NOTES ON CONNECTIONS ON BUNDLES

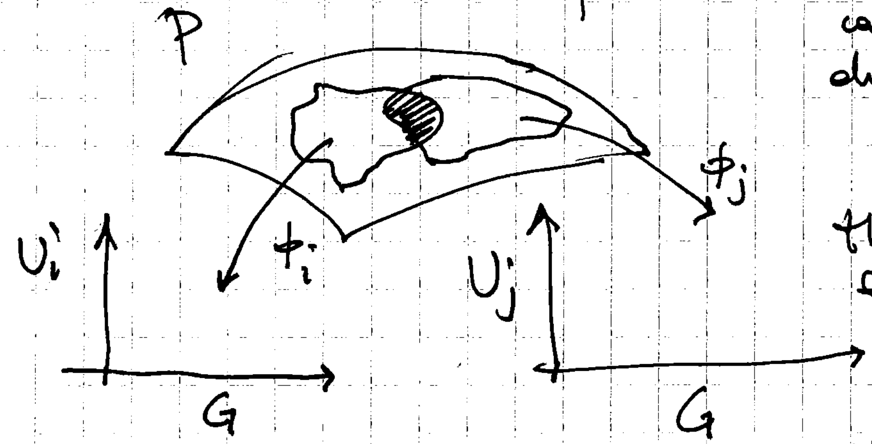
Before getting into the main topic of these notes, let me remind few things on principal bundles which we'll be using throughout

PRINCIPAL BUNDLES

* We will indicate the principal bundle as P. Recall the main features:
  1. P is the total space
  2. M is the base manifold. $\exists\, \pi : P \to M$ & π is called, for obvious reasons, the PROJECTION MAP
  3. G which is at the same time the STRUCTURE GROUP & the FIBER of the bundle. In general $P \not\cong M \times G$. We'll always assume G is a Lie Group.

On this space there are two further important maps:

4. $\phi_i : P \to U_i \times G$ called "LOCAL TRIVIALIZATION" which basically make our bundle treatable in the sense that we can do calculation with it. ϕ_i is the analog of a chart in the case of manifold. As usual if $\exists\, \phi_i : U_j \cap U_i \neq \emptyset$ then we have two ways of describing the same chunk of bundle. In particular there will be a point $u \in P$ which can be decomposed in two different ways:



$$\phi_i(u) = (x, g_i)$$
$$\phi_j(u) = (x, g_j)$$

the reason why there is the same x is because to identify x we have the map π:

$$\pi(u) = x$$

which is then a "trivialization independent" statement. It follows that the difference in the two descriptions can only be in the G part. Then we can connect the two descriptions by introducing the * TRANSITION FUNCTIONS:

$$\phi_i(P_{U_j \cap U_i}) = t_{ij}(U_j \cap U_i)\, \phi_j(P_{U_j \cap U_i})$$

where $P_{U_j \cap U_i}$ has the obvious meaning of being the chunk of principal bundle "above" $U_j \cap U_i$ [formally $P_{U_j \cap U_i} = \pi^{-1}(U_j \cap U_i)$]

⑤ Last but not least we have sections of the bundle:

$\sigma : M \longrightarrow P$ the importance of σ is that it defines a natural local trivialization, ϕ^σ [the i refers to the fact that σ is actually a map$_i$ from $U_i \longrightarrow \pi^{-1}(U_i)$].

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ϕ_i^σ acts as follows: $\phi_i^\sigma(u) = (x, g)$ where $u = \sigma(x) g$.

or better $\phi_i^\sigma(\sigma(x)) = (x, \mathbb{1})$

RIGHT ACTION of G on P

To understand the above formula we have to introduce a right action of the structure group on the bundle ~~from the right~~. ~~To do that we'll use~~

$$\phi_i(u') = \phi_i(ug) = (x, \tilde{g} g) \quad \text{where } \phi_i(u) = (x, \tilde{g})$$

this definition is intrinsic in the sense that we do not need any specific local trivialization. This is very important. The right action of the group, gives us an ~~way~~ intrinsic way of identifying the motion along the fiber. We in fact know that $\tilde{u} : \{u \in P : u = \tilde{u} g \ \forall g \in G\}$ is isomorphic to G [in other words the right action of the group "spans" the fiber].

SUMMARY

$P \rightarrow$ principal bundle

$M \rightarrow$ base manifold

$G \rightarrow$ fiber + structure group

$\phi_i^\sigma \rightarrow$ local trivialization. We'll always use ϕ_i^σ the ones induced by σ. $\phi_i^\sigma(u) = (x, g)$ where $\sigma(x) g = u$

or $\phi_i^\sigma(\sigma(x)) = (x, \mathbb{1})$.

$\sigma \rightarrow$ the section is only defined locally $\sigma_i : U_i \rightarrow \pi^{-1}(U_i)$ on $U_i \cap U_j$ they obviously differ by elements of G so $\sigma_i(x) = \sigma_j(x) g_{ji}(x)$

CONNECTION, HEURISTIC

AIM: Define the notion of "moving along the base manifold."

PROBLEMS: You might think that that is very easy. A curve on P which is just along the base manifold is one which, while moving, keeps the "group variable" constant, formally:

$$P \ni \tilde{\gamma}(t) \text{ is "along } M\text{" if } \phi_1^{\sigma}(\tilde{\gamma}(t)) = (\gamma(t), g)$$

where g does not dep. on t.
Is this def. good (meaning intrinsic)?
Obviously not! Suppose we use another section $\tilde{\sigma}$ to trivialize our bundle: $\tilde{\sigma}(\gamma(t)) = \sigma(\gamma(t))\tilde{g}(t)$
then $\phi^{\tilde{\sigma}}$ sets the "origin" in the group coordinates (that is $\phi_2^{\tilde{\sigma}}(\tilde{\sigma}(t)) = (\gamma(t), \mathbb{1})$ ...) differently than ϕ_1^{σ}, explicitely in the new coordinates:

$$\phi_2^{\tilde{\sigma}}(\tilde{\gamma}(t)) = \phi_2^{\tilde{\sigma}}(\sigma(t)g) = \phi_2^{\tilde{\sigma}}(\tilde{\sigma}(t)\tilde{g}^{-1}(t)g)$$

by our def. of right action:

$$\phi_1^{\tilde{\sigma}}(\tilde{\sigma}(t)\tilde{g}^{-1}(t)g) = (\gamma(t), \tilde{g}^{-1}(t)g)$$

so now $\tilde{\gamma}(t)$ does move along G!

