"FIBER BUNDLES FOR PHENO BUMBLERS"

THIS WILL BE A VERY HEURISTIC TALK HIGHLIGHTING SOME BASIC APPLICATIONS OF THE BUNDLE FRAMEWORK TO PARTICLE PHYSICS. WE WILL AVOID MATHEMATICAL RIGOR AND INSTEAD TRY TO MOTIVATE WHY ONE SHOULD CARE ABOUT GEOMETRY IN OUR LINE OF WORK.

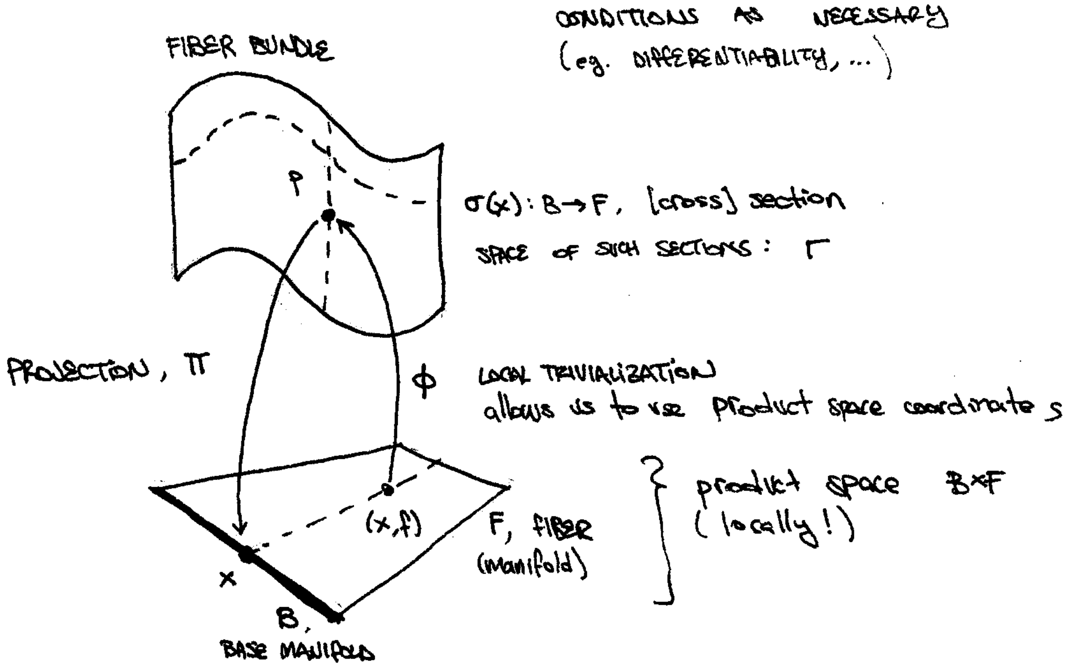
[HOPEFULLY SOME OF THESE TOPICS WILL BE EXPLORED MORE THOROUGHLY IN FUTURE JOURNAL CLUB TALKS!]

MY FAVORITE REFS

- GÖCKELER + SCHÜCKER Diff Geo, Gauge th., Grav.
- BERTLMANN Anomalies in QFT
- DANIEL + VIALLET 1980
- NAKAHARA
- GREEN, SCHWARZ, WITTEN vol II
- MORITA Geometry + Diff forms
- FRANKEL The Geom. of Phys.
- COLLINUCCI + WIJNS
- GIELEN + PERRY : Applications of Diff Geo to Phys.
- MAYER Intro to the fiber-bundle approach to gauge thy
- DRECHSLER Gauge thy ... formulated on a fiber bundle of Cartan type
- FLIP'S A-EXAM ON INSTANTONS

