Mario : CHERN-SIMONS 18 FEB 2013

1. INTRO TO CS TERMS
2. PARITY ANOMALY
3. 3D SUSY DUALITIES : Aharony vs. Giveon-Kutasov

REFERENCES

Phys. Rev. D29 10 ('84) → Parity anomaly
hep-th/9703110 → "5 PERSON PAPER"
thesis by Brian Willett → systematic treatment (but advanced

CHERN SIMONS : FEATURE of ODD-DIMENSIONAL SPACETIMES

CONSIDER THE j^{th} CHERN CHARACTER

shows up in index thm

$$\int_M \underbrace{\partial_\mu j_5^\mu}_{\text{AXIAL}} \, d^{\overbrace{2j}^{\text{even}}} x = \int ch_j(\underbrace{F}_{\text{USUAL } F_{\mu\nu}})$$

← j^{th} CHERN CHARACTER

$$\boxed{ch_j(F) = \frac{1}{j!} \mathrm{Tr} \left(\frac{iF}{2\pi}\right)^j} \qquad (F = dA + A \wedge A)$$

2j-form, must be integrated over 2j-dim SPACE

FACT : $d \, ch_j(F) = 0$

pf/ BIANCHI IDENTITY ← constraint from NonAbelian gauge thy.

$$DF \equiv dF + A \wedge F - F \wedge A = 0 \qquad (\text{ABELIAN: } dF = 0)$$

$$d\,\mathrm{Tr}\,F^j = j\,\mathrm{Tr}[dF \wedge F \wedge \cdots \wedge F] \equiv j\,\mathrm{Tr}[\underbrace{DF}_{=0} \wedge \cdots \wedge F]$$

by using cyclicity of trace:

$$\mathrm{Tr}[A \wedge F - A \wedge F] = \mathrm{Tr}[A \wedge F - F \wedge A] \qquad \square$$

So $d\,ch_j(F) = 0$

$\Rightarrow \exists\, \Omega_j$, A $(2j-1)$ FORM SUCH THAT

$$\boxed{ch_j(F) = d\Omega_j}$$

Ω_j is the CHERN-SIMONS FORM in $(2j-1)$ dim

observe: it is topological

We can write an explicit expression: ($\forall$ dim, odd)

$$\Omega_j = \frac{2j-1}{2}\int_0^1 dt\; \mathrm{Tr}[A \wedge \mathcal{F}_t^{\,j-1}]$$

$$\mathcal{F}_t = t\,dA + t^2 A^2$$

rest of talk

SPECIFICALLY, in 3d:

$$\Omega_2 = \mathrm{Tr}\left[A \wedge dA + \tfrac{2}{3} A^3\right]$$

now this looks familiar!

IN 3D, I CAN USE THIS TO GIVE AN ACTION THAT IS NOT AN INTEGRAL OVER SPACETIME:

$$S_{CS} = \frac{K}{4\pi^2} \int d^3x\, \varepsilon^{\mu\nu\lambda}\, \mathrm{Tr}\left[A_\mu \partial_\nu A_\lambda + \tfrac{2}{3} A_\mu A_\nu A_\lambda\right]$$

$\int \Omega_2$

K is the LEVEL of the theory ($k \sim \theta_{QCD}$)

CLASSICALLY, EOM $\Rightarrow \frac{K}{2\pi} F = 0$

IS Ω_2 GAUGE INVARIANT?

$ch_j(F)$ IS GAUGE INV'T, TRACE OF F_s & $F \to gFg^{-1}$

BUT WHAT ABOUT Ω_2?

