Introduction

The original form of the AdS/CFT-correspondance was given by Maldacena where he showed the duality between $\mathcal{N}=4$ superconformal Yang-Mills theory and $AdS_5 \times S^5$ supergravity with N D3-branes.

We can use the duality to calculate strong coupling observables in the CFT using weak coupling observables in the SUGRA. Now we want to use this prescription to calculate observables in more realistic gauge theories; loosing conformality and supersymmetry. On the gravity side this corresponds to modyfying the geometry and matter content. This will lead to a more general Gauge/Gravity duality.

Here we show how:

* Varying geometry $\leftrightarrow$ breaking of SUSY
* Adding fluxes $\leftrightarrow$ breaking of conformality

History

The story outlined above was taken in many steps:

i) Maldacena: $\mathcal{N}=4$ SYM (conformal)
$AdS_5 \times S^5$ + D3s & F_5 flux

ii) KW: $\mathcal{N}=1$ SYM (conformal)
$AdS_5 \times T^{1,1}$ + D3s & F_5 flux

iii) KT: $\mathcal{N}=1$ SYM ($\cancel{\text{conformal}}$)

$AdS_5 \times T^{1,1}$ + D3s & F_5 flux + D5s & F_3 flux

iv) KS: $\mathcal{N}=1$ SYM ($\cancel{\text{conformal}}$)

KT in the UV, deformed in the IR

i) Maldacena

We now consider Maldacenas original setup. On the gravity side we have gravity with N D3-branes (3+1 worldvolumes) and generalized gauge fields C_{MNPQ} with field strengths $F_5 = dC_4$.

<u>Note</u>: Remember how the photon A_M couples to the 0+1 worldline of a point particle.

The action is given by parts of the IIB action, see 12.1.31 Polchinski Vol II

$$S_{IIB} \ni \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g} \left(R - \frac{1}{4 \cdot 5!} F_{MNPQR} F^{MNPQR} \right) + S_{loc}$$

where S_{loc} is the action for the localized sources

$$S_{loc} = -\frac{T_3}{g_s} \int_{D3} d^4x \sqrt{-g_4} + i T_3 \int_{D3} C_4$$

Note: This is just the generalisation of Maxwell-Einstein theory

$$S = \frac{1}{16\pi G}\int d^{4}x\sqrt{-g}R - \frac{1}{4}\int d^4x\sqrt{-g}F^2 + M\int ds + q\int A$$

The solution to the field equations is given by

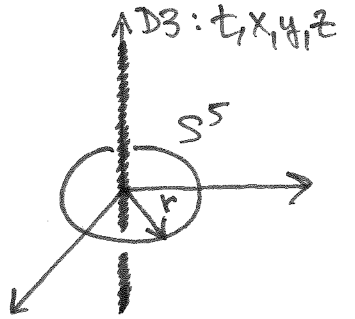
$$ds^2 = \frac{1}{\sqrt{h(r)}}\left(-dt^2 + d\vec{x}^{\,2}\right) + \sqrt{h(r)}\left(dr^2 + r^2 d\Omega_5^2\right)$$

with warp factor $h(r) = 1 + \frac{L^4}{r^4}$, $L^4 = 4\pi g_s N (\alpha')^2$

and $F_5 = \mathcal{F}_5 + *\mathcal{F}_5$, $\mathcal{F}_5 = 16\pi^{4}(\alpha')^2 N \,\mathrm{vol}(S^5)$

The normalisation of F_5 is such that the D3-brane charge is quantized in the following way

$$Q_{D3} = \frac{1}{(4\pi^2\alpha')^2}\int_{S^5} *F_5 = n \,, \quad \text{with } n = N.$$



We integrate over an S^5 around the D3-brane which stretches along the t, x, y, z volume.