<u>OUR ATTEMPT FAILED!</u>

VERTICAL DIRECTION: There is no problem instead to def. a curve just along the fiber G (we'll call it vertical direction...). This is so since we can use the intrinsic projection map to get Trivialization independent statements.
In fact $\tilde{\gamma}_v(t)$ is along the fiber if:

$$\pi(\gamma_v(t)) = x_0 \in M \quad \forall t \in [0,1]$$

that is if its projection map is constant.
We said already that changing trivialization only affect the group part $(x, g) \to (x, \tilde{g})$

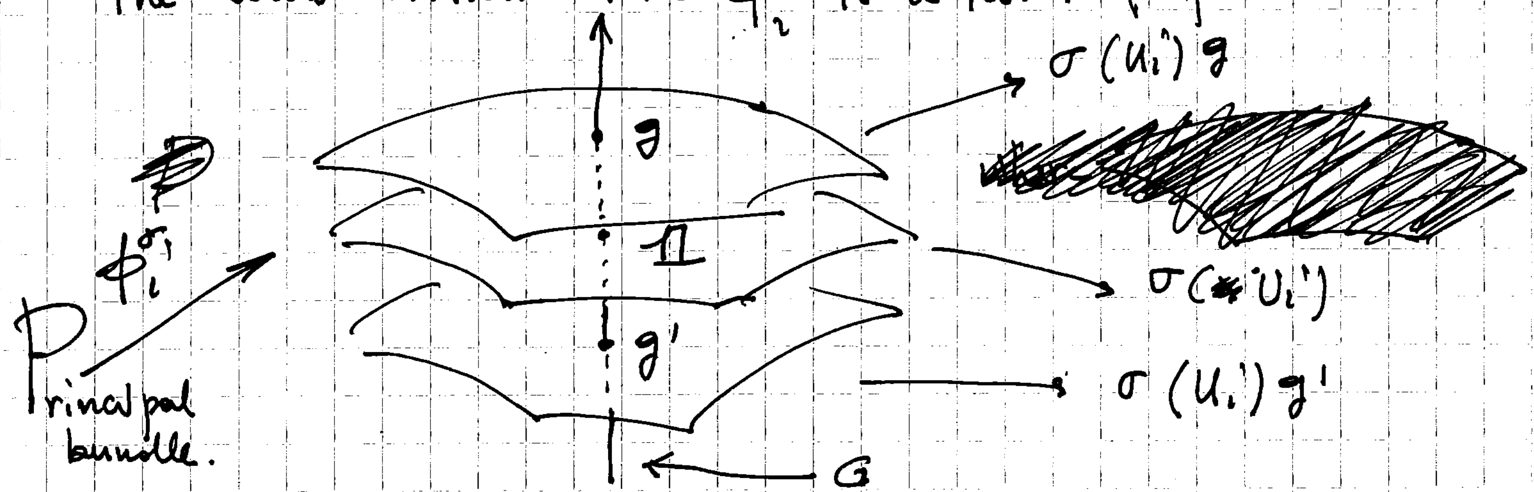
so if x_0 is constant in one trivialization is constant in any other: $\phi_i^{\sigma}(\gamma_v(t)) = (x_0, g(t))$