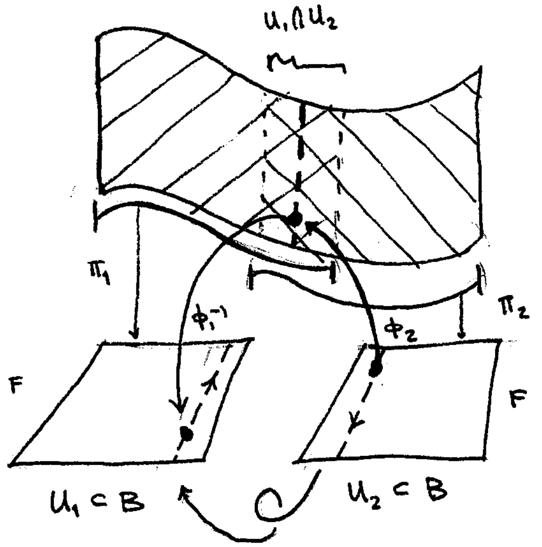
REVIEW OF THE BASIC IDEA

WE WON'T BE MATHEMATICALLY RIGOROUS. IMPLICITLY ASSUME ALL "NICENESS" CONDITIONS AS NECESSARY (eg. DIFFERENTIABILITY, ...)



CONSISTENCY: $\pi \circ \phi(x,f) = x$

LOCALLY TRIVIAL, GLOBALLY NON-TRIVIAL (Grass roots movement: ACT LOCALLY, THINK GLOBALLY!)

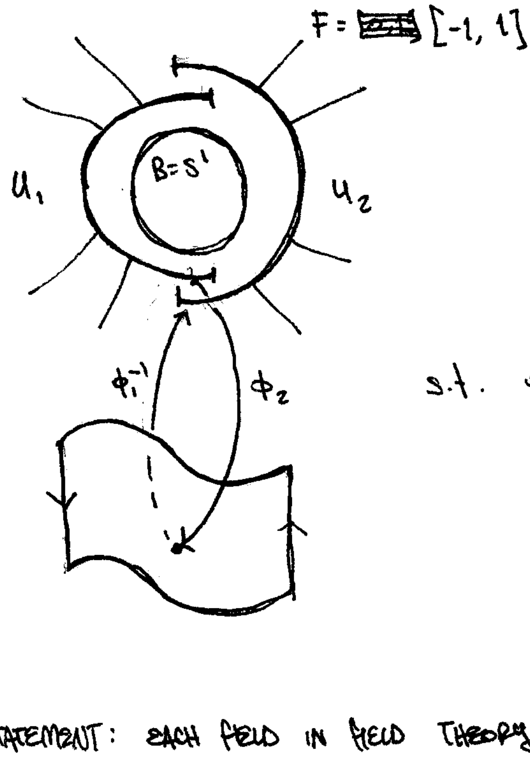


$$\phi_1(x,f) = \phi_2(x, \underbrace{\phi_2^{-1}\circ\phi_1(x)}_{\equiv g_{21}(x)} f)$$

[where $\phi(x) f = \phi(x,f)$]

$\phi_1^{-1}\circ\phi_2 = g_{12} \in G$, STRUCTURE GROUP (assumed: representation which acts on F) } TELLS US HOW LOCAL PATCHES SHOULD BE GLUED TOGETHER ↳ info ABOUT 'TWISTS'

IF YOU'RE CONFUSED, THINK OF THE MOBIUS STRIP



s.t. $\phi_1^{-1}\circ\phi_2(x) \in \mathbb{Z}_2$ ACTING ON THE INTERVAL BY $f \to -f$.

TRIVIAL STATEMENT: EACH FIELD IN FIELD THEORY IS A BUNDLE OVER MINKOWSKI SPACETIME

eg. A SCALAR FIELD IS A $\mathbb{R}$ BUNDLE OVER $M^{3,1}$
$\mathbb{C}$ —— " —— IS A $\mathbb{C}$ LINE BUNDLE OVER $M^{3,1}$
SPINOR FIELD IS A SPIN BUNDLE OVER $M^{3,1}$ (more on this later)
} technically, a SECTION of THESE BUNDLES, $f(x)$.

Q: WHEN DO WE GET NONTRIVIAL TWISTING?

SOME GENERAL IDEAS TO KEEP IN MIND

ONE of THE THEMES HERE IS THE RELATION BETWEEN <u>LOCAL</u> & <u>GLOBAL</u> PROPERTIES.

<u>LOCAL</u> $\downarrow$	<u>GLOBAL</u> $\downarrow$
Geometry	Topology
CURVATURE	TWISTS
CALCULUS	BOUNDARIES
"CONTINUOUS"	INVARIANTS, INDICES

$\uparrow$

the LANGUAGE of PHYSICS! [YOU SHOULD ALREADY BE FAMILIAR W/ THE FORMULATION of CLASSICAL E&M IN TERMS of DIFFERENTIAL FORMS.]

