Lectures on String Theory

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Abstract

This is a set of LaTeX'ed notes on String Theory from Liam McAllister's Physics 7683: String Theory course at Cornell University in Spring 2010. This is a working draft and is currently a set of personal notes. The lectures as given were flawless, all errors contained herein reflect solely the student's typographical and/or intellectual deficiencies.

Contents

1 Course details

> The content in this section are based on Liam McAllister's course bulletin and the introductory part of the first lecture.

This course will cover classic results in string theory that are relevant for contemporary research. It will differ from other courses on the subject in that our goal will be to cover special topics that developed over the past decade that have not yet found themselves in standard textbooks. We will necessarily have to tread more briefly and lightly on more traditional topics in perturbative string theory.

1.1 Prerequisites

The course will begin with a brief introduction to perturbative string theory; prior acquaintance with this subject is preferable, but not essential. Students are expected to have a solid background in quantum field theory and general relativity. Familiarity with supersymmetry will be necessary. In particular, the three aspects of supersymmetry that will be especially relevant are (i) BPS states, (ii) nonrenormalization theorems, and (iii) holomorphy. For an undergraduate-level introduction one can consult the text by Zwiebach [1]; the first twelve chapters roughly correspond to the first chapter or so of Polchinski.

1.2 References

The primary references for this course will be original papers, but it may be useful to consult the textbooks by Polchinski [2, 3]; Green, Schwarz, and Witten [4, 5]; and Becker, Becker and Schwarz [6]. The typist adds that a particularly good pedagogical treatment of a 'traditional' string theory course can be found in David Tong's lectures [7] and the accompanying course website [8].

1.3 Topics

After a brief introduction to the quantization of the bosonic string, topics will include important achievements from the 1980s, such as heterotic compactifications and non-renormalization theorems; the 1990s, including mirror symmetry, dualities, M-theory, matrix theory, and the AdS/CFT correspondence; and the past decade, including geometric transitions and flux compactifications. We will focus particularly on the AdS/CFT correspondence and its many applications, including dual descriptions of warped compactifications for models of electroweak symmetry breaking.

1.4 Additional sources

In some parts of these notes the typist has included discussions from other sources. These include the Durham University Centre for Particle Theory MSc course on Superstrings and D-Branes taught by Simon Ross in 2008, lecture notes from the 2008 version of the present course, Clifford Johnson's *D-Branes* text, Bailin and Love's SUSY and strings text (one of the original textbooks) [9], Dine's textbook [10], and various other sources as necessary.

Comment boxes like these will be scaterred around the document highlighting supplementary topics that were not directly covered in the lectures.

2 Introduction: What is string theory?

String theory is a quantum theory of 1D objects called strings. These strings come in open (free endpoints) and closed (connected endpoints) varieties. Slightly more rigorously, it can be defined as a quantum field theory on the (1+1) dimensional worldsheet of the string, $S = \int d^2\sigma\ \mathcal{L}_{\text{string}}$. There exist many such quantum field theories and so there exist many string theories. Further, for some string theories the strings themselves arise from wrapped higher-dimensional objects and hence can have some internal structure.

To whet our appetites and motivate our exploration of the subject, we will see that:

- All closed string theories contain a massless spin-2 particle. General arguments say that the only consistent couplings of such a particle are those of a graviton. Open string theories always contain closed strings, and thus string theory is a theory of quantum gravity. We will see further that it is in fact a *finite* theory of quantum gravity.

- Spacetime is treated as a target space of quantum fields. Consistency at the quantum level requires that the dimension of spacetime is $D > 3+1$. Bosonic strings require $D = 26$ while superstrings 'only' require $D = 10$. One can find other values for more exotic theories.

- The metric on the target space obeys the Einstein equations. This is surprising and amazing.

- Open strings often contain non-abelian gauge fields and chiral fermions. Both of these are important ingredients for the Standard Model. String theory naturally exists in $D > 4$ and can be readily supersymmetrized so that one might hope that string theory could be the UV completion of of the Standard Model and its most popular extensions.

Even though string theory has its origins in "dual resonance models" of hadrons in the pre-QCD era, much of its allure is its potential as a consistent theory of quantum gravity. Why should this be interesting? Recall in quantum field theory, e.g. a toy model of scalars,

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \lambda_4\phi^4 - \frac{\lambda_6}{M^2}\phi^6 + \cdots \tag{2.1}$$

that interactions λ_k with

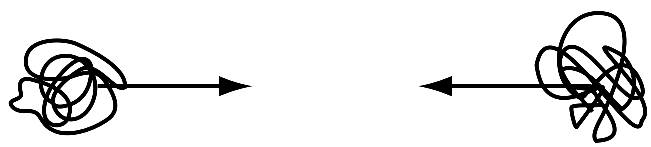
- positive mass dimension are super-renormalizable

- zero mass dimension are renormalizable

- negative mass dimension are non-renormalizable.

Recall further that the Newton constant has dimension $[G_N] = -2$ so that gravity is non-renormalizable. At high energies, higher order corrections become important. General relativity is badly behaved in the ultraviolet. We can understand this intuitively. Consider a high-energy collision between two point particles in GR+QFT:

At high-enough energies these can produce a microscopic black hole. The fact that we can have a black hole as an intermediate state tells us that the process can be very badly nonlinear. On the other hand, let us consider a cartoon picture of string scattering:



Higher energy strings have more oscillations, so our picture of a very high energy collision now looks like the collision of two bird nests. When the two bundles hit they produce another bird nest. This resulting tangled ball of yarn is larger than the Schwarzchild radius so that the collision is actually very soft. *Voilá*, no poor UV behavior!

