta_1 \to X$ maps ⊷ to $\gamma_{a,b}$

$\lambda^2_A : \Delta_2 \to X$ maps  to patch

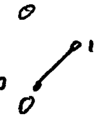
 on torus surface.

↳ Define singular p-chain group $S_p(X) = \{ \sum \lambda_i^p g_i \mid g \in \text{abelian } G \}$ (same as C_p)

- Define boundary map : $\partial \lambda^p = \sum\limits_{r=0}^{p} (-1)^r \underbrace{\lambda^p \circ F^r}_{r^{th} \text{ face of } \lambda^p}$

If $\lambda_A : \Delta_2 \to X$ takes  to 

$\lambda_A \circ F^1 : \Delta_1 \to X$ takes 0 — 2 to 

$\lambda_A \circ F^2 : \Delta_1 \to X$ takes  to 

etc

> Formal Def'n:
> F^r embeds Δ_{p-1} in Δ_p s.t. it Δ_{p-1} maps to r^{th} face of $\Delta_p \Rightarrow F_r : \Delta_{p-1} \to \Delta_p$
> $\lambda^p \circ F^r : \Delta_{p-1} \to X$ is just λ^p restricted to the r^{th} face of Δ_p

- Define singular homology for arbitrary top space X as

$$\boxed{H_p(X, G) = \frac{\mathrm{Ker}[\partial : S_p \to S_{p-1}]}{\mathrm{Im}[\partial : S_{p+1} \to S_p]}}$$

5. Cohomology

- Preliminary Remarks:
  - ↳ we introduced homology group $H_p(X, G)$. Previously we used $G = \mathbb{Z}$. Now let $G = \mathbb{R}$

infinitely ctsly diff'ble

  - ↳ Let X be a differentiable Manifold M. Require $f_p : \Delta_p \to M$ to be C^∞
    The group of C^∞ p-chains is $C_p(M)$.
  - ↳ Recall differential forms: $\omega = T_{i_1 \ldots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p} \in \Omega^p(M)$

exterior derivative $d\omega = \dfrac{\partial T_{i_1 \ldots}}{\partial x^{i_{p+1}}} dx^{i_{p+1}} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_p} \in \Omega^{p+1}(M)$

↳ $\boxed{d^2 = 0}$

Stokes' Thm $\displaystyle\int_M d\omega = \int_{\partial M} \omega$

pullback/pushforward: given a map $\phi : M \to N$,

vectors can be pushed forward $\phi_* : T_p M \to T_p N$

$(\phi_* X) f = X f(\phi(p))$

forms can be pulled back $\phi^* : T_p^* N \to T_p^* M$

$\langle \phi^* \omega, X \rangle = \langle \omega, \phi_* X \rangle$

closed form : $d\omega = 0$

exact form : $\exists \eta$ s.t. $\omega = d\eta$

- There is a duality between homology & cohomology, but coh. is more powerful (ring structure) and easier to use (local info)
  - ↳ Don't even need homology to construct cohomology groups!

- Integrate over a p-chain: for $c = \sum a_i \lambda_i \in C_p(M)$, $\omega \in \Omega^p(M)$,

define $\int_c \omega = \sum a_i \int_{\Delta_p} \boxed{\underset{\uparrow \text{pullback}}{\lambda_i^* \omega}}$ ← p-form on Δ_d

⟹ Denote $C^p(M)$ = set of all p-forms (<u>co-chains</u>) on $M = \Omega^p(M)$

Then $\omega \in C^p$ defines a map $\omega: C_p \longrightarrow \mathbb{R}$

$$\langle \omega, c \rangle = \int_c \omega$$

↳ boundary operator adjoint to ext. derivative by Stokes' Thm

$$\langle d\omega, c \rangle = \langle \omega, \partial c \rangle$$

⟹ <u>Homology</u>

$C_p(M)$ = set of all C^∞ p-chains on M

$Z_p(M) = \ker \partial_p$ = p-cycles

$B_p(M) = \operatorname{Im} \partial_{p+1}$ = p-boundaries

$H_p(M,\mathbb{R}) = \dfrac{Z_p}{B_p}$ = p^{th} homology group with real coeffs

<u>Cohomology</u>

$C^p(M) = \Omega^p(M)$

$Z^p(M) = \ker d_{p+1}$ = closed forms

$B^p(M) = \operatorname{Im} d_p$ = exact forms

$H^p(M,\mathbb{R}) = \dfrac{Z^p}{B^p}$ = p^{th} deRahm cohomology group with real coeffs.

- Duality Action: $([\omega],[c]) = \langle \omega, c \rangle$ for $[\omega] \in H^p$, $[c] \in H_p$.

- $\boxed{\dim H^p(M,\mathbb{R}) = \#\text{ of } (p+1)\text{-dim Holes in } M}$

- Spheres: $H^p(S^n,\mathbb{R}) = \begin{cases} \mathbb{R} & p=0,n \\ \{0\} & \text{else} \end{cases}$ Tori: $H^p(T^n,\mathbb{R}) = \mathbb{R}^\alpha$ with $\alpha = \binom{n}{p}$

- Define $H^*(M,\mathbb{R}) = \bigoplus_{p\geq 0} H^p(M,\mathbb{R})$. For $[\omega] \in H^p$, $[\nu] \in H^q$, the CUP PRODUCT $[\omega] \cup [\nu] = [\omega \wedge \nu] \in H^{p+q}$ makes H^* a <u>RING</u> ($+,\circ$, no mult. inverse)

  ↳ Two spaces might have same cohomology groups but different ring structure!

Sample Calculation: show $H^1(S^1,\mathbb{R}) = \mathbb{R}$

- We'll need Poincaré's Lemma: If M is contractible to a pt, all closed forms on M are exact.

- S^1 has dim 1, so if $\omega \in \Omega^1(M)$, $d\omega \in \Omega^2(M) \Rightarrow d\omega = 0$
  $\Rightarrow$ all 1-forms on S^1 are closed.

- ω exact ($\omega = d\eta$) $\Rightarrow \int_{S^1} \omega = 0$ by Stoke's Thm

- Claim: $\int_{S^1} \omega = 0 \Rightarrow \omega$ exact

  $\hookrightarrow$ Proof: Consider S^1 with one point deleted: $\tilde{S}^1 = S^1 - \{p\}$.

  $\int_{S^1} \omega = 0 \Rightarrow \int_{S^1 - \{p\}} \omega = 0$ since $\{p\}$ has measure 0

  Since $\tilde{S}^1$ is contractible, $\omega = df$ on $\tilde{S}^1$. Since f is C^∞, it must be periodic. Hence $\omega = df$ is defined on whole S^1 $\square$

- Classify members of $H^1(S^1,\mathbb{R})$: say ω, ω' are closed but not exact, ie $d\omega \neq 0 \Rightarrow \int_C \omega \neq 0 \Rightarrow$ can define $c = \dfrac{\int_{S^1}\omega}{\int_{S^1}\omega'} \in \mathbb{R}$

Then $\int (\omega - c\omega') = 0 \Rightarrow$ different elements of $H^1(S^1,\mathbb{R})$ differ only by a "real-number"

$$\Rightarrow \boxed{H^1(S^1,\mathbb{R}) = \mathbb{R}}$$