so $\phi_i^{\tilde{\sigma}}(\gamma_v(t)) = (x_0, \tilde{g}^{-1}(x_0)\, g(t))$

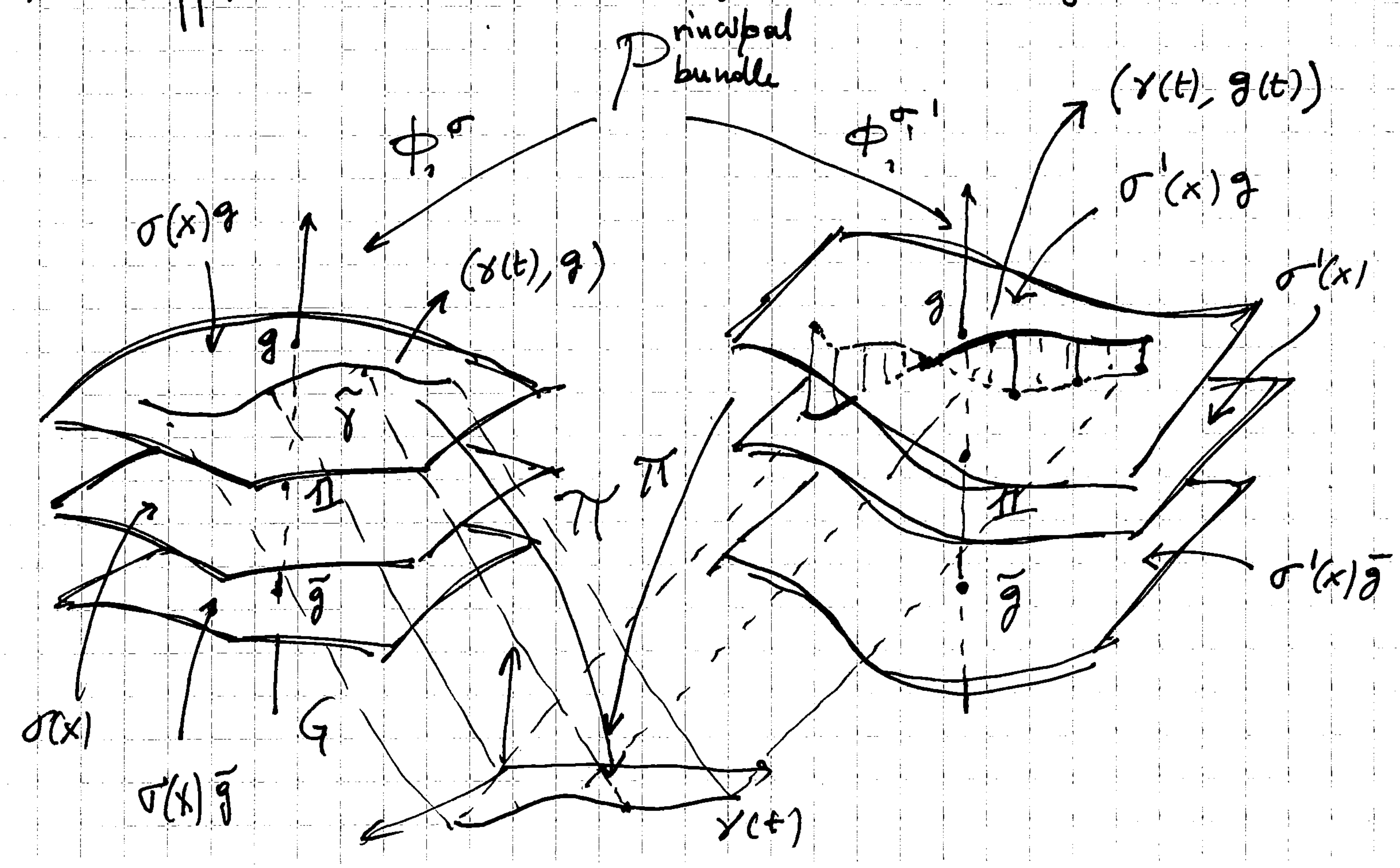
where $\tilde{\sigma}(x_0)\, g(x_0) = \sigma(x_0)$.

PICTURES, let's try to understand it better with some draws:

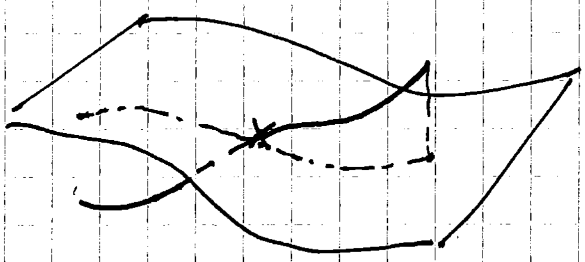
The local trivialization ϕ_i^{σ} is a kind of foliation:



now suppose we have σ & σ' & $\tilde{\gamma}$ is constant along G w.r.t. σ:



In the first case $\tilde\gamma$ lies fully on one "slice" so g is constant but in the second case $\tilde\gamma$ "punctures" the $g\,\sigma$' slice:



so it is not constant along the fiber G anymore indicate previously as the z axis.

In order to define uniquely when a curve in P moves only along the manifold we need a further mathematical structure: <u>A CONNECTION</u>.

HOW DOES A CONNECTION LOOK LIKE?

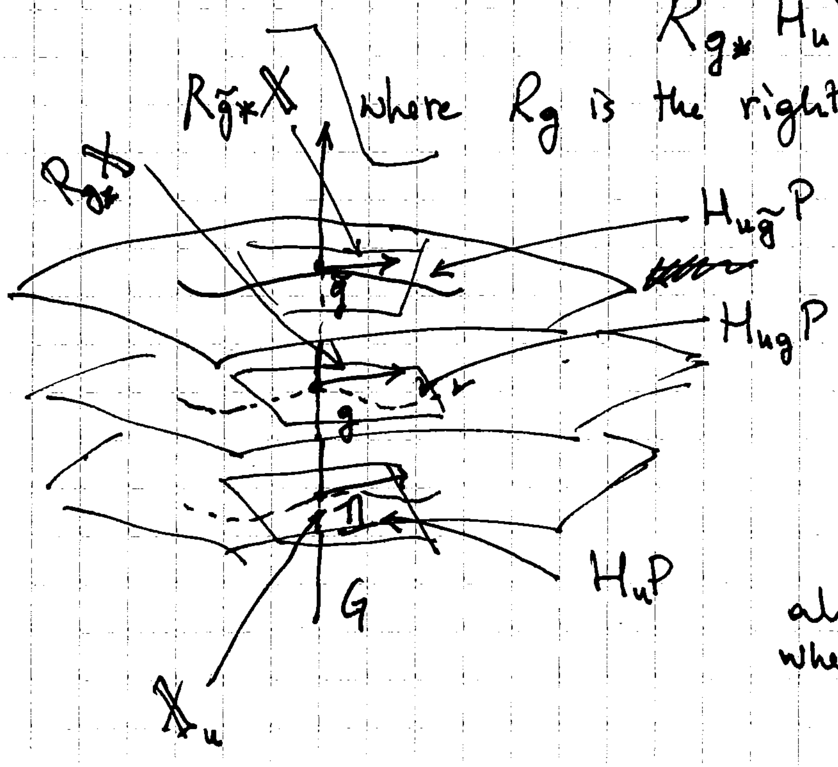
First of all we should formalize better what we are asking for. If we could split the tangent space of P in vertical & horizontal subspaces we could call the ~~direction~~ horizontal direction as the one "along the manifold". So we want an object so that:

$$T_uP = V_uP \oplus H_uP$$

where V_uP & H_uP are respectively the <u>VERTICAL</u> & <u>HORIZONTAL SUBSPACES</u>. We also expect that ~~no matter~~ the notion of being horizontal is independent of the coordinate along the group. This requirement is formulated by saying that

$$R_{g*}H_uP = H_{ug}P$$

where R_g is the right action by g: $R_g u = ug$



recall that each vector X_u can be thought as the tangent vector of a curve $\gamma(t)$ at a point u. As you see in the picture we expect that if X_u is horizontal the only thing that happens by moving along G is to change the point where the vector is applied. So

$$R_{g*}H_uP = H_{ug}P.$$

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WHY A ONE-FORM?