THIS RELATION BETWEEN LOCAL & GLOBAL IS WELL KNOWN FROM THE GENERAL STOKES' THEOREM:

$$\int_B d\omega = \int_{\partial B} \omega$$

INTEGRAL OVER A SPACE $\rightarrow$ INTEGRAL OVER THE BOUNDARY

A "CONVERSELY" RELATED IDEA IS <u>de Rham COHOMOLOGY</u>.

RECALL: ~~POINCARÉ LEMMA~~: $d^2 = 0 \Rightarrow$ ALL <u>EXACT</u> FORMS ($F = dA$) ARE CLOSED ($dF = 0$)

<u>POINCARÉ LEMMA</u>: FOR <u>$\star$-SHAPED</u> REGIONS, CLOSED $\Leftrightarrow$ EXACT

TOPOLOGY!

IN GENERAL THIS IS NOT TRUE. THE EXTENT TO WHICH THIS IS NOT TRUE IS GIVEN BY THE de ~~Rham~~ Rham COHOMOLOGY GROUP:

$$H^p = \text{CLOSED FORMS} / \text{EXACT FORMS} \qquad (\text{OVER A DOMAIN})$$

THIS WILL TURN OUT TO BE IMPORTANT FOR CLASSIFYING TOPOLOGICAL INVARIANTS! eg. $\dim(H^p) = p^{th}$ <u>BETTI NUMBER</u> w $\sum_{p=0}^{n} (-1)^p \dim(H^p) =$ <u>EULER CHARACTERISTIC</u> (HOLE-Y-NESS OF M)

eg: FOR $J_M = 0$ (no mag monopoles), $F = dA$. F IS EXACT $\Rightarrow$ CLOSED.

BUT FOR $J_M \neq 0$, WE KNOW WE GET DIRAC QUANTIZATION FROM AHARONOV-BOHM PHASE $\exp(i\int_\gamma A) = \exp(i\int_\sigma F)$. NOW $dF = J_M$, NOT CLOSED. $\nexists$ SMOOTH 1-FORM A GLOBALLY $\longrightarrow \exists$ 'HOLES' IN DOMAIN OF A SINGLE 'GLOBAL' GAUGE FIELD.

$\hookrightarrow$ MONOPOLES ARE THE de Rham COHOMOLOGY OF THE EM POTENTIAL.

GAUGE THEORY : the PRINCIPAL BUNDLE AND ITS FRIENDS.

Physics: the GAUGE degree of freedom is a redundancy in our description of a physical system; such a redundancy is convenient because it allows us to write fields in nice representations of the Lorentz group & write nice actions.

But: at the end of the day the physical system is the gauge description modulo the gauge redundancy.

Mathematics: the gauge redundancy will be a fiber over our spacetime manifold. PHYSICS IS 'INSENSITIVE' TO THE FIBER (WE WILL CLARIFY THIS). GAUGE FIXING = A SECTION OF THE BUNDLE.

Q: WHAT IS THE FIBER DESCRIBING GAUGE REDUNDANCY?
A: THE GAUGE GROUP ITSELF.

PRINCIPAL FIBER BUNDLE: THE FIBER IS THE STRUCTURE GROUP.
LIE GROUPS ARE MANIFOLDS
TANGENT SPACE ~ LIE ALGEBRA.

DOES THIS MAKE SENSE? YES: THE STRUCTURE GROUP MUST ACT ON THE FIBER. THIS IS GIVEN BY THE USUAL LEFT MULTIPLICATION $G \times G \to G$.

BUT WE ALSO GET SOMETHING ELSE FOR FREE: RIGHT MULTIPLICATION, WHICH COMMUTES w/ LEFT MULTIPLICATION.

eg. SUPERSPACE IS A PRINCIPAL FIBER BUNDLE WITH GRASSMAN FIBER (RING w/ $\theta, \bar{\theta}$). LEFT MULTIPLICATION GENERATES MOTION ALONG THE FIBERS $(Q, \bar{Q})$, WHILE RIGHT MULTIPLICATION IS THE SUSY COVARIANT DERIVATIVE. WE WILL MENTION CONNECTIONS ON A FIBER BUNDLE SHORTLY.