So is that all? String theory gives us hope for a finite theory of quantum gravity with some prospect of lower-energy model building? No! The extremely rich structure of string theory has led to many important insights into a range of topics, such as

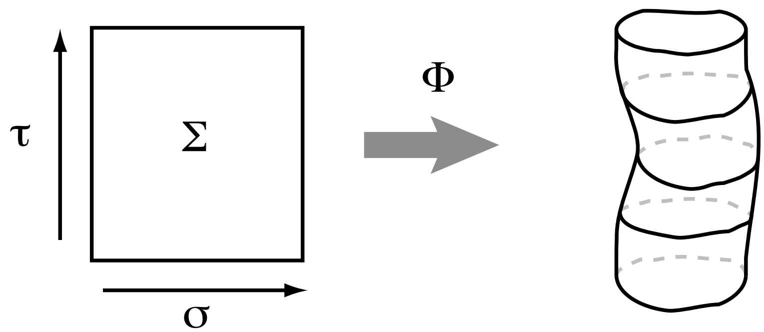
- nonperturbative dualities
- gauge theories (e.g. at strong coupling)
- mathematics (e.g. algebraic geometry)
- black holes (e.g. entropy from microscopic counting of states)
- holography, i.e. the AdS/CFT correspondence
- theories on branes.

In this course we will not spend too much time talking about strings as a fundamental theory. Instead, we will focus on topics that are the 'bonuses' that we get from studying a quantum theory of strings. These bonus topics have become a major reason for the growing appeal of string theory to a broad audience, including mathematicians, particle phenomenologists, and condensed matter theorists.

Now let's gets our hands dirty. *Allons-y!*

3 A classical theory of relativistic strings

We will begin by considering a *classical* theory of *relativistic* strings. Let us define the **target space** M to be a spacetime with metric $g_{\mu\nu}$ and coordinates X^μ. The string's **worldsheet** Σ is a two dimensional spacetime with coordinates $\xi_1 = \tau$, $\xi_2 = \sigma$ where we treat τ as a time direction. For now we will assume Minkowski signature for both spaces, though we will soon wick rotate into Euclidean coordinates on the worldsheet.



We would like to study embeddings $\Phi : \Sigma \to M$ of the worldsheet into the target space,

$$\Phi : \xi_\alpha \to X^\mu(\xi_\alpha). \tag{3.1}$$

Let us consider the pullback of the spacetime metric by the embedding Φ,

$$\Phi^* g \equiv h_{\alpha\beta} = \frac{\partial X^\mu}{\partial \xi^\alpha}\frac{\partial X^\nu}{\partial \xi^\beta} g_{\mu\nu}. \tag{3.2}$$

The natural measure for the size of the worldsheet is the usual volume form

$$\mathrm{Vol}(\xi_2) = \int d^2\xi \, \sqrt{-\det h_{\alpha\beta}}. \tag{3.3}$$

As we know from the derivation of an action principle in general relativity, this is also a natural candidate for the action of the string. Let us study this instructive analogy more thoroughly.

3.1 An analogy to the relativistic point particle

For the point particle we would like to consider the world*line* with a single parameter $\xi = \tau$. The pullback of the metric is simply

$$h_{00} = \frac{\partial X^\mu}{\partial \tau}\frac{\partial X^\nu}{\partial \tau} g_{\mu\nu} \tag{3.4}$$

so that the volume of the worldline's embedding into the target spacetime is

$$\mathrm{Vol}(\xi_1) = \int d\tau \, \sqrt{\frac{\partial X^\mu}{\partial \tau}\frac{\partial X^\nu}{\partial \tau} g_{\mu\nu}} = \frac{1}{m} S_{\text{point}}. \tag{3.5}$$

We are thus inspired to use $\mathrm{Vol}(\xi_2)$ as an action for the relativistic string. This is called the **Nambu-Goto** action,

$$S_{\mathrm{NG}} = -T \cdot \mathrm{Vol}(\xi_2). \tag{3.6}$$

The constant of proportionality (which was the particle mass for a point particle) is the string **tension** which has dimensions of mass per length, or $[T] = 2$. At this point one can treat this action as simply a choice that we can make.

Quantum field theorists should be unsettled. When we want to treat our relativistic particle as a quantum object we know that this volume form action is *not* the one we would use to quantize the theory. Actions with a square root are notoriously hard to quantize. We would like to write down an *equivalent* action that is easier to quantize. By this we mean we would like an action which will ultimately give us the same equation of motion (EOM) for the to-be-quantized fields, X:

$$\frac{\partial}{\partial\tau}\left(\frac{m\dot{X}^\mu}{\sqrt{\dot{X}^\mu\dot{X}_\mu}}\right) = 0 \qquad \Leftrightarrow \qquad \frac{m\dot{X}^\mu}{\sqrt{\dot{X}^\mu\dot{X}_\mu}} = \text{const.} \tag{3.7}$$

where we write the dot to mean a derivative with respect to τ.

So what is a likely action that would reproduce this equation of motion? As quantum field theorists we could write down our favorite action

$$S_{\text{wrong}} = \int d\tau \, \left(\dot{X}^\mu\dot{X}_\mu + m^2\right). \tag{3.8}$$

It sure has that familiar ring to it, but this doesn't work. A clever way to see this is to note that the action (3.5) obeys a symmetry

$$\tau \longrightarrow \lambda\tau. \tag{3.9}$$

This is simply a reparameterization invariance that tells us that it doesn't matter how we parameterized the worldline. We can see that our guess for a nice action (3.8) does not obey this reparameterization invariance and so cannot be correct. Let's try to fix up this action by introducing a compensator field e,

$$S' = \int d\tau \, \left(e^{-1}\dot{X}^\mu\dot{X}_\mu + em^2\right). \tag{3.10}$$

One can think of this compensator as a Lagrange multiplier, or more usefully (and with some foresight) as the metric on the worldline. If we assume that under reparameterizations we have

$$\tau \longrightarrow \lambda\tau \tag{3.11}$$
$$e \longrightarrow \lambda^{-1}e, \tag{3.12}$$

then our action S' successfully is successfully reparameterization invariant. The key thing to check is that we now get the same equation of motion. The EOM for the compensator field is

$$-\frac{1}{e^2}\dot{X}^\mu\dot{X}_\mu + m^2 = 0 \Rightarrow e = \frac{1}{m}\sqrt{\dot{X}^\mu\dot{X}_\mu}. \tag{3.13}$$

Similarly, the EOM for the X^μ field (the field of interest) is

$$\frac{d}{d\tau}\left(e^{-1}\dot{X}^\mu\right) = 0. \tag{3.14}$$

When we plug in the compensator EOM, we get precisely (3.7). We've thus found a quantization-friendly action for the relativistic point particle which reproduces the correct equation of motion.