Recall that a one-form acts on vectors through an inner product $\langle \omega, X \rangle = \omega_\mu X^\mu$ which reminds us of a scalar product. So by ~~identifying~~ imposing $\omega(X) = 0$ is like we are saying X is perpendicular to ω. Since roughly $H_u P \perp V_u P$, it is exactly what we need.

Specifically our connection will be a one-form with values in $\mathfrak{g}$, that is the Lie algebra of G which spans instead $V_u P$.

"FORMAL DEFINITION"

We define a <u>CONNECTION 1-FORM</u> over a principal bundle P a Lie algebra valued one form on P which fulfill the following properties:

$$\omega : T_p P \longrightarrow \mathfrak{g} \quad \text{or} \quad \omega \in T^*P \otimes \mathfrak{g}$$

$$① \quad \omega(A^*) = A \qquad [A \in \mathfrak{g} \text{ the Lie algebra of } G]$$

$$② \quad R_g^* \, \omega = g^{-1} \omega g.$$

this two properties are enough to define at each point the decomposition of the tangent space we were talking about before:

$$i) \quad T_u P = V_u P \oplus H_u P$$

$$ii) \quad R_{g*} H_u P = H_{ug} P$$

let's first have a look at properties ① & ②. First we define the vector A^*:

$$A_u^*(f) = \frac{d}{dt} f(u \exp(iAt)) \Big|_{t=0}$$

that basically is tangent to the curve (along the fiber) $\exp[iAt]$.

The fact that ω spits out A means that "recognizes" the vertical directions where curves are moving.

Property ② is pretty reasonable. ω takes value in $\mathfrak{g}$ & R_g^* is the action of G on it. We do know that G acts on its Lie algebra through the adjoint repr. which is exactly what ② says.

let's prove that if ω satisfies ① & ② then i) & ii) are true.
At each point u the vectors of the form $A^{\#}$ $\forall A \in \mathfrak{g}$ span V_uP whereas H_uP is the kernel of ω:

$$X \in H_uP \text{ iff. } \omega(X) = 0$$

we should now prove that $R_{g*} H_uP = H_{ug}P$, that is if $X \in H_uP$ then $R_{g*}X \in H_{ug}P$. This follows immediately since by def:

$$\omega(R_{g*}X) = R_g^* \omega(X) = g^{-1}\omega(X)g = 0$$ since we assumed $X \in H_uP$ or $\omega(X) = 0$.

LOCAL FORM OF THE CONNECTION

We heard often that the gauge fields are connections on bundles but so far nothing looks ~~also~~ at all alike!
The main problem is that ω is def over T^*P, ~~?~~ that means is a function of P whereas we know that the gauge field A_μ is a function on space-time. If we have a section $\sigma: U_i \to P$ (which always exists locally) we can pull-back ω on M.
We define our gauge field as the local form of ω:

$$\mathcal{A} = \sigma^*\omega \quad \text{so } \mathcal{A}: TM \to \mathfrak{g} \text{ or } \mathcal{A} \in T^*M \otimes \mathfrak{g}$$

we will convince ourselves that $\mathcal{A}$ does have the property of a gauge field. Obviously $\mathcal{A}$ can be written as:

$$\mathcal{A} = A_\mu^a \lambda_a dx^\mu \qquad \lambda_a \in \mathfrak{g}$$

It can be proven that ~~in coordinates~~ if we have a local trivialization ϕ_i^σ ω looks:

$$\omega_i = g_i^{-1} \pi^* \mathcal{A}_i g_i + g_i^{-1} dg_i$$

where π^* just makes sure that $\mathcal{A}_i \in T^*_u M$ gets pulled-back in a "trivial" way to T^*_uP & can act on $X \in TP$ ($\pi: P \to M$ so $\pi^*: T^*M \to T^*P$).
The last piece is called the MAURER-CARTAN 1-form

$g^{-1}dg$. Josh will tell about it much more...

IMPORTANT, ω is defined GLOBALLY on the Bundle, what might be defined only locally is $\mathcal{A}$ since we need a section to pull ω back & we have a global section only iff P is trivial!

GAUGE TRANSFORMATIONS

Often we are in a situation where we need to describe the bundle with more than 1 section (if the bundle is non-trivial) so we need to understand how two local descriptions relates on $U_i \cap U_j$

$$\mathcal{A}_i = \sigma_i^{*}\omega \quad \& \quad \mathcal{A}_j = \sigma_j^{*}\omega$$

recall that by moving from $\sigma_i \to \sigma_j$ we get a multiplication from the left by the transition function $t_{ij}(x) \equiv h(x)$