$A_\mu \to g^{-1} A_\mu g + g^{-1} \partial_\mu g$

$$S_{CS} \mapsto S_{CS} + \tfrac{1}{3} \int d^3x\, \mathrm{Tr}\left[\varepsilon^{\mu\nu\lambda} (g^{-1}\partial_\mu g)(g^{-1}\partial_\nu g)(g^{-1}\partial_\lambda g)\right]$$

+ {SURFACE TERMS}

$\int d^3x \longmapsto \int_{S^3} \sqrt{g}\, d^3x$ integ over S^3

Using geometry write this as

$$S_{CS} \mapsto S_{CS} + \underbrace{\tfrac{1}{3}\int \mathrm{Tr}(g^{-1}dg)^3}_{\text{WINDING \# OF } g}.$$

claim: WINDING # OF g, $g \in \pi_3(g)$

NOW DO SOME THINGS CSABA-STYLE.
(CONNECT TO CSABA'S HOMEWORK)

FINAL CLAIM: Ω_2 is only not-gauge invariant for "large gauge transformations"

cannot continuously deform to identity.

Pf//

$$g(x) = e^{iT^a f_a(x)} \qquad g : S^3 \mapsto G$$

$$g^{-1}(x)\partial_\mu g(x) = iT^a \partial_\mu f_a(x)$$

$$\Rightarrow \int_{S^3} \sqrt{g(\phi)}\, d\phi_1 d\phi_2 d\phi_3\, \varepsilon^{\mu\nu\lambda}\, \mathrm{Tr}[T^a T^b T^c]\, \partial_\mu f^a \partial_\nu f^b \partial_\lambda f^c$$

FACT: $\forall$ simple Lie group, G, all continuous mappings from $S^3 \to G$ can be continuously deformed into maps from $S^3 \to SU(2)$ subgroup.

$\Rightarrow$ so effectively, G is $SU(2) \sim S^3$.

$\Rightarrow T^a \sim \sigma^a$

$g: \frac{\partial}{\partial \phi^\mu} f^a$

$\Rightarrow \mathrm{Tr}[T^a T^b T^c] = \varepsilon^{abc}$

$$\int \longrightarrow \int \sqrt{g(\phi)}\, d\phi_1 \cdots d\phi_3\, \varepsilon^{\mu\nu\lambda} \varepsilon^{abc}\, \partial_\mu f^a\, \partial_\nu f^b\, \partial_\lambda f^c$$

FACT: (def) $\det(M) = \varepsilon^{\mu_1 \cdots \mu_n} \varepsilon_{\nu_1 \cdots \nu_n} M^{\nu_1}_{\mu_1} \cdots M^{\nu_n}_{\mu_n}$

$$\Rightarrow \int \mathrm{Tr}(g\,dg)^3 = \boxed{\int \sqrt{g(\phi)}\, d\phi_1 \cdots d\phi_3 \det\left(\partial f / \partial \phi\right)}$$

= winding number of a map $f: S^3 \to S^3$ of CSABA's HW #2 $\quad\square$

so: winding # of f ($\leftrightarrow$ "largeness of gauge trans") is the non-gauge invariance of S.

ok. we'll use this non-gauge invariance shortly.

Parity Anomaly

Motivation: Anomalies are useful tools for checking ~~[illegible]~~ DUALITIES. eg. 't Hooft anomaly matching conditions.

In odd dimensions, there are no $U(1)_A$ or GAUGE anomalies!

CHIRAL ANOMALY: no chirality in odd dim

GAUGE ANOMALY: related to ~~[illegible]~~ (n+2) dim $U(1)$ anomaly.

SO INSTEAD OF THESE, WE USE WHAT WE HAVE FOR ANOMALY MATCHING: Parity anomaly.

Parity anomaly in 3d is related to $U(1)$ in 4d.

$\hookrightarrow$ SCHEMATICALLY: $\downarrow$ $U(1)$ IN $2n$-dim

PARITY IN $(2n-1)$-dim

GAUGE IN $(2n-2)$-dim

consider the partition function

$$Z = \int \mathcal{D}A\, \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\, \exp\left[i \int d^3x\, \tfrac{1}{2} \mathrm{Tr} F^2 - i \bar{\psi}(\not{\partial} + \not{A})\psi \right]$$

$$= \int \mathcal{D}A\, \exp\left[i \int d^3x \left(\tfrac{1}{2}\mathrm{Tr} F^2\right) - i \ln \det\, (\not{\partial} + \not{A}) \right]$$

$$= \int \mathcal{D}A\, e^{i\int d^3x F^2 + S_{eff}} \qquad S_{eff} \equiv \int d^3x\, I_{eff}$$