Note: For the point particle in 4D we have $F_2 = -\frac{Q}{4\pi}\frac{1}{r^2}\, dt\wedge dr$ and $*F_2 = \frac{\sqrt{-g}}{2}\epsilon^{tr}{}_{\alpha\beta}\left(-\frac{Q}{4\pi}\frac{1}{r^2}\right)dx^\alpha\wedge dx^\beta = \frac{Q}{4\pi}\, d\theta\wedge d\varphi \sin\theta$

so that we get

$$\int_{S^2} *F_2 = \int \frac{Q}{4\pi}\sin\theta\, d\theta\wedge d\varphi = Q \,!$$

We are now going to discuss the theory on the branes at $r=0$. In the limit $r \to 0$ we have $h \sim L^4/r^4$ and $\sqrt{h} \sim L^2/r^2$. Defining $z = L^2/r$ the metric reduces to

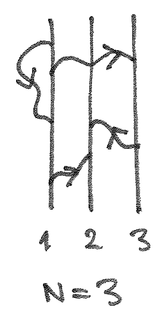
$$ds^2 = \frac{L^2}{z^2}\left(-dt^2 + d\vec{x}^2 + dz^2\right) + L^2 d\Omega_5^2$$

This is the metric of $AdS_5 \times S_5$ with equal radii of curvature L. For lecture notes on AdS, see Ingemar Bengtsson's lecture notes.

The theory on the branes is given by 4D superconformal $\mathcal{N}=4$ $SU(N)$ Yang-Mills theory. I now present very brief arguments for A) $SU(N)$ and B) $\mathcal{N}=4$.

A) The $SU(N)$ comes from the fact that there are $N \times N$ different strings stretching in between the N branes.

A state is described by $|ij\rangle$, an $N \times N$ matrix, denoting on which brane the string starts and ends.



You can find a short review on this subject in Zwiebach, or go to Polchinski!

We now go onto describe the action on the D3 world volume.

The actions we get from generalizing the two localized terms described before.

$$S_{D3} = S_{DBI} + S_{WZ}$$

Here S_{DBI} denotes the Dirac-Born-Infeld action, see 13.3.4 Polchinski vol II

$$S_{DBI} = -T_3 \int_{D3} d^4x \, \mathrm{tr} \left\{ e^{-\Phi} \left[-\det (g_{\mu\nu} + B_{\mu\nu} + 2\pi\alpha' F_{\mu\nu}) \right]^{1/2} \right\}$$

and S_{WZ} denotes the Wess-Zumino action, see 13.3.18 Polchinski vol II

$$S_{WZ} = i T_3 \int_{D3} \mathrm{tr} \left\{ \exp \left(2\pi\alpha' F_2 + B_2 \right) \wedge \sum_q C_q \right\}$$

Above $g_{\mu\nu}$, $B_{\mu\nu}$, Φ descibes the closed string fields described in Liams class; $F_{\mu\nu}$ is the Yang-Mills field; C_q denotes the generalized gauge fields corresponding to the $D(q-1)$-branes in your theory.

From the actions above we can read off the couplings of the Yang-Mills theory.

$$S_{DBI} \ni -\frac{1}{4g_{YM}^2} \int d^4x \, F_{ab} F^{ab} , \qquad \frac{1}{g_{YM}^2} \sim e^{-\Phi} \equiv \frac{1}{g_s}$$

$$S_{WZ} \ni -\frac{\theta_{YM}}{32\pi} \int d^4x \, F_{ab} \tilde{F}^{ab} , \qquad \theta_{YM} \sim i C_0 \equiv i \chi$$

B) I will not discuss how the geometry determines the SUSY of the theory, see introduction to chapter 17 in Polchinski or introduction to chapter 15 in GSW.

Though I would like to stress that both the dual theories have the same global symmetries:

- $AdS_5 \times S^5$ has an $SO(6)$ symmetry of S^5
- $\mathcal{N}=4$ has an $SU(4) \cong SO(6)$ R-symmetry

Note: Remember that the SUSY-algebra for $\mathcal{N}$ charges

$$\{ Q^I_\alpha , Q^\dagger_{\dot\alpha J} \} = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu \delta^I_J \quad \text{is invariant}$$

under $U(\mathcal{N})_R$ transformations $Q^I \to M^I{}_J Q^J$.