We can derive how $\mathcal{A}_i$ & $\mathcal{A}_j$ transform by asking for ω to be invariant. Let's do it:

$$\omega_i = g^{-1}\mathcal{A}_i g + g^{-1}dg \quad \& \quad \omega_j = [h(x)g]^{-1}\mathcal{A}_j[h(x)g] + [h(x)g]^{-1}d(h(x)g)$$

we did not care about the π^* & we'll now drop the x ~~depend~~ dependence of h. We also have $d(hg) = h\,dg + (dh)g$

so $$g^{-1}\mathcal{A}_i g + g^{-1}dg = g h^{-1}\mathcal{A}_j h g + g^{-1}dg + g^{-1}h^{-1}dh\,g$$

or $$\mathcal{A}_j = h\mathcal{A}_i h^{-1} - dh\,h^{-1} = h\mathcal{A}_i h^{-1} + h\,dh^{-1} \quad \text{since } d(hh^{-1}) = 0.$$

so we get that from going to the σ_i description to the σ_j one the local form $\mathcal{A}$ undergoes a gauge Transf.

$$\boxed{\mathcal{A}_j = h\mathcal{A}_i h^{-1} + h\,dh^{-1}}$$

We can now understand what is the meaning of fixing the gauge: we have a theory which lives on a principal bundle over which a connection ω is defined. There are many different ways to "represent" it on $\mathcal{M}$, each one of them is connected to the other by a gauge Transf.

CAN WE FIX ALWAYS THE GAUGE GLOBALLY?

If the bundle is NOT TRIVIAL we do not have a global section so we must define various different local forms

$$\mathcal{A}_i, \mathcal{A}_j, \mathcal{A}_k \dots$$

the gauge then cannot be fixed globally!!!

HORIZONTAL LIFT & PARALLEL TRANSPORT

We are now ready to define the HORIZONTAL LIFT of a curve $\gamma(t)$ in $\mathcal{M}$. Ultimately we would like to define the notion of parallel transport, that is we want to move objects "as straight as possible" & check whether or not they pick up a contribution along the group as well.
All the objects we are interested in, in physics, live on space-time, that is the base manifold $\mathcal{M}$. So we want to move such an object along a curve in $\mathcal{M}$, the problem is how to embed such a curve γ into the principal bundle. The horizontal lift $\tilde{\gamma}$ of a curve γ is then defined uniquely as:

$\tilde{\gamma}$: $\tilde{X}$, vector field tangent to it, has no-vertical components: $\omega(\tilde{X})=0$
& obviously $\pi(\tilde{\gamma}) = \gamma(t)$.

Given a local section σ, $\tilde{\gamma}(t) = \sigma(\gamma(t))\, g(t)$. So the task is now to find the expression for $g(t)$. I will not go in many details. The crucial steps are:

① write down an expression for $\tilde{X}$ in terms of X, vector field tangent to $\gamma(t)$: $\tilde{X} = \left[R_{g(t)*} \circ \sigma_* X\right]$

② impose $\omega(\tilde{X}) = 0 \Leftrightarrow g^{-1}(t)\, \omega(\sigma_* X)\, g(t) + g^{-1}(t)\, dg(t) = 0$

③ recall that $\omega(\sigma_{*i} X) = \sigma_i^* \omega(X) = \mathcal{A}_i(X)$

④ solve the eq. $\dfrac{dg(t)}{dt} = -\mathcal{A}_i(X)\, g_i(t)$
with initial condition $g_i(0) = \mathbb{1}$ [we have chosen $\sigma_i : \sigma_i(\gamma(0)) = \tilde{\gamma}(0)$.]

⑤ The expression for g is then:

$$g(\gamma(t)) = \mathcal{P} \exp\left[-\int_{\gamma(0)}^{\gamma(t)} \mathcal{A}_{i\mu}(\gamma(t))\, dx^\mu\right]$$

We can now define the PARALLEL TRANSPORT of a point $u_0 \in P$ along $\gamma(t) \in \mathcal{M}$. {obviously in order to transport u_0 along $\gamma(t)$, we need $\pi(u_0) = \gamma(0)$}.
So u_1, the parallel transported of u_0 along $\gamma(t)$, it is just the ending point of the unique horizontal lift $\tilde{\gamma}_{u_0}$ of $\gamma(t)$ such that $\tilde{\gamma}(0) = u_0$. So

$$\gamma : [0,1] \to \mathcal{M}, \quad u_1 = \tilde{\gamma}(1) = \sigma(1)\, g(1) = \sigma_i(1)\, \mathcal{P} \exp\left[-\int_{\gamma(0)}^{\gamma(1)} \mathcal{A}_{i\mu}(\gamma(t))\, dx^\mu\right]$$

HOLONOMY GROUP

Suppose now that $\gamma(t) \in M$ is a closed loop. Then $\gamma(0) = \gamma(1)$. & the parallel transport of $u_0 \in \pi^{-1}(\gamma(0))$ is a map from $\pi^{-1}(\gamma_0) \to \pi^{-1}(\gamma_1 = \gamma_0)$ so u_0 & u_1 belong to the same fiber & they have to differ by an element of G, so $\exists \bar{g} \in G : u_0 = u_1 \bar{g}$. We can indicate such a map $P \to G$ as τ_γ:

$$\tau_\gamma(u_0) = u_0 \bar{g}$$

there is a well-known composition map among curves, $\gamma_1(t)$ & $\gamma_2(t)$, which satisfy $\gamma_2(0) = \gamma_1(1)$ [trivially true for loops applied at the same point]. Such a map is the one that defines the group structure in the case of HOMOTOPY GROUPS & it is defined as:

$$\bar{\gamma}(t) = (\gamma_1 * \gamma_2)[t] := \begin{cases} \gamma_1(2t) & 0 \le t \le 1/2 \\ \gamma_2(2t-1) & 1/2 \le t \le 1 \end{cases}$$

it can be shown that in the case of parallel transport:

$$\tau_{\bar{\gamma}}(u) = \tau_{\gamma_2} \circ \tau_{\gamma_1}(u)$$

that is if $\tau_{\gamma_1}(u) = u g_1$ & $\tau_{\gamma_2}(u) = u g_2$

then $\tau_{\bar{\gamma}}(u) = \tau_{\gamma_1 * \gamma_2}(u) = \tau_{\gamma_2} \circ \tau_{\gamma_1}(u) = u g_1 g_2$

So the set of maps τ_γ for any $\gamma : \gamma_0 = \pi(u_0)$ [γ is a loop] form a group, in particular it is a subgroup of G.

$$\Phi_{u_0} : \{ g \in G \mid \tau_\gamma(u_0) = ug,\ \gamma \text{ loop based on } \pi(u_0) \}$$

As usual (like it is the case for homotopy groups) if M is connected then Φ_{u_0} is the same no mater where we choose the based point u_0.