<u>consider large gauge transformations:</u>

<u>CLAIM:</u> $S_{eff} \longmapsto S_{eff} + \underbrace{\delta S}_{}$

not gauge inv't!
(S_{cs} will help)

$$\delta S = \int_M ch_i(F) = \frac{1}{2}\int \frac{1}{4\pi^2} \mathrm{Tr}(F \wedge F)$$

<u>BIG PIC:</u> δS is a chern-simons term which is canceled when you regulate the theory, eg by Pauli-Villars, which gives the cs term to cancel this gauge-non-invariance.

Q: where the FUCK does 4th dim come from?

Dirac

write: $\det(\not\partial + \not A) = \det(\sigma^i(\partial_i + A_i)) \equiv \det(D_3[A])$

HOW DO I TEST THE BEHAVIOR UNDER LARGE GAUGE TR?

Define:

$$A_i(x,\tau) : A_i(x,-\infty) = 0$$
$$A_i(x,+\infty) = \underbrace{g^{-1}\partial_r g}_{\text{w/ } g \text{ large}}$$

BOTH PURE GAUGE, BUT A_{∞} IS LARGE. $A_{-\infty}$ IS TRIVIAL

$$\det(D_3[A])_{-\infty} = (-)^n \det(D_3[A])_{+\infty}$$

n is # eigenvalues which flip signs as I go from $-\infty \to +\infty$

large gauge: cannot deform $A_{-\infty} \mapsto A_{+\infty}$ w/o passing through gauge config which are not pure gauge. Along the way, eigenvalues will change sign.

CLAIM: $n = n_+ - n_-$ of $D_4[A] = \gamma_\mu(\partial^\mu + A^\mu)$

ZERO MODES wrt D_3

ZERO MODES wrt D_4

$\gamma_0 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$ $\gamma_i = i\begin{pmatrix} & \sigma_i \\ -\sigma_i & \end{pmatrix}$

$A_\mu = (A_i(x,\tau), 0)$

CONSTRUCTED $D_4[A]$ OUT OF A_i

$$\int_M ch_j(F) = \int_{\partial M} \Omega_j$$

$$\delta S = \frac{1}{2} \int_{\mathbb{R}\times S^3} ch_2(F) = \frac{1}{2}\frac{1}{4\pi^2} \int_{\partial M} \Omega_j$$

$$\partial(\mathbb{R}\times S^3) = \cancel{(-\infty,+\infty)}\ (-\infty \cup +\infty) \times S^3$$

$$= \frac{1}{2}\frac{1}{4\pi^2}\left[\int_{S^3} AdA + \frac{2}{3} A\wedge A\wedge A \Big|_{\tau=+\infty} - \int_{S^3} \Omega_2 \Big|_{\tau-\infty}\right]$$

$= 1/2 \times$ (WINDING #) ← from LG GAUGE TR.

w/ LEVEL $= 1/2$

BIG PIC AGAIN:

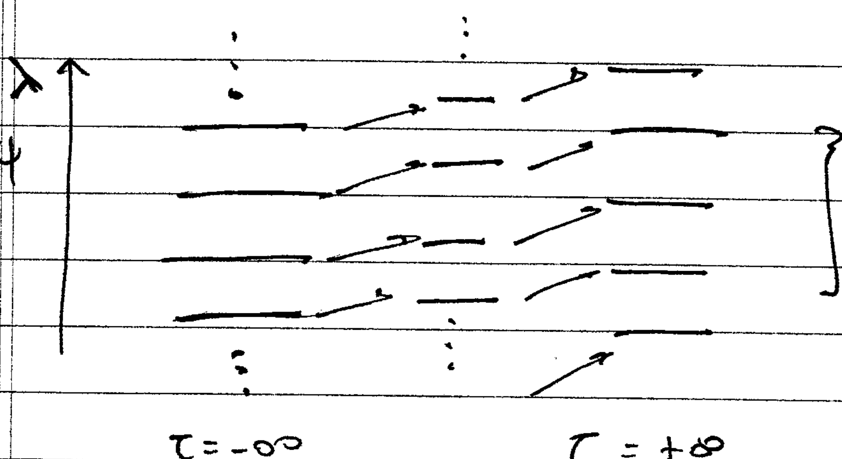
BTWN $\tau = -\infty$ & $\tau = +\infty$