SEE: · WESS + BAGGER CH IV FOR BASIC PICTURE.
· GIERES Geometry of SUSY Gauge Theories FOR A MORE FORMAL DESCRIPTION.
· KUZENKO + BUCHBINDER.

eg. (Nakahara 9.7) THE DIRAC MONOPOLE IS A PRINCIPAL $U(1)$ BUNDLE OVER S^2, SPATIAL INFINITY.

IN ORDER TO DESCRIBE GAUGE FIELDS WE NEED MORE GEOMETRIC MACHINERY: <u>CONNECTIONS ON PRINCIPAL FIBER BUNDLES</u>.

YOU ALREADY KNOW WHAT THIS MEANS IN GR
IN FACT, WE ALREADY KNOW THAT THE YANG-MILLS FIELD MUST SOMEHOW BE IDENTIFIED WITH THIS.

<u>Refresher & Motivation</u>: CONNECTION IN GR

IN GR WE WERE INTERESTED IN THE TRANSFORMATION OF VECTORS & 1-forms AS THEY WERE DRAGGED ALONG A SPACETIME MANIFOLD. IN BUNDLE LANGUAGE:

WE SEPARATED "PHASE SPACE" INTO A FLAT TANGENT PLANE T_pM @ EACH POINT IN SPACETIME $p \in M$. THIS IS THE TANGENT BUNDLE. THE CONNECTION TOLD US HOW TO SEPARATE THE FIBER (TM) FROM THE BASE (M) AND DESCRIBE THE TRANSFORMATION ON THE FIBER AS IT IS PARALLEL TRANSPORTED ALONG THE BASE.

[THE <u>CONNECTION</u> IS THE FUNDAMENTAL OBJECT + IS DEFINED EVEN WHEN THE MANIFOLD DOESN'T HAVE A METRIC!]

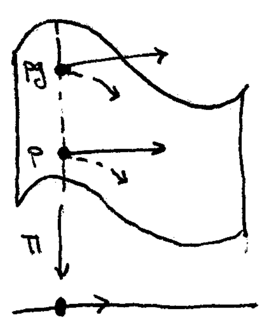
FOR A PRINCIPAL BUNDLE, A CONNECTION IS THE SEPARATION OF

$$T_pP = \underbrace{V_pP}_{\substack{\text{VERTICAL} \\ \sim \text{fiber}}} \oplus \underbrace{H_pP}_{\substack{\text{HORIZONTAL} \\ \sim \text{base}}} \qquad \forall p \quad (\text{smoothly})$$

FURTHER (cf SUPERSPACE), WE USE THE <u>RIGHT</u> MULTIPLICATION DEFINED FOR OUR STRUCTURE GROUP / FIBER TO FORCE ALL POINTS IN $\pi^{-1}(x)$ TO BE PARALLEL TRANSPORTED IN THE SAME WAY:

$$H_{pg}P = \underbrace{R_{g*}}_{} H_pP$$

MAP THAT SENDS TANGENT VECTORS @ p TO TANGENT VECTORS @ pg



THE IDEA IS THAT: WHILE OUR (LOCAL) TRIVIALIZATION HAS A WELL DEFINED SEPARATION BETWEEN THE BASE & THE FIBER, THE FIBER BUNDLE ITSELF IS JUST SOME MANIFOLD. HOW DO WE SEPARATE FIBER FROM BASE?

ON A PRINCIPAL BUNDLE WE SEPARATE BASED ON THE ACTION OF THE GROUP G; THE HORIZONTAL TANGENT SPACE IS UNAFFECTED BY RIGHT-MULTIPLICATION BY ELEMENTS OF G.

BY THE WAY: THE VERTICAL TANGENT SPACE IS ISOMORPHIC TO THE LIE ALGEBRA (tautologically).

OUR GAUGE FIELDS TAKE VALUES HERE... WE'RE ON THE RIGHT TRACK.

this isomorphism is made manifest via the MAURER-CARTAN form

$g^{-1}dg$; "LIE-ALGEBRA-VALUED ONE-FORM"

SYSTEMATIC SEPARATION OF $V_pP \oplus H_pP$ IS GIVEN BY THE CONNECTION 1-FORM

ω SUCH THAT (1) $\omega(X_{v_i}) = T^i$

LIE ALG VALUED

VECTOR FIELD ON P GENERATED BY $T^i \in \mathcal{L}(G)$, LIE ALG OF G

$$X \in T_pP \quad \longrightarrow \quad (2) \quad (R_g)^* \omega_p(X) = \underbrace{g^{-1}\omega(X)g}_{\text{ACTS AS THE ADJOINT}}$$