SUMMARY

1. The gauge field A is the local form of the connection 1-form ω on the bundle P. Choosing which section to use to pull-ω back, $\sigma^*\omega$, consists in choosing the gauge.
2. The HORIZONTAL LIFT of $\gamma(t) \in M$ is defined as the curve in P : $\pi(\tilde{\gamma}(t)) = \gamma(t)$ & its tangent vector field is always horizontal
3. The parallel transport of a point $u_0 \in P : \pi(u_0) = \gamma(0)$ along $\gamma(t) \in M$, is the ending point of $\tilde{\gamma}(t)$, horizontal lift of γ starting at u_0
4. If γ is a loop, the parallel transp. of u_0 is just an element of G. All these element togheter, $\forall \gamma$, form the HOLONOMY

CURVATURE TWO-FORM & EXTERIOR [redacted] → [redacted] ON BUNDLES
DERIVATION

Once we define a connection one-form, we can define another object ~~which~~ $\Omega = D\omega$ which might gives us information on how two infinitesimal horizontal displacement combine. (This is simply coming from the fact that Ω is a two-form & therefore acts on two vector fields). D is not the usual exterior derivative map [redacted] which we use on n-forms. There are two heuristic ways to see why we have to change the notion of exterior derivatives:

TRANSF. UNDER THE GROUP

Suppose that ω is a n-form that transforms under [redacted] a group [redacted] G as $\omega' = g^{-1}\omega g$

We want $D\omega \to (D\omega)' = g^{-1} D\omega g$

this is obviously not true for $D \equiv d$, in fact:

$$(d\omega)' = d\left[g^{-1}\omega g\right] = g^{-1}d\omega g - \frac{dg}{g^2}\omega g + g^{-1}\omega dg$$

"PHYSICAL" MEANING

We [redacted] hear often that the curvature two-forms measures the commutativity of [redacted] two covariant derivatives over a bundle. [redacted] Which basically reduces to measures the difference in vertical component of moving along X & [redacted] then Y or the other way around. Since the only "physical direction" is the horizontal one, we would like Ω just to [redacted] act [redacted] upon the horizontal component of X^H & Y^H.

Having said few [redacted] motivations we can give the formal definition:

$\omega \in \Lambda^n P$ [redacted] then $D\omega(X_1, \ldots X_{n+1}) = d\omega(X^H_1, \ldots, X^H_{n+1})$

where $d\omega$ is just the usual exterior derivative on P.

Specifically if ω is the <u>CONNECTION 1-FORM</u> we define the <u>CURVATURE 2-FORM</u> as: $\Omega = D\omega$.

Let's look at the geometrical meaning of Ω:

$$\Omega(X, Y) = D\omega(X, Y) := d\omega(X^H, Y^H)$$

following the def. of exterior derivative [Nakahara P. 199 eq (5.71)]

$$d\omega(X^H_1, Y^H) = X^H(\omega(Y^H)) - Y^H(\omega(X^H)) + \omega([X^H, Y^H])$$

few remarks are in order:

i) $\omega(V^H) = 0$ by def. of horizontal vectors

ii) $[V^H, V'^H]$ [redacted] $\notin HP$ that is by taking the commutator of two horizontal vectors we might get a vertical comp.

Finally we conclude that $\Omega(X, Y) = \omega([X^H, Y^H])$

that is Ω exactly measures the "amount" of vertical component that we get by doing two infinitesimal horizontal transformations in opposite order. That's exactly the geometric meaning of the Riemann tensor in GR!

FIELD - STRENGTH

As A_μ is the local form of ω, we will def $F_{\mu\nu}$ as the local form of Ω: $\quad F_i = \sigma_i^* \Omega$.

It is possible to show that the covariant exterior derivative D acts as: $\quad D\omega = d\omega + \omega\wedge\omega \quad$ where $\omega\wedge\omega$ [which would be normally 0 if ω was just a one-form] for $\mathfrak{g}$ valued one-forms is defined as:

$$\omega = \omega_a \lambda^a \quad \text{so} \quad \omega\wedge\omega = [\lambda^a, \lambda^b]\, \omega_a \wedge \omega_b = f^{ab}{}_c \lambda^c\, \omega_a \wedge \omega_b .$$

So the field strength takes the usual form:

$$F_i = dA_i + A\wedge A \quad \text{or explicitely} \quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^a{}_{bc} A^b_\mu A^c_\nu$$

ASSOCIATED VECTOR BUNDLES

So far we dealt with principal bundle which describe the structure of pure gauge theories. How about matter?

Matter fields are in general vector valued fields so in order to describe them we need to introduce a new concept: an ASSOCIATED VECTOR BUNDLE to A PRINCIPAL BUNDLE P

HEURISTIC IDEA

We'll call the new bundle E. Heuristically E is a bundle with the same kind of structure as P but with a vector space as fiber instead of the group G.
How do we keep track of the structure of P? It is completely embodded in the transition functions from one local-trivialization to another. In fact the structure group of E is exactly G (the group which constitutes the fiber of P) & the vector space on E provides a left repr. of the group G.