3.2 The Polyakov action

We would now like to find the analogous quantization-friendly action for the string. This is called the **Polyakov action**. It takes the form

$$S_{\text{P}} = -\frac{T}{2}\int d^2\xi\, \sqrt{-\gamma}\,\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}, \tag{3.15}$$

where we are specializing to the case $g_{\mu\nu} = \eta_{\mu\nu}$ for now. This won't change the treatment here, but it will make quantization much smoother. (It is still a messy business to quantize string theory for a general target space metric.) The quantity $\gamma_{\alpha\beta}$ is our compensator field which now manifestly takes the form of a worldsheet metric. We've written $\gamma = \det\gamma_{\alpha\beta}$ so that our Polyakov action looks like a two dimensional σ model. This is an important analogy to keep in mind. Recall the usual σ model in 4D QFT on a general background spacetime with metric $g_{\mu\nu}$:

$$S_\sigma = \int d^4x\, \sqrt{g} g^{\mu\nu}\partial_\mu \phi^M \partial_\nu \phi^N G_{MN}, \tag{3.16}$$

where G_{MN} is a metric on the field space. To go back and forth from the Polyakov action for string theory and our σ model analogy, we note

$$X^\mu \leftrightarrow \phi^M \tag{3.17}$$

$$\eta_{\mu\nu} \leftrightarrow G_{MN} \tag{3.18}$$

$$\gamma_{\alpha\beta} \leftrightarrow g_{\mu\nu}. \tag{3.19}$$

In particular, don't confuse X^μ (which corresponds to a field to be quantized) with coordinates on the space (these are the ξ_αs or more properly $\Phi(\xi_\alpha)$), and don't confuse the worldsheet metric with the target space metric. This is a little tricky the first time you see it since the target space metric (i.e. the G_{MN} in our analogy) will end up being the spacetime metric that we're used to from general relativity. If you want you could take the Polyakov action as the starting point for string theory and treat everything prior to this as motivation.

We are working with a (1+1) dimensional quantum field theory with the metric $\gamma_{\alpha\beta}$ treated, in some sense, as a physical field. Does this mean that we're actually doing two dimensional general relativity? (Or, more appropriately: two dimensional GR coupled to scalar fields X^μ.) The answer is no. The worldsheet metric isn't *really* physical since currently it's just playing the role of the compensator from our analogy to the relativistic point particle. We know it's not physical because it doesn't have a kinetic term (like the auxilliary fields in supersymmetry). In other words, our action does not contain an Einstein-Hilbert term. In fact, general relativity in (1+1) dimensions is nearly trivial.

GR in (1+1)D. The Einstein-Hilbert action gives us the Einstein equations

$$S_{\text{EH}} = \frac{1}{2\kappa^2}\int d^2\xi\,\sqrt{-\gamma}R \quad \Rightarrow \quad R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = T_{\alpha\beta}, \tag{3.20}$$

which vanishes in vacuum. However, let us note that the Riemann tensor obeys

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\delta\gamma} = R_{\gamma\delta\alpha\beta} \tag{3.21}$$

so that since $\alpha, \beta, \gamma, \delta \in \{0, 1\}$ we can write

$$R_{\alpha\beta\gamma\delta} = f\,\epsilon_{\alpha\beta}\epsilon_{\gamma\delta}. \tag{3.22}$$

Contracting indices to get the Ricci tensor we find

$$R = f\,\epsilon_{\alpha\beta}\epsilon_{\gamma\delta}g^{\alpha\gamma}g^{\beta\delta} = R_{\alpha\beta\gamma\delta}f\,g^{\alpha\gamma}\left({\epsilon_\alpha}^\delta\epsilon_{\gamma\delta}\right) = f\,\epsilon^{\gamma\delta}\epsilon_{\gamma\delta} = 2f, \tag{3.23}$$

where we've used ${\epsilon_\alpha}^\delta\epsilon_{\gamma\delta} = g_{\alpha\gamma}$. Plugging this back into the Riemann tensor we obtain

$$R_{\alpha\beta\gamma\delta} = \frac{R}{2}\epsilon_{\alpha\beta}\epsilon_{\gamma\delta} \tag{3.24}$$

which we can again plug back into the Ricci tensor to find (in vacuum)

$$R_{\alpha\gamma} \equiv R_{\alpha\beta\gamma\delta}g^{\beta\delta} = \frac{1}{2}Rg_{\alpha\beta}, \tag{3.25}$$

which is precisely the Einstein equation. In other words, the equations of motion from the 2D Einstein-Hilbert action are trivially satisfied.

We want to take the variation of the action with respect to the worldsheet metric $\gamma_{\alpha\beta}$. In particular, we would like to write out the tricky term $\delta\sqrt{-\gamma}/\delta\gamma_{\alpha\beta}$, recalling that $\gamma = \det\gamma_{\rho\sigma}$. We can invoke a well-known result for a general metric g_{ab},

$$\delta g = gg^{ab}\delta g_{ab}, \tag{3.26}$$

which one can prove by using $\det M = \exp\operatorname{Tr}\ln M$. The important thing to keep track of are signs relative to the height of the indices. For example, by swapping the heights of the indices we can see that we pick up a negative sign since

$$\delta(g^{ab}g_{ab}) = 0 \qquad \Rightarrow \qquad \delta g = -gg_{ab}\delta g^{ab}. \tag{3.27}$$

We thus have

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{ab}\delta g_{ab} = -\frac{1}{2}\sqrt{-g}g_{ab}\delta g^{ab}. \tag{3.28}$$

The variation of our action with respect to $\gamma^{\rho\sigma}$ is then

$$\delta S = -\frac{T}{2}\int d^2\xi \,\left\{-\frac{1}{2}\sqrt{-\gamma}\,\gamma_{\sigma\rho}\,\delta\gamma^{\sigma\rho}\,\gamma^{\alpha\beta}\,h_{\alpha\beta} + \sqrt{-\gamma}\,h_{\sigma\rho}\,\delta\gamma^{\sigma\rho}\right\} \tag{3.29}$$

$$\frac{\delta S}{\delta\gamma^{\rho\sigma}} = -\frac{T}{2}\int d^2\xi\,\sqrt{-\gamma}\left\{h_{\rho\sigma} - \frac{1}{2}\gamma_{\rho\sigma}\,\gamma^{\alpha\beta}h_{\alpha\beta}\right\} \tag{3.30}$$

with the equation of motion

$$h_{\rho\sigma} = \frac{1}{2}\gamma_{\rho\sigma}\,\gamma^{\alpha\beta}\,h_{\alpha\beta}. \tag{3.31}$$