$R_g : p \mapsto pg$

$(R_g)^*$ TAKES 1-FORMS AT pg AND GIVES 1-FORMS AT p.

$$(R_g)^* \omega_p(X) = \omega_{pg}(\underbrace{(R_g)_*}_{\text{SENDS } X \text{ TO ITS VALUE AT } pg}X)$$

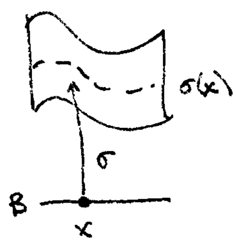
THE HORIZONTAL TANGENT SPACE IS THE COMPLEMENT OF V_pP.

SANITY CHECK: IF $X_H \in H_pP \subset T_pP$, THEN $R_{g*}X_H \in H_{pg}P$

↳ follows from (2): $(R_g)^* \omega_p(X_H) = g^{-1}\underbrace{\omega(X_H)}_{=0}g = 0$ ✓

NOW WE GET TO THE GAUGE FIELDS THAT YOU'RE FAMILIAR WITH.

DEFINE A LOCAL SECTION FOR EACH LOCAL TRIVIALIZATION

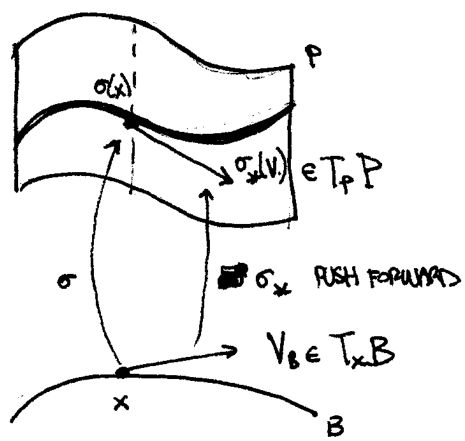


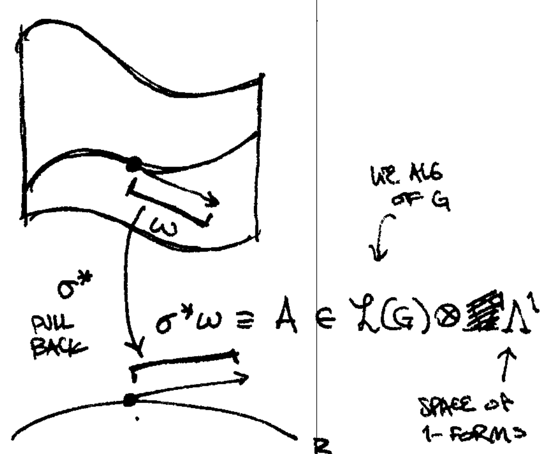
THIS IS A CHOICE OF <u>GAUGE</u>.

THE <u>GAUGE POTENTIAL</u> IS: $\boxed{A \doteq \sigma^* \omega}$

ω: LIE-ALGEBRA-VALUED 1-FORM THAT TAKES TANGENT VECTORS OF P

A: LIE-ALGEBRA-VALUED 1-FORM THAT TAKES TANGENT VECTORS OF B.





<u>FACTS</u> (FOR PROOFS: SEE REFS, OR MAYBE MARIO'S TALK)

- GIVEN $A, \sigma \longrightarrow \exists$ <u>UNIQUE</u> CONNECTION 1-FORM ω
  - $\omega_i = \omega_j$ ON INTERSECTION OF TWO LOCAL TRIVIALIZATIONS
- THIS ENFORCES COMPATIBILITY CONDITIONS ON THE A_i

$$A_j = g_{ij}^{-1} (A_i + d) g_{ij}$$

→ GAUGE TRANSFORMATION!
PHYSICS CONTENT: WE CAN DESCRIBE DIFFERENT PATCHES W/ DIFF DESCRIPTIONS, BUT THEY CAN ONLY DIFFER BY A GAUGE TRANSFORMATION.