TRANSITION FROM $P \longrightarrow E$

In order to get insights on the formal def. let's look at the structure of the physical theory in the U(1) for example. We know that once we have a certain field configuration:

$$\tilde{\psi}(x)$$

there is a "ray" of solutions at each point which describe the exact same configuration, in fact

$$\tilde{\psi}(\bar{x}) \sim e^{i\lambda}\,\tilde{\psi}(\bar{x})$$

describe the same physics. Somewhat our physics lives in a space in which each point (intended as inequivalent fields configurations) is in fact a ray obtained by the left action of the group. That's obviously very rough and imprecise but it gives the feeling of what we have to do to construct the associated bundle.

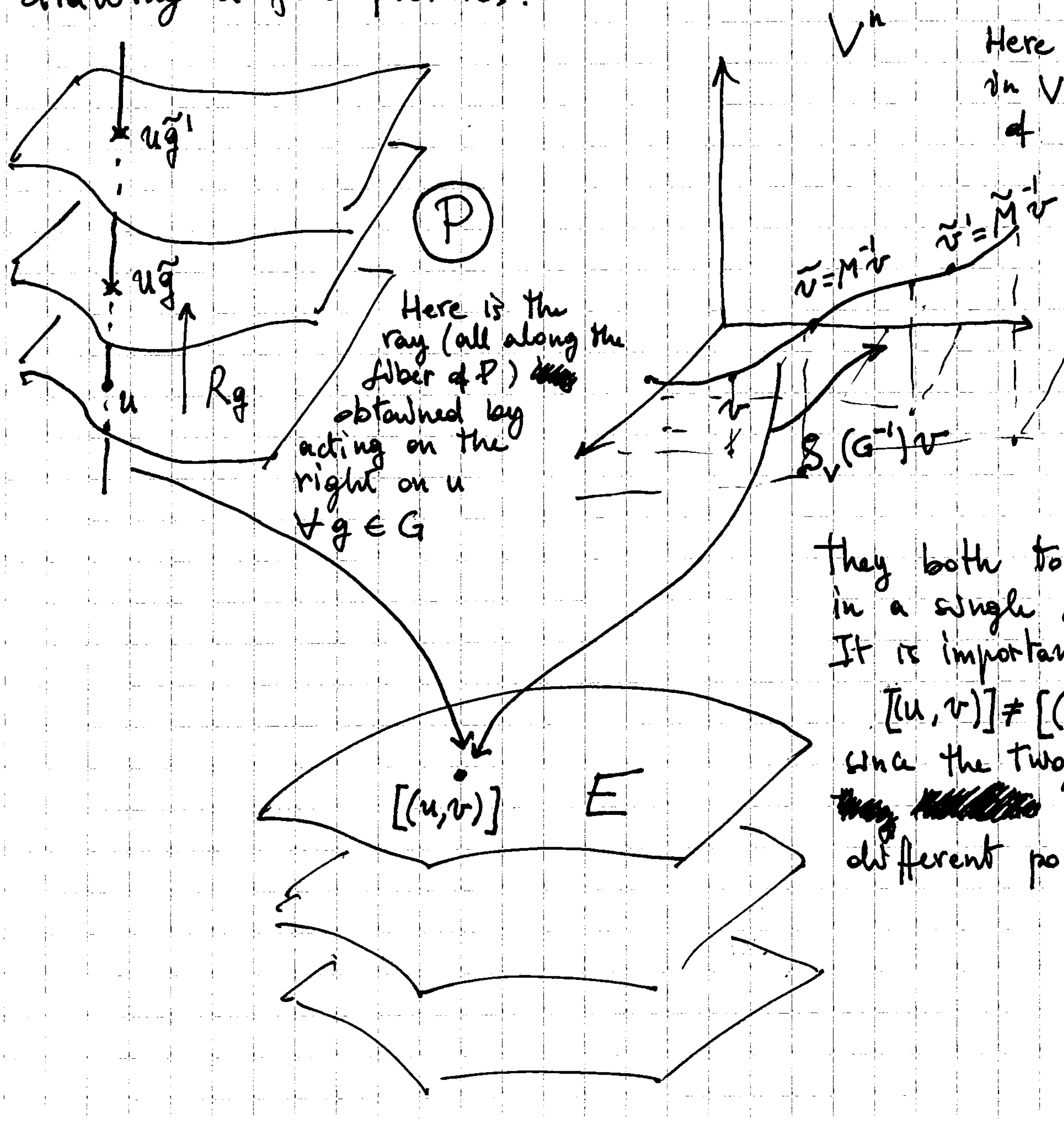
FORMAL DEFINITION

Let's P be a principal bundle over a base manifold $\mathcal{M}$ & with fiber G. Consider a vector space V^n providing a left repr. of G, call it ϱ_V, then the associated vector bundle E is defined as:

$$E = P \times_\varrho V := \{ [(u, v)] \in P \times V^n \mid (u, v) \sim (ug, \varrho_V(g^{-1}) v) \}$$

that is E is the space of equivalent classes where (u, v) on $P \times V^n$ is identified with $(ug, \varrho_V(g^{-1})v)$ $\forall g \in G$.

It can be proven that a bundle defined in this way is in fact a G-vector bundle (in the sence that G is its structure group) on the same base manifolds $\mathcal{M}$. Let's try to understand it better by drawing a few pictures:



They both together collapse in a single point on E.
It is important to notice that

$$[(u, v)] \neq [(ug, v)]$$

since the two rays (on P & V^n) "start" from different points.