When these EOM are satisfied, one can take a determinant to prove

$$\gamma^{\alpha\beta}h_{\alpha\beta} = 2\frac{\sqrt{-h}}{\sqrt{-\gamma}}. \tag{3.32}$$

To prove this just take the determinant of both sides and obtain an expression for $\sqrt{-h}$ then explicitly use these equations to write out $h_{\alpha\beta}/\sqrt{-h}$. The factor of two comes from a trace of $\gamma_{\alpha\beta}$. Plugging this handy equation back into the Polyakov action, we get

$$S_{\mathrm{P}}|_{\mathrm{EOM}} = -T\int d^2\xi\,\sqrt{-h} = S_{\mathrm{NG}}, \tag{3.33}$$

and so we have successfully shown that the Polyakov action is the same as the Nambu-Goto action once we plug in the equation of motion for the compensator (worldsheet metric). It may be worth remarking that even though the Polyakov action contains a square root of a field ($\sqrt{-\gamma}$), we are perfectly happy with this since it is not the field that we are quantizing. This is the analogy of doing quantum field theory on a curved background.

Variation of the action. To prove equations (3.27) and (3.28) we use the two matrix identities

$$\det M = \exp \mathrm{Tr}\ln M \qquad\qquad (M+\delta M)_{ab} = M_{ac}\left(\delta^c{}_b + (M^{-1})^{cd}\,\delta M_{db}\right). \tag{3.34}$$

From this we have

$$\det(M+\delta M) = \det M\,\det\left(\mathbb{1} + M^{-1}\delta M\right) \tag{3.35}$$

$$= \det M\,\exp\mathrm{Tr}\ln\left(\mathbb{1} + M^{-1}\delta M\right) \tag{3.36}$$

$$= \det M\,\left(\mathbb{1} + \mathrm{Tr}M^{-1}\delta M\right), \tag{3.37}$$

so that the variation of the determinant is

$$\delta\det M = \det M\,\mathrm{Tr}M^{-1}\delta M \tag{3.38}$$

$$= \det M\,\left(M^{-1}\right)^{ab}\,\delta M_{ab}. \tag{3.39}$$

For $M_{ab} = g_{ab}$ we get precisely (3.27), from which (3.28) follows readily as explained above.

3.3 Symmetries of the Polyakov action

Before we work out the classical equations of motion, let's pause to think about the Polyakov action. As every theorist knows, symmetries are very powerful tools. We saw this when we noted the reparameterization invariance of the relativistic point particle action. Let's go over the symmetries of the Polyakov action.

1. **Poincaré symmetry**. This is the symmetry of Lorentz transformations and translations. This means that our action is invariant if our fields transform as

$$X^\mu(\tau,\sigma) \to \Lambda^\mu{}_\nu X^\nu(\tau,\sigma) + a^\mu. \tag{3.40}$$

2. **Diffeomorphism invariance** in 2D, i.e. reparameterization invariance. This is the symmetry of general relativity that tells us that we could choose coordinates however we like:

$$\tau \to \tau'(\tau,\sigma) \tag{3.41}$$

$$\sigma \to \sigma'(\tau,\sigma) \tag{3.42}$$

   where we remind ourselves that the worldsheet metric in different coordinates is

$$\gamma_{\alpha\beta}(\tau,\sigma) = \frac{\partial\xi'^\rho}{\partial\xi^\alpha}\frac{\partial\xi'^\sigma}{\partial\xi^\beta}\gamma'_{\rho\sigma}(\tau',\sigma'). \tag{3.43}$$

3. **Weyl rescaling**. This is a particularly important symmetry which we can see in the Polyakov action but not in the Nambu-Goto action. We may rescale the metric via

$$\gamma_{\alpha\beta}(\tau,\sigma) \to e^{2\omega(\tau,\sigma)}\gamma_{\alpha\beta}(\tau,\sigma). \tag{3.44}$$

   It is clear that this is related to conformal invariance, which will be very handy for us. That Weyl invariance should appear in S_{P} and not S_{NG} is a sign that local dilations are an additional redundancy of the Polyakov actions. It is worth noting that $\omega(\tau,\sigma)$ is not a physical field and has no degree of freedom associated with it.

Diffeomorphism and Weyl invariance are *local* symmetries of the Polyakov actions, in other words we can think of them as gauge symmetries. (And here the mnemonic 'gauge symmetry = gauge *redundancy*' is handy.) Poincaré symmetry, on the other hand, is a global, internal symmetry.

3.4 The X^μ equation of motion

We can work out that the functional derivative of the Polyakov Lagrangian with respect to the derivative of the dynamical field is

$$\frac{\delta\mathcal{L}_P}{\delta\partial_\alpha X^\mu} = -\frac{T}{2}\sqrt{-\gamma}\cdot 2\gamma^{\alpha\beta}\partial_\beta X^\nu\eta_{\mu\nu}. \tag{3.45}$$

Thus the Euler-Lagrange equations give

$$\partial_\alpha\left(\sqrt{-\gamma}\,\gamma^{\alpha\beta}\partial_\beta X_\mu\right) = 0. \tag{3.46}$$

Does this look familiar? One should recall (or prove for one's self or otherwise accept on some kind of religious faith) that the expression for the Laplacian for a general metric g_{ab} is

$$\nabla^2 f \equiv \frac{1}{\sqrt{g}}\partial_a\left(\sqrt{g}\, g^{ab}\partial_b f\right). \tag{3.47}$$

Thus we see that our Euler-Lagrange equation is just telling us that $\nabla^2 X^\mu = 0$, i.e. the strings (unsurprisingly) obey a wave equation. Since our worldsheet (for now) is Minkowski, this we should really be talking about the d'Alembertian, but since we'll be Wick rotating there's no need to make a distinction.