Remark: DERRICK'S THM: NO SOLITONIC SOLUTIONS FOR $d>1$

↳ WICK ROTATE & LOOK @ TUNNELING SOLUTIONS
NOT POSSIBLE: $\prod_{\substack{\infty\,\#\\ \text{of osc.}}} e^{-\Delta E} = 0$

THE WAY OUT: GAUGE REDUNDANCY ALLOWS US AN UNPHYSICAL DEGREE OF FREEDOM TO WRAP & TWIST OUR FIELD CONFIGURATIONS.

CURVATURE & FIELD STRENGTH

USUAL EXTERIOR DERIVATIVE

EXTERIOR COVARIANT DERIVATIVE: $D\alpha(X_1, \dots, X_{p+1}) = d\alpha(\underbrace{X_1^H, \dots, X_{p+1}^H}_{\text{PROJECT TO HORIZONTAL SUBSPACE!}})$

takes p-FORM → (p+1) FORM

CURVATURE 2-FORM: $\Omega \doteq D\omega$ ← LIE-ALG-VALUED 2-FORM

CONNECTION 1-FORM

TRANSFORMATION OF Ω: $R_g^* \Omega = g^{-1}\Omega g$ ← ADJOINT.
(pf: SEE REFS)

CARTAN'S STRUCTURE EQN: $\Omega = d\omega + \omega\wedge\omega = d\omega + \frac{1}{2}[\omega,\omega]$

FIELD STRENGTH: $F = \underbrace{\sigma^*\Omega}_{\text{of } A \text{ from } \omega} = dA + A\wedge A$

ACTION: $S = \frac{1}{g^2}\int F\wedge * F$

Remark: WE CAN NOW CONSIDER HOLONOMIES AS WE MAP LOOPS IN B TO TRANSFORMATIONS ON P.
WE KNOW THESE BETTER AS WILSON LOOPS AND SUCH STRUCTURES PLAY A KEY ROLE IN LATTICE QCD!

(SOMEWHAT SURPRISING "INTUITIVELY" SINCE LATTICE SEEMS TO DO VIOLENCE TO OUR GEOMETRY.)
[the point: only SPACETIME GEOMETRY IS MESSED UP — GAUGE GEOMETRY IS LEFT INTACT!]

SO FAR, WE'VE DESCRIBED A WORLD OF GLUE.
TO INCLUDE MATTER FIELDS WE MUST INTRODUCE ANOTHER TYPE OF BUNDLE.

ASSOCIATED <u>VECTOR BUNDLE</u> $\approx$ (PRINCIPAL BUNDLE) $\times$ (VECTOR SPACE)

FIBER IS A VECTOR SPACE

ASSOCIATED TO A PRINCIPAL BUNDLE:
THE GROUP G ACTS ON THE VECTOR SPACE ON THE LEFT

the REP OF G ACTING ON THE FIBER GIVES REP OF MATTER FIELD

eg. $P \times V$ w/ ELEMENT (p, v)

ACTION OF G ON $P \times V$ IS $(p,v)\cdot g = (pg, \mathrm{rep}(g^{-1})\, v)$

WE WANT TO MOD OUT BY THE GAUGE REDUNDANCY.
SO THE <u>ASSOCIATED BUNDLE</u> IS

$$P \times V / G$$

GROUP ACTION ON $P \times V$ DEF ABOVE

$$\pi_{P\times V/G}\left([(p,v)]\right) = \pi_P(p)$$

EQUIVALENCE CLASS
UNDER G ORBIT

ASIDE: GRAVITY AS A GAUGE THEORY – (EINSTEIN-CARTAN GRAVITY)

Physics: LOCALLY, CURVED SPACETIME IS DESCRIBED BY FLAT INERTIAL FRAMES

WE CAN DEFINE A FRAME BUNDLE OVER SPACETIME → ea frame related by a $GL(n,\mathbb{R})$ transf.

LOCALLY FLAT INERTIAL COORDINATES
BUT THE BASIS VECTORS CHANGE w/ POSITION
"REPERE MOBILE" – moving frame

FOR A METRIC SPACE (AS IN GR), WE CAN IMPOSE AN ORTHONORMAL FRAME
→ BASIS of TANGENT VECTORS $e^a_\mu(x)$ ← "CONVERTS SPACETIME INDEX TO TANGENT SPACE INDEX."

a : tangent space index (FLAT)
μ : SPACETIME index (CURVED)
x : POSITION ON SPACETIME

"SQUARE ROOT" OF METRIC

$$\text{s.t.}\quad e^a_\mu(x)\, e^{b\mu} = \eta^{ab}, \qquad e^a_\mu(x)\, e_{a\nu}(x) = g_{\mu\nu}(x)$$

THESE ARE CALLED VIELBEINS OR TETRADS.