LOCAL TRIVIALIZATIONS ON E

Ultimately this complicated object made of equivalence classes, is, we said, a vector bundle with fiber V^n over a base manifold M. We should then expect to be able to define local trivializations which describe each equivalence class (points on E) by a pair of "numbers" namely a point on M & a vector in V^n:

$$\phi_E\{[(u,v)]\} = (x,\bar{v}) \in M \times V^n$$

we also suspect that, being E obtained by P & being local triv. in P determined by local sections $\sigma_i : M \to P$, σ_i could induce local trivialization on E as well. It is in fact the case. Remember that using ϕ^σ we can describe $\phi^\sigma(u) = \phi^\sigma(\sigma(x)g) = (x,g)$

so
$$\phi^\sigma_E\{[(u,v)]\} = \phi^\sigma_E\{[((x,g),v)]\} = \phi^\sigma_E\{[((x,g)g^{-1}, gv)]\}$$
$$= \phi^\sigma_E\{[((x,\mathbb{1}), gv)]\} = (x, gv) \in M \times V^n$$

where by gv we mean $\rho_V(g)v$.
Again the section σ allows us to choose a reference frame with respect to which fix our coordinates on E.

Since choosing one particular σ_i on P corresponds to fix the gauge we expect that our matter fields will also undergo a gauge transf. when we move from a σ_i to a σ_j description. We'll check that after having defined what a matter field correspond to.

MATTER FIELDS & GAUGE TRANSF. Let's take the example of spinorial (Weyl or Dirac) fields. Mathematically speaking they are just maps from space-time (M) to either $\mathbb{C}^2$ or $\mathbb{C}^4$ (both vector spaces). Once we notice that becomes obvious what such fields are in the bundle setting:

- Weyl: they are sections of an $\mathbb{C}^2$-vector bundle over M
- Dirac: sections of a $\mathbb{C}^4$-vector bundle over M.

Sections of vector bundles are actually vector fields, but this is just terminology & it does not matter.
We should now turn into the coordinate repr. of sections of vector bundle which we'll bring to "gauge transformations" for matter fields.

A section $\psi(x)$ should be thought as a smooth assignment of elements in E as function of x. So basically:

$$\Psi(x) = [(u, v)](x)$$

If we introduce a system of coordinates induced by a section σ: $u(x) = \sigma(x) g(x)$. Then the coordinates of ψ will look as:

$$\Psi(x) = [(\sigma(x) g(x), v(x))] = (x, g\, v(x)) = (x, \psi(x))$$

where $\psi(x)$ represents the actual elements in V^n.
If we now change from $\sigma \to \tilde{\sigma}(x) = \sigma(x)\tilde{g}(x)$ then the induced coordinate transformation is:

$$\Psi(x) = [(\sigma(x), g(x) v(x))] = [(\sigma(x)\tilde{g}(x)\tilde{g}^{-1}(x), \tilde{g}(x)\psi(x))] =$$

$$= (x, \tilde{g}(x)\psi(x))$$

So by changing the local trivialization $\psi(x)$ undergoes exactly a gauge transf. In the $U(1)$ case for instance as we change description from $\sigma_i \to \tilde{\sigma}_i = \sigma_i e^{i\Lambda(x)}$, we get:

$$A_j = A_i + e^{+i\Lambda(x)} d\left(e^{-i\Lambda(x)}\right) = A_i - \partial_\mu \Lambda(x)$$

$$\psi_j = e^{i\Lambda(x)} \psi_i$$

So once again fixing the gauge just means choosing a particular section σ to trivialize our bundle.

PARALLEL TRANSPORT on E

We have now only a couple of things left. One of them is finding a way to parallel transport our matter fields along a curve $\gamma(t) \in M$.
The notion of parallel transport is again connected with the idea that while moving along $\gamma(t)$ the element in the fiber through $\Psi(x)$ stays as constant as possible.
Again we can try to define a section to be parallel transported along a curve $\gamma(t) \in M$ if the vector field $v(\gamma(t)) \in V^n$ does not depend on t in some particular local trivialization:

$$\Psi_{\gamma}(t) = [(u(t), v_0)] = (x, v_0)$$

but again such an attempt fails since it depends upon the local trivialization chosen. We have seen in fact that had we chosen a different section $\tilde{\sigma}(t) = \sigma(t)\, g(t)$ then Ψ would depend on t:

$$\Psi_{\tilde{\sigma}}(t) = (x, \tilde{g}(t) v_0) = (x, \tilde{v}(t))$$

In the exact same way as it happened in the principal bundle case, the notion of horizontal lift comes as a solution.
We'll say that $\Psi(t)$ is parallel transported along $\gamma(t) \in M$ if it is constant with respect to the horizontal lift of $\tilde{\gamma}(t)$.
The horizontal lift is not unique if we don't specify the initial point $\tilde{\gamma}(0) = u_0$ & in this case it would be meaningless to do it since in E u_0 is identified with the whole fiber (definitely in a non trivial way through the $[(u, v)]$ equivalence classes but the statement is still true...). This is not a problem since two horizontal lift $\tilde{\gamma}_1$ & $\tilde{\gamma}_2$ such that $\tilde{\gamma}_1(0) = u_1$ & $\tilde{\gamma}_2(0) = u_2$ they differ by a constant group element g ($u_1 = u_2 g \mapsto \tilde{\gamma}_1(t) = \tilde{\gamma}_2(t)$

Let's see that this def. of parallel transport is intrinsic:

$$\text{if } \Psi_{\tilde{\gamma}}(t) = [(u, v)](t) = [(u(t), v(t))] = [(\tilde{\gamma}(t)\tilde{g}(t), v(t))] =$$

$$= [(x, \tilde{g}(t) v(t))] = [(x, v_0)]$$

is constant in V^n with respect to $\tilde{\gamma}(t)$ then it will be constant for any other horizontal lift $\tilde{\gamma}'(t) = \tilde{\gamma}(t) g$ [obviously both project down to the initial curve $\gamma(t) \in M$], in fact:

$$\Psi_{\tilde{\gamma}'}(t) = [(\tilde{\gamma}(t) g^{-1} \tilde{g}(t), v(t))] = [(x, g^{-1} v_0)] = [(x, \tilde{v}_0)]$$