Final remark

Note that our supergravity solution is valid for small curvature, i.e. large radius $L^4 = 4\pi g_s N (\alpha')^2$. Thus we conclude that the SUGRA is reliable in the limit $g_s N >> 1$. This is the large N limit.

ii) <u>Klebanov-Witten</u> (KW)

In this work the background geometry $AdS_5 \times S^5$ is changed to some more general background $AdS_5 \times X_5$, i.e. the extra dimensions take on some more general cone structure $ds_6^2 = dr^2 + r^2 ds_{X_5}^2$, where X_5 is some manifold with reduced symmetries. This will give us different dual gauge theory with other symmetries.

The case studied in KW is the conifold $\mathcal{C}$ defined by

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0 \quad , \quad z_i \in \mathbb{C}$$

This geometry is a cone since it is invariant under the rescaling $z_i \to t z_i$, $t \in \mathbb{R}_+^*$. (It also has a $U(1)$ symmetry and an $SO(4)$).

The base we get from modding out the rescalings $(X_5 =)\ T^{1,1} \equiv \mathcal{C}/\mathbb{R}_+^*$. $T^{1,1}$ has a $U(1) \times SO(4) \cong U(1) \times SU(2) \times SU(2)$ symmetry and is topologically an S^1 bundle over $S^2 \times S^2$ with metric

$$ds_{T^{1,1}}^2 = \frac{1}{9}\left(d\psi + \cos\theta_1 d\varphi_1 + \cos\theta_2 d\varphi_2\right)^2 + \frac{1}{6}\left(d\theta_1^2 + \sin^2\theta_1 d\varphi_1^2 + d\theta_2^2 + \sin^2\theta_2 d\varphi_2^2\right)$$

with $\psi \in S^1$, $(\theta_1, \varphi_1) \in S^2$, $(\theta_2, \varphi_2) \in S^2$

Again placing N D3s at the conical singularity (the apex of the cone) we find a warped metric

$$ds^2 = \frac{1}{\sqrt{h(r)}}(-dt^2 + d\vec{x}^2) + \sqrt{h(r)}\,(dr^2 + r^2 ds^2_{T^{1,1}})$$

$$h(r) = 1 + \frac{L^4}{r^4}, \quad L^4 = 4\pi g_s N (\alpha')^2 \cdot \frac{27}{16}$$

$$\left(\mathrm{VOL}(T^{1,1}) = \frac{16}{27}\pi^3, \quad \mathrm{VOL}(S^5) = \pi^3\right)$$

The dual gauge theory is a superconformal $\mathcal{N}=1$ Yang-Mills theory with gauge group $SU(N)\times SU(N)$ and two chiral $SU(2)$ doublets A_i, B_i, $i=1,2$, in $(N,\bar{N})$, $(\bar{N},N)$ respectively. R-anomaly cancellation fixes $R_A = R_B = 1/2$.

The fact that $\mathcal{N}=1$ comes from the fact that C is Calabi-Yau and preserves $1/4$ of the original SUSYs. The mattercontent matches the global symmetries $U(1)\times SU(2)\times SU(2)$ of $T^{1,1}$.

The unique superpotential with these symmetries is given by $W = \mathrm{tr}(A_i B_k A_j B_l)\,\epsilon^{ij}\epsilon^{kl}$.

The gauge couplings are determined by

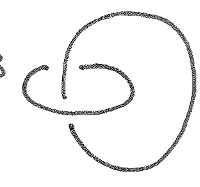
$$\begin{cases} \dfrac{4\pi^2}{g_1^2} + \dfrac{4\pi^2}{g_2^2} = \dfrac{\pi}{g_s e^{\Phi}} \\[2ex] \left[\dfrac{4\pi^2}{g_1^2} - \dfrac{4\pi^2}{g_2^2}\right] g_s e^{\Phi} = \dfrac{1}{4\pi\alpha'}\displaystyle\int B_2 \end{cases}$$

and since $\Phi = B_2 = 0$ there is no running, i.e. KW is a conformal theory.

iii) Klebanov–Tseytlin (KT)