TRIVIAL PURE GAUGE

HOW DOES SPECTRUM CHANGE?

@ $\tau = -\infty$, $\tau = +\infty$ JUST RELATED BY GAUGE REDUNDANCY

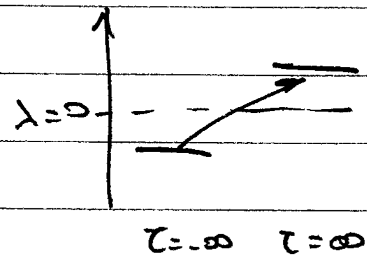
$D_3[A]\psi = \lambda\psi$



SAME SPECTRUM GETS SHIFTED BY INTEGER, BUT SAME B/C ∞ LADDER

BUT HOW DOES $\det(D_3[A])$ change?

eg one (or n) eigenvalue changes from minus to plus, so pick up $(-)^n$



all w/rt 3D Dirac. τ JUST A PARAM.

$$\Rightarrow \det(D_3[A])_{-\infty} = (-)^n \det(D_3[A])_{+\infty} \checkmark$$

how many eigenvals actually make this jump?

$$\hookrightarrow n = \underbrace{n_+ - n_-}$$

4D

3D

NEVER CARED ABOUT ZERO MODES HERE (NO CHIRAL ANOM)

ZERO MODES OF D_4

$$(\partial\!\!\!/ + A\!\!\!/)_4 \psi = 0$$

$$\downarrow$$

$$\frac{d\psi}{d\tau} = -\gamma_0 \gamma_i (\partial^i + A^i)\psi$$

$$= -\underbrace{\begin{pmatrix} i\sigma_i & \\ & -i\sigma_i \end{pmatrix} (\partial_i + A_i)}\psi$$

All 3D quantities

RECALL: ψ IS ZERO MODE OF D_4, BUT <u>NOT</u> OF D_3

NOW CHOOSE $\gamma_5 \psi_\pm = \pm \psi_\pm \to \pm u_\pm$

(*) $$\begin{cases} \dfrac{du_+}{d\tau} = i\sigma^i(\partial_i + A_i)u_+ \\ \dfrac{du_-}{d\tau} = i\sigma^i(\partial_i + A_i)u_- \end{cases}$$

want to solve this: $u_\pm(\vec{x},\tau) = f_\pm(\tau)\,\phi_\tau(x)$

ADIABATIC APPROX: $i D_3[A]\phi(x) = \lambda(\tau)\phi_\tau(x)$

plug into (*), get eqn for $f_\pm$:

$$\frac{df_\pm}{d\tau} = \mp\lambda(\tau) f_\pm(\tau)$$

$$\Rightarrow f_+(\tau) = f_+(0)\exp\left[-\int_0^\tau d\tau'\,\lambda(\tau')\right]$$

$$f_-(\tau) = f_-(0)\exp\left[\int_0^\tau d\tau'\,\lambda(\tau')\right]$$

CLAIM: f_+ is ONLY NORMALIZABLE IF $\lambda(+\infty) > 0$ & $\lambda(-\infty) < 0$.

ie only those that flip sign.

so for each $\lambda(\tau)$ we have def an f.

so: # of states of D_3 whose λ crossed 0 (changes sign), n, is equal to # zero modes of D_4 $(n_+ - n_-)$.