3.5 Boundary conditions

Now we get into the business of boundary terms. Usually in QFT we derive the equations of motion and discard the boundary terms since we assume spacetime to be infinite and that all relevant quantities die off sufficiently quickly. In string theory, however, boundaries matter. The variation of the Polyakov action includes boundary terms

$$\delta S_P = -\frac{T}{2}\cdot 2\int d\tau\, d\sigma\, \sqrt{-\gamma}\,\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta\left(\delta X_\mu\right). \tag{3.48}$$

Recall that for any operator $\mathcal{O}$

$$\int d\xi_\alpha\, \mathcal{O}\,\partial_\alpha \delta X_\mu = -\int d\xi_\alpha\left(\partial_\alpha \mathcal{O}\right)\delta X_\mu + \mathcal{O}\,\delta X_\mu\Big|_{\xi_\alpha^{\text{init}}}^{\xi_\alpha^{\text{fin}}}. \tag{3.49}$$

We will treat τ as a timelike coordinate and σ as a spacelike coordinate with the integration regions

$$-\infty < \tau < \infty \tag{3.50}$$

$$0 \le \sigma \le \pi. \tag{3.51}$$

Thus the boundary terms for τ really do vanish, while we are left with a boundary term for σ,

$$\delta S_P = -T\int d\tau\, \sqrt{-\gamma}\,\partial^\sigma X^\mu \delta X_\mu\Big|_{\sigma=0}^{\sigma=\pi}. \tag{3.52}$$

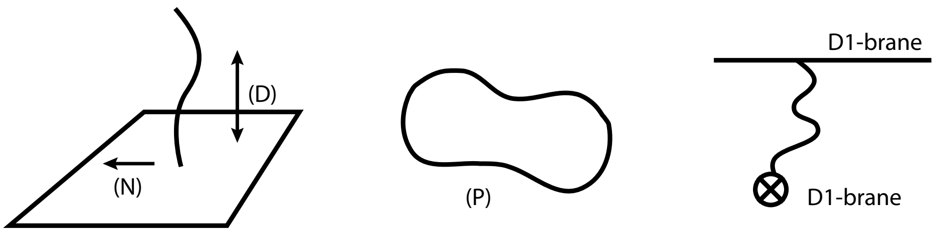
This boundary term vanishes if they are

(a) **Neumann**: $\partial^\sigma X^\mu(\tau,0) = \partial^\sigma X^\mu(\tau,\pi) = 0$

(b) **Dirichlet**: $X^\mu(\tau,0) = \text{const}$ and $X^\mu(\tau,\pi) = \text{const}$

(c) **Periodic**: $X^\mu(\tau,0) = X^\mu(\tau,\pi)$, $\partial^\sigma X^\mu(\tau,0) = \partial^\sigma X^\mu(\tau,\pi)$, $\gamma(\tau,0) = \gamma(\tau,\pi)$.

The periodic boundary conditions correspond to a closed string for somewhat obvious reasons (see picture below). For an open string one may have Neumann boundary conditions in some directions and Dirichlet boundary conditions in others. This maps out surfaces in the target space that the strings can move along. One of the unexpected things we will find in our study of string theory is that it is not only a theory of strings, but also of extended objects called **D-branes**. We shall denote the dimensionality of a brane by saying that a Dp-brane is an object with p space dimensions. Here are some pictures illustrating Neumann and Dirichlet boundary conditions defining a Dp-brane, the periodic boundary conditions for a closed string, and an example of how an open string may have endpoints which are Neumann/Dirichlet in different directions on each endpoint.



3.6 The energy-momentum tensor

Recall the equation of motion for the $\gamma_{\alpha\beta}$,

$$\frac{\delta S_P}{\delta \gamma_{\alpha\beta}} = 0, \tag{3.53}$$

where we computed the left-hand side in (3.30). We know how to understand this quantity from appealing to general relativity, where we know that

$$T^{ab} = \frac{\text{const.}}{\sqrt{-g}} \frac{\delta S_{\cancel{EH}}}{\delta g_{ab}}, \tag{3.54}$$

where $S_{\cancel{EH}}$ means everything in S that is *not* the Einstein-Hilbert action, i.e. the matter action. The constant is usually -2 but we will write it as 4π to follow the usual notation from Polchinski. We end up with

$$T_{\alpha\beta} = -2\pi T \left\{ \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2} \gamma_{\alpha\beta} \partial^\alpha X^\mu \partial^\beta X_\mu \right\}, \tag{3.55}$$

from which we can see manifestly that $T_{\alpha\beta}$ is traceless: $\gamma^{\alpha\beta} T_{\alpha\beta} = 0$. There is also a definition of the energy-momentum tensor from the Noether procedure which agrees. Since we'll be using Noether's theorem, it is prescient to review this now. If, under a transformation of fields

$$\phi^a \to \phi^a + \delta\phi^a(\xi^a) \tag{3.56}$$

the Lagrangian density transforms as

$$\delta \mathcal{L}(\phi^a, \partial_\alpha \phi^a) = \partial_\alpha V_a^\alpha, \tag{3.57}$$

then we have (plugging in the Euler-Lagrange equations)

$$\partial_\alpha V_a^\alpha = \frac{\delta \mathcal{L}}{\delta \phi^a}\delta\phi^a + \frac{\delta\mathcal{L}}{\delta(\partial_a \phi^a)}\delta(\partial_a\phi^a) \tag{3.58}$$

$$= \partial_\alpha\left(\frac{\delta\mathcal{L}}{\delta(\partial_\alpha\phi^a)}\right)\delta\phi^a + \left(\frac{\delta\mathcal{L}}{\delta(\partial_\alpha\phi^a)}\right)\partial_a\delta\phi^a \tag{3.59}$$

$$= \partial_\alpha\left[\left(\frac{\delta\mathcal{L}}{\delta(\partial_\alpha\phi^a)}\right)\delta\phi^a\right]. \tag{3.60}$$

This tells us that there is conserved current j_a^α, **[CHECK:** a indices?]

$$\partial_\alpha j_a^\alpha \equiv \partial_\alpha\left[\left(\frac{\delta\mathcal{L}}{\delta(\partial_\alpha\phi^a)}\right)\delta\phi^a - V_a^\alpha\right] = 0. \tag{3.61}$$