We now add M fractional D3s, that is M wrapped D5s on a S^2 of $T^{1,1}$. Now a D5 couples to a C_6 with field strength $F_7 = dC_6$ and magnetic dual $F_3 = *F_7$ thus we have magnetic F_3-flux:

$$\frac{1}{4\pi\alpha'}\int_{S^3} F_3 = M \quad ,$$



The full action for IIB with $g_{MN}, B_{MN}, \Phi; C_0, C_2, C_4$ is now given by

$$S_{IIB} = \frac{1}{2\kappa_{10}^2}\int d^{10}x \sqrt{-g}\left(R + \frac{\partial_M \bar{\tau}\partial^M \tau}{2(\mathrm{Im}\,\tau)^2} - \frac{1}{2}M_{ij}F_3^i \cdot F_3^j - \frac{1}{4}|\tilde{F}_5|^2\right) - \frac{\epsilon_{ij}}{8\kappa_{10}^2}\int C_4 \wedge F_3^i \wedge F_3^j$$

with $\tau = C_0 + ie^{-\Phi}$, $F_3^i = \begin{pmatrix} H_3 \\ F_3 \end{pmatrix}$, $H_3 = dB_2$, see 12.1.31 Polchinski Vol II.

This time also B_2, C_2 together with C_4 takes on non-zero values, and B_2 will start running with r.

The metric is given by,

$$ds^2 = \frac{1}{\sqrt{h(r)}}(-dt^2 + d\vec{x}^2) + \sqrt{h(r)}\,(dr^2 + r^2 ds^2_{T^{11}})$$

This time h is not $1 + \frac{L^4}{r^4}$ but the three-form fluxes feeds the einstein equations such that

$$R = \frac{1}{24}(H_3^2 + g_s^2 F_3^2)$$

<u>Note</u>: We only have to use the trace equation since we have assumed only one variable h in the metric. Also F_5 does not enter since $F_5^2 = 0$ from the self-duality.

The solution is given by

$$h(r) = \frac{27\pi(\alpha')^2}{4r^4}\left(g_s N + a(g_s M)^2 \cdot \left[\ln\frac{r}{r_0} + \frac{1}{4}\right]\right)$$

with $a = 3/2\pi$.

Now $\Phi = \text{const}$ but B_2 runs with r so that

$$\begin{cases} \dfrac{4\pi^2}{g_1^2} + \dfrac{4\pi^2}{g_2^2} = \text{const} \\[2ex] \dfrac{8\pi^2}{g_1^2} - \dfrac{8\pi^2}{g_2^2} = 6M \ln\dfrac{r}{r_s} + \text{const} \end{cases}$$

$\therefore$ Identifying $\ln\frac{r}{r_s} = \ln\frac{\Lambda}{\mu}$ we find running in the dual theory!

We now show that the dual theory has exactly this running.

Now in the case of M additional fractional D3s the gauge group is enhanced from $SU(N) \times SU(N)$ to $SU(N+M) \times SU(N)$.

Now using the NSVZ result for the β-function

$$\frac{d}{d \ln \frac{\Lambda}{\mu}} \left(\frac{8\pi^2}{g^2} \right) = 3T(Ad) - \sum_i T(r_i)(1-\gamma_i)$$

(note the absence of the factor $\left(1 - T(Ad)\frac{g^2}{8\pi^2}\right)^{-1}$ on the left hand side which is a matter of conventions, see footnote 3 in original reference)

we have

$$\left(\frac{8\pi^2}{g_1^2}\right)' = 3(N+M) - \frac{1}{2}(1-\gamma) \cdot \overset{\substack{SU(2)\\ \downarrow}}{2} \cdot \overset{\substack{A,B\\ \downarrow}}{2} \cdot \overset{\substack{SU(N)\\ \downarrow}}{N}$$

$$\left(\frac{8\pi^2}{g_2^2}\right)' = 3 \cdot N - \frac{1}{2}(1-\gamma) \cdot 2 \cdot 2 \cdot (N+M)$$

$$\Rightarrow \left(\frac{8\pi^2}{g_1^2} - \frac{8\pi^2}{g_2^2} \right)' = M\left(3 + 2(1-\gamma)\right)$$

Now in this theory one can argue that $\gamma = -\frac{1}{2} + \mathcal{O}\left(\left(\frac{M}{N}\right)^{2n}\right)$ which gives us

$$\therefore \frac{8\pi^2}{g_1^2} - \frac{8\pi^2}{g_2^2} = 6 \ln \frac{\Lambda}{\mu}$$

which matches the SUGRA result exactly!

Referenser

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