& we've proved by construction. □

so: in 3D: $S_{eff} \longmapsto S_{eff} + \delta S$

where $\delta S = \frac{1}{2}\int_{\mathbb{R}\times S^3} \mathrm{Tr}\, F\wedge F$

$$= \frac{1}{2}\int_{S^3}\left[\Omega_2(+\infty) - \Omega_2(-\infty)\right]$$

$$= \frac{1}{2}\,(\text{winding number}) \leftarrow \text{from beginning.}$$

so: S_{eff} NOT invariant under large gauge

BUT: $\det(\partial\!\!\!/ + A\!\!\!/)$ should be regulated.
say, PAULI-VILLARS: add $\bar{\chi}\, i\partial\!\!\!/\chi + iM\bar{\chi}\chi$, $M\to\infty$

recall: $S_{eff} = \int d^3x\, I_{eff}$

$$I^R = I(M=0) - I(M=\infty)$$
$$I_{REG} = I(\text{finite}, M=0) \mp \frac{M}{|M|}\frac{1}{2}\Omega_2$$

so: integrate out Majorana fermion χ,
then you get this $(M/|M|)\frac{1}{2}\Omega_2$ term.

then $S^R_{eff} = \int d^3x\, I^R_{eff}$

$$S_{eff} \xrightarrow{\text{GAUGE}} S_{eff} + \delta_S$$

$$S^R_{eff} = S_{eff,fin} + \int \frac{M}{|M|} \frac{1}{2} \Omega_2$$

$$\downarrow \text{GAUGE} \qquad\qquad \downarrow \text{GAUGE}$$

$$S_{eff,fin} + \cancel{\delta S} \qquad \int \frac{M}{|M|} \frac{1}{2} R_2 \cancel{- \delta S}$$

GAUGE NON-INVARIANCE CANCEL.

parity anomaly: Ω_2 is not parity invariant. //

Briefly: DUALITY

Aharony duality: SEIBERG DUALITY IN 3D

$$U(N)\ ,\ F(Q_a, \tilde{Q}^b) \quad w/ \ \underbrace{V_\pm = e^{\pm \Sigma}\ ,\ \Sigma = \Phi + i\gamma}_{\text{cf FLIP'S TALK}}$$

COULOMB BRANCH

HIGGS BRANCH: $M^a_b = Q_a \tilde{Q}^b$

$\Downarrow$ [DUAL TO]

$$U(F-N)\ ,\ F(q_a, \tilde{q}^b)\ ,\ F \text{ singlets } M^a_b$$

ADDITIONAL SINGLETS: $\hat{V}_\pm$

tilde of magnetic dual.

$$W = q_a M^b_a \tilde{q}^b + \hat{V}^+ \tilde{\hat{V}}^- + \hat{V}^- \tilde{\hat{V}}^+$$

DERIVE NEW DUALITY BY NAVIGATING MODULI SPACE

SO TAKE $U(N)$ w/ $(F+k)\ (Q_a, \tilde{Q}^b)$

INTRO. "R MASS" $M|Q_a|^2 + M|\tilde{Q}|^2$ for $a = F+1, \cdots, F+k$

↳ integ out Majorana fermions. each contributes $\frac{1}{2}\Omega_2$

$$U(N), \ (F+k)\ (Q_a, \tilde{Q}^b) \quad \swarrow k=0$$

INTEGRATE OUT κ

$$U(N)_k, \ F(Q_a, \tilde{Q}^b)$$ COULOMB BRANCHES LIFTED

explicit k CS term

<u>subtle</u>: how to map IR masses?

$$M|Q_a|^2 \xrightarrow{\text{GAMM}} -M|q_a|^2$$

$$(\text{cf } MQ\tilde{Q} \to M\,\text{Meson})$$

$$U(F+k-N)\ ;\ N_f+k\ (q, \tilde{q})$$

$$U(F+k-N)_{-k} \quad F(q,\tilde{q}) \quad W = qM\tilde{q}$$

b/c of $-M|q|^2$! no monopoles

SO: ★ IS CALLED <u>GIVEON-KUTASOV</u> DUALITY