Let's consider a concrete application of this. If $\mathcal{L}$ does not depend explicitly on ξ^α (e.g. if our action is reparameterization invariant) then under $\xi'^\alpha = \xi^\alpha + \epsilon^\alpha$, we have

$$\phi^a(\xi^\alpha + \epsilon^\alpha) \approx \phi^a(\xi^\alpha) + \epsilon^\alpha\partial_\alpha\dot{\phi}^a. \tag{3.62}$$

We can then show that

$$\delta\mathcal{L} = \epsilon^\beta\frac{\partial}{\partial\xi^\alpha}\left(\delta^\alpha{}_\beta\mathcal{L}\right). \tag{3.63}$$

Thus there exist conserved currents

$$T^{\alpha\beta} \equiv j^{\alpha\beta} = \frac{\delta\mathcal{L}}{\delta(\partial_\alpha X^\mu)}\partial^\alpha X_\mu - \gamma^{\alpha\beta}\mathcal{L} \tag{3.64}$$

$$= -\frac{T}{2}\left[2\partial^\beta X^\mu\partial^\alpha X_\mu - \gamma^{\alpha\beta}\partial_{\alpha'}X^\mu\partial_{\beta'}X_\mu\gamma^{\alpha'\beta'}\right] \tag{3.65}$$

$$= -T\left[\partial^\alpha X^\mu\partial^\beta X_\mu - \frac{1}{2}\gamma^{\alpha\beta}\partial^\gamma X^\mu\partial_\gamma X_\mu\right]. \tag{3.66}$$

where we can identify this with the stress-energy tensor $j^{\alpha\beta} \Leftrightarrow T^{\alpha\beta}$ so that one can see that the GR and Noether definitions of $T^{\alpha\beta}$ agree.

It is also useful to notice that under constant [target] spacetime translations $X^\mu \to X^\mu + \epsilon^\mu$ the action is invariant with $V^\mu = 0$. This tells us that

$$j_\mu^\alpha = \frac{\delta\mathcal{L}}{\delta(\partial_\alpha X^\mu)} \tag{3.67}$$

$$j_\mu^\tau = \frac{\delta\mathcal{L}}{\delta\dot{X}^\mu} \tag{3.68}$$

$$j_\mu^\sigma = \frac{\delta\mathcal{L}}{\delta X'^\mu}. \tag{3.69}$$

The space integral of the time component gives the conserved charge,

$$Q = \int d^{d-1}x\ j^0 = \int d\sigma\ j_\tau^\mu, \tag{3.70}$$

which is just spacetime momentum.

3.7 Summary so far

We've seen that from the principle that the action is proportional to the volume form $S \propto \mathrm{Vol}(\Phi(\Sigma))$ we can get two actions:

$$S_{\mathrm{NG}} = -T \int d^2\xi \, \sqrt{-h} \tag{3.71}$$

$$S_{\mathrm{P}} = -\frac{T}{2} \int d^2\xi \, \sqrt{-\gamma}\, \gamma^{\alpha\beta} h_{\alpha\beta}. \tag{3.72}$$

The Nambu-Goto action is really ugly since it has a square root of the dynamical fields. We can get rid of the square root by transitioning to the Polyakov action. This form is less ugly, but is still somewhat formidable. We'll try to simplify this even further in just a bit. The equations of motion are

$$\nabla^2 X^\mu = 0 \tag{3.73}$$

$$T^{\alpha\beta} = 0. \tag{3.74}$$

We have three sets of symmetries: D-dimensional Poincaré, two dimensional diffeomorphism, and Weyl. We also have three kinds of boundary conditions: Dirichlet, Neumann, and periodic. Using our calculation for $\delta S_P / \delta \gamma_{\alpha\beta}$, we have

$$T_{\alpha\beta} = -2\pi T \left\{ \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2} \gamma_{\alpha\beta} \partial^\rho X^\mu \partial_\sigma X_\mu \right\}, \tag{3.75}$$

from which it is clear that

$$\gamma^{\alpha\beta} T_{\alpha\beta} = 0, \tag{3.76}$$

i.e. our energy-momentum tensor is traceless. You know what that means: our theory is scale invariant. (This is another harbinger of conformal methods to come.)

3.8 Gauge fixing

We would like to use our symmetries to even further simplify our Polyakov action before we quantize, i.e. we'd like to gauge fix. A good analogy is the quantization of QED. We have to be a bit careful since we must remember to impose the "lost" equations of motion (the EOM before gauge fixing) as constraints to our gauge-fixed theory. This is best clarified with an example.

The worldsheet metric $\gamma_{\alpha\beta}$ is a symmetric 2×2 matrix and thus has 3 degrees of freedom. Diffeomorphism invariance gives us two functions to fix two degrees of freedom while Weyl invariance gives an additional function to fix a degree of freedom. Thus it looks like we can completely gauge away γ, i.e. we can perform transformations such that

$$\gamma_{\alpha\beta} \xrightarrow[\text{Diff}]{} e^{2\omega} \eta_{\alpha\beta} \xrightarrow[\text{Weyl}]{} \eta_{\alpha\beta}. \tag{3.77}$$

We call this the **unit gauge**. However, the equation of motion for the $\gamma_{\alpha\beta}$ tells us that $T_{\alpha\beta} = 0$. This piece of information is lost when we gauge fix $\gamma_{\alpha\beta}$. We've lost a total of two degrees of freedom.

Why two? From the symmetry of the energy-momentum tensor we can count three independent degrees of freedom: $T_{01} = T_{10}$, T_{11}, and T_{00}. Additionally, we know from the tracelessness of the energy-momentum tensor that two of these are related via $T_{00} - T_{11} = 0$. This leaves us with two equations of motion in $T_{\alpha\beta} = 0$ that have to be applied as a constraint.

We continue to use a dot to mean a $\partial/\partial\tau$ derivative and introduce the notation of a prime to mean a $\partial/\partial\sigma$ derivative. The constraints that have to be applied are thus

$$0 = T_{00} \propto \frac{1}{2}\left(\dot{X}^2 + X'^2\right) \tag{3.78}$$

$$0 = T_{01} \propto \dot{X} \cdot X'. \tag{3.79}$$

These are called the **Virasoro constraints**. So in summary, we can use our symmetries to dramatically simplify our general worldsheet metric $\gamma_{\alpha\beta}$ by sending it to the flat metric $\gamma_{\alpha\beta} \to \eta_{\alpha\beta}$, but we have to additionally impose $\dot{X}^2 = X'^2 = \dot{X} \cdot X' = 0$.