EVEN THOUGH WE IMPOSE ORTHONORMALITY, THERE IS STILL A FREEDOM IN HOW WE DEFINE THE FRAME AT EACH SPACETIME POINT: LORENTZ TRANSFORMATIONS, $SO(n-1,1)$

OUR ORTHONORMAL FRAME BUNDLE HAS STRUCTURE GROUP $SO(n-1,1)$
→ this is a gauge theory, with a gauge field.
WHAT IS IT? → "SPIN CONNECTION" → important in RS ☺

THE SAME MACHINERY AS GAUGE THEORY CARRIES OVER.
→ CARTAN'S STRUCTURE EQNS

$$S \sim \int F \wedge * F \;\longrightarrow\; \int R^\mu{}_\nu \wedge * \left(e^\nu_a dx^a \wedge e_{\mu b} dx^b\right)$$

TOPOLOGY: GAUSS-BONNET THM RELATES $\int$ CURVATURE ~ EULER #

- CAN COUPLE SPIN CONNECTION TO MATTER (ASSOCIATED BUNDLE) ← not seen in $GL(n,\mathbb{R})$!
  FOR SPINORS, MUST TAKE SPIN REP OF LORENTZ GROUP
  TECHNICALLY, MUST INTRODUCE SPINOR BUNDLE WHOSE STRUCTURE GROUP IS UNIVERSAL COVER OF LORENTZ.

NOW I SHOULD GO ON & SHOW WHAT KINDS OF THINGS YOU CAN DO WITH THE BUNDLE FORMULATION OF GAUGE THEORY... BUT AFTER WINTER CAMP YOU'RE ALL ALREADY FAMILIAR WITH THE OBVIOUS EXAMPLES!

↳ MONOPOLES
STRINGS
DOMAIN WALLS
INSTANTONS

very interesting, but now ∃ many experts in our group.

THESE ARE ALL VARIATIONS OF THE SAME THEME
- NONTRIVIAL GAUGE FIELD CONFIGURATION
- CLASSIFY VACUA BY TOPOLOGICAL CHARGE
- INTERSECTION OF TWO PATCHES ($\sim S^n$), FIELD CONFIG CAN ONLY DIFFER BY A GAUGE TRANSFORMATION.

INSTEAD OF DWELLING ON THESE, LET'S MENTION (W/O MUCH DISCUSSION) SOME INTERESTING "ADVANCED" APPLICATIONS

- WE WANT A WAY TO CLASSIFY GLOBAL INVARIANTS OF OUR BUNDLES.
  ↳ <u>CHARACTERISTIC CLASS</u>: POLYNOMIALS IN THE CURVATURE WHICH ARE CLOSED FORMS, BUT NOT <u>GLOBALLY</u> EXACT.
  ⇒ POINCARÉ LEMMA: ONLY LOCALLY EXACT ON EA PATCH
  ↳ INTEGRALS OF THESE OBJECTS DEPEND ONLY ON THE BUNDLE'S TRANSITION FUNCTIONS!

  THIS IS A CLASSIFICATION BASED ON de RHAM COHOMOLOGY

- A SIMILAR IDEA IS THE <u>CHERN-SIMONS FORM</u>
  ⇒ WE GET OBJECTS CALLED CHERN CLASSES. A VERY FAMILIAR OBJECT:
  2^{ND} CHERN CLASS HAS A CS FORM GIVEN BY:

$$\omega_3(A) = \frac{1}{8\pi^2} \operatorname{Tr}\underbrace{(A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A)}_{\text{very familiar from } \underline{\text{ANOMALIES}}}$$

- THE <u>ATIYAH-SINGER INDEX THEOREM</u> RELATES CHAR. CLASSES TO THE INDEX OF ELLIPTIC (FREDHOLM) OPERATORS, LIKE THE DIRAC OPERATOR.
  ⇒ TOPOLOGICAL ORIGIN OF ANOMALIES

- GEOMETRY OF BRS COHOMOLOGY
- GEOMETRY OF SUPERGRAVITY
- TOPOLOGICAL FIELD THEORY