We can now start thinking a couple steps ahead and consider our options for quantizing this classical theory. We have three options:

1. Old **covariant quantization**. We can quantize the theory first and then impose the Virasoro constraints. The Hilbert space after quantization is too big and includes negative norm states (it is not manifestly unitary, and the Schrodinger equation doesn't generate time evolution) so that imposing the Virasoro constraints corresponds to projecting onto a physical Hilbert subspace. The nice feature of this approach is that we never have to mention gauge choices when quantizing and so it is manifestly covariant.

2. **Lightcone gauge**. Alternately, *before* we quantize we can use our symmetries immediately to immediately pick a particular gauge that solves the Virasoro constraints. This is manifestly unitary but also not covariant and obscures Lorentz invariance. It has the benefit of being a quick way to quantize the string.

3. **BRST quantization** ('new' covariant quantization). Unfortunately we do not have the time (or perhaps patience) to cover this rich subject in our course.

We will highlight some features of the covariant approach and point out where things become hairy. Upon the first sign of distress we'll immediately retreat to light cone gauge and follow through.

4 Quantizing the string

We will proceed by focusing on the case of an open string with Neumann boundary conditions. We will mention the other two boundary conditions (Dirichlet, periodic) at the end of this section.

4.1 Solving the classical theory

Before we jump into quantizing the string, let's make use of all the pieces we've gathered to explicitly solve the classical theory. Let's remind ourselves once again that the that the Polyakov action is

$$S_{\mathrm{P}} = -\frac{T}{2} \int d^2\xi \sqrt{-\gamma}\, \gamma^{\alpha\beta}\, \partial_\alpha X^\mu \partial_\beta X_\mu. \tag{4.1}$$

Furhter, we recall that we've used our diffeomorphism and Weyl invariance to send our worldsheet metric to a canonical form, (3.77). The equation of motion for the X^μ fields is

$$\partial_\alpha \partial^\alpha X^\mu = 0. \tag{4.2}$$

We also have the additional constraint $T_{\alpha\beta} = 0$ arising as a relic of gauge fixing, but we'll get to this in due course. The solution to the X^μ equation of motion is easy since this is just the wave equation. We know that the general solution is written as the sum of left-moving (X_L^μ) and right-moving (X_R^μ) modes,

$$X^\mu(\tau, \sigma) = X_L^\mu(\tau + \sigma) + X_R^\mu(\tau - \sigma). \tag{4.3}$$

4.2 Neumann boundary conditions

Let's now consider the case of Neumann boundary conditions. This requires imposing the following constraints:

$$\frac{\partial X^\mu}{\partial \sigma}(\tau, 0) = 0 \tag{4.4}$$

$$\frac{\partial X^\mu}{\partial \sigma}(\tau, \ell) = 0. \tag{4.5}$$

From the first equation we get the constraint (up to an overall constant which can be absorbed into the definition of X_L^μ)

$$X_L'^\mu(u) = X_R'^\mu(u) \equiv f^\mu(u), \tag{4.6}$$

where prime ($'$) denotes a derivative with respect to the argument, e.g. $\partial/\partial(\tau + \sigma)$ for X_L^μ. Thus

$$X^\mu = \frac{1}{2}\left[f^\mu(\tau + \sigma) + f^\mu(\tau - \sigma)\right]. \tag{4.7}$$

The second boundary condition then reduces to

$$f'^\mu(\tau + \ell) = f'^\mu(\tau - \ell), \tag{4.8}$$

which tells us that f'^μ is periodic with period 2ℓ. The most useful normalization is to set $\ell = \pi$ so that $\sigma \in [0, \pi]$. We can now straightforwardly expand this function into a Fourier series,

$$f^\mu(u) = f_0^\mu = f_1^\mu u + \sum_{n=1}^{\infty} \left(A_n^\mu \cos nu + B_n^\mu \sin nu\right). \tag{4.9}$$

Plugging into our X^μ general solution (4.6) we have

$$X^\mu = f_0^\mu + f_1^\mu \tau + \sum_{n-1}^{\infty} A_n^\mu \cos[n(\tau+\sigma)] + B_n^\mu \sin[n(\tau+\sigma)]$$
$$+A_n^\mu \cos[n(\tau-\sigma)] + B_n^\mu \sin[n(\tau-\sigma)] \tag{4.10}$$
$$= f_0^\mu + f_1^\mu \tau + \sum_{n-1}^{\infty} \left(A_n^\mu \cos n\tau + B_n^\mu \sin n\tau\right) \cos n\sigma. \tag{4.11}$$

The f_1^μ is clearly the momentum P^μ. Now to connect to the literature and for later convenience, let us define an additional set of coefficients a_n^μ such that

$$A_n^\mu \cos n\tau + B_n^\mu \sin n\tau \equiv i\sqrt{\frac{2}{n}} \frac{1}{\sqrt{2\pi T}} \left(a_n^{\mu *} e^{in\tau} - a_n^\mu e^{-in\tau}\right), \tag{4.12}$$

where we can also introduce the ubiquitous **inverse tension**, $\alpha' = 1/2\pi T$ which has dimensions of (length)^2. These a_n Fourier coefficients will be the canonically normalized raising and lowering operators in our quantized theory. Instead of these perfectly reasonable objects, we will be somewhat perverse and introduce yet another set of coefficients, α,

$$\alpha_0^\mu = P^\mu / \sqrt{2\alpha'} \tag{4.13}$$
$$\alpha_n^\mu = a_n^\mu \sqrt{n} \tag{4.14}$$
$$\alpha_{-n}^\mu = (\alpha_n^\mu)^* \sqrt{n}, \tag{4.15}$$

where $n \geq 1$. Under no circumstances should These αs be confused with the constant α'. The *raison d'être* for these oddly-normalized coefficients will not be clear until equation (4.69) when a τ derivative will bring down a factor of n to cancel a $1/n$ in the Virasoro constraint. For now have faith that it's a somewhat arbitrary definition of our coefficients with no physical content but that will eventually simplify some expressions. Plugging in all of this jazz into our expression for X^μ,

$$X^\mu = X_0^\mu + \sqrt{2\alpha'}\alpha_0^\mu \tau - \sqrt{\frac{2}{n}} \frac{1}{\sqrt{2\pi T}} \sum_{n=1}^{\infty} \left(a_n^{\mu *} e^{in\tau} - a_n^\mu e^{-in\tau}\right) \cos n\sigma \tag{4.16}$$
$$= X_0^\mu + \sqrt{2\alpha'}\alpha_0^\mu \tau - i\sqrt{2\alpha'} \sum_{n=1}^{\infty} \frac{1}{n} \left(\alpha_n^\mu e^{in\tau} - \alpha_n^\mu e^{-in\tau}\right) \cos n\sigma \tag{4.17}$$
$$= X_0^\mu + \sqrt{2\alpha'}\alpha_0^\mu \tau + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos n\sigma. \tag{4.18}$$

Note that all we've done is shuffle around how we define coefficients in a way that will turn out to be helpful. One can check that this is the still the most general solution to the X^μ equation of motion subject to Neumann boundary conditions. This expression is completely classical and we have not yet imposed the very important constraint $T_{\alpha\beta} = 0$ coming from our specialization to unit gauge. Thus we have not yet found a correct solution.

4.3 Quantizing the open string incorrectly

Being bold men and women of science, we won't let matters of correctness get in the way of pedagogy. Let's go ahead and try to quantize what we have to see where things fall apart. Recall the usual prescription: from the Lagrangian, we find the canonical momenta π to each field ϕ via

$$\pi = \frac{\delta \mathcal{L}}{\delta(\partial_t \phi)} \tag{4.19}$$

and then impose the equal-time canonical commutation relations

$$[\phi(t,x), \pi(t,x')] = i\delta(x-x'). \tag{4.20}$$

In our [overly] convenient unit gauge our action takes the form

$$S = -\frac{T}{2}\int d\tau\, d\sigma\ \partial_\tau X^\mu \partial_\tau X_\mu - \partial_\sigma X^\mu \partial_\sigma X_\mu \tag{4.21}$$

and so our momenta are

$$P^\mu(\tau,\sigma) = -T\partial_\tau X^\mu(\tau,\sigma). \tag{4.22}$$

Our equal-time canonical commutation relations take the form

$$[X^\mu(\tau,\sigma), P^\mu(\tau,\sigma')] = i\eta^{\mu\nu}\delta(\sigma-\sigma'). \tag{4.23}$$

Let us remark first that $\eta^{\mu\nu}$ should be understood to be the metric on the target space of the fields X^μ rather than as a spacetime metric. While both statements are correct, it is important to emphasize that we are doing field theory on the two-dimensional string worldsheet, *not* the D-dimensional spacetime. Next, one might ask why this factor of $\eta^{\mu\nu}$ should appear at all in the commutation relations. This comes from the covariance with respect to the global symmetry of the X^μ target space (which happens to be Lorentz). Substituting our naïve mode expansion (4.18), we get

$$[\alpha^\mu_m, \alpha^\nu_n] = m\eta^{\mu\nu}\delta_{m,-n}. \tag{4.24}$$

This, however, is *terrible*! Consider, for example, the $\mu=\nu=0$ relation,

$$[\alpha^0_m, \alpha^0_{-m}] = m\eta^{00} = -m. \tag{4.25}$$

We can consider a vacuum state $|0\rangle$ which is acted upon by the raising operators $\alpha^\mu_{-m} = \alpha^{\mu\dagger}_m$. then this commutation relation tells us that

$$||\alpha^0_{-m}|0\rangle||^2 = \langle 0|\alpha^0_m\alpha^0_{-m}|0\rangle = -m\langle 0|0\rangle < 0. \tag{4.26}$$

This state has a negative norm! We can now scream, pull out our hair, and say things like 'not unitary', 'negative probability', 'ghost,' and 'apocalypse.' Of course, it's not really the end of the world since we know that we've been going about this whole business very naïvely since we haven't taken into account the all-important Virasoro constraints, $T_{\alpha\beta}=0$.

4.4 A leftover gauge freedom

Like all good fairy tales, our problem will be solved (to some extent) when we incorporate the Virasoro constraints. We know from Section 3.8 that there are a few ways of doing this. Our problematic undertaking above was a half-hearted attempt at covariant quantization where Lorentz symmetry was manifest at each step. In other words, each equation was written in terms of objects with well-defined Lorentz structure and we never had to 'break up' the μ, ν indices into non-covariant components. We could work harder and fix up this attempt, but instead we will sacrifice covariance and retreat to a simpler 'quick-and-dirty' quantization procedure in **light cone gauge**.

We can always write down our coordinates/fields in light cone *coordinates*,

$$x^{\pm} = \frac{1}{\sqrt{2}} \left(x^0 \pm x^1 \right). \tag{4.27}$$

This is just a trivial choice of coordinates and we haven't actually 'done anything' with respect to our theory. Light cone *gauge* is different and involves actually fixing a remaining gauge freedom in our theory.

Wait, *what*? Haven't we already used up our entire diffeomorphism and Weyl (diff×Weyl) gauge invariance to set $\gamma_{\alpha\beta} \to \eta_{\alpha\beta}$? Somewhat surprisingly, we have not. We have a leftover 'hidden' gauge symmetry. Actually, there's nothing particularly mysterious about this. The implicit assumption we've made is that diffeomorphism and Weyl invariance are two separate sets of gauge symmetries. In fact there is a small set of reparameterizations that are *both* diffeomorphisms *and* Weyl rescalings, i.e. the set diff ∩Weyl is nonzero. The transformations that we have 'used up' to set $\gamma_{\alpha\beta} = \eta_{\alpha\beta}$ live in

$$\frac{\text{diff} \times \text{Weyl}}{\text{diff} \cap \text{Weyl}}. \tag{4.28}$$

We still have the leftover freedom of transformations that belong to diff ∩Weyl. For these special reparameterizations, one can perform a diffeomorphism and then undo its transformation of $\gamma_{\alpha\beta}$ by a subsequent Weyl rescaling. Note, however, that while we can use a diffeomorphism to undo the transformation of the worldsheet metric, the effect of the diffeomorphism on the ξ^α coordinates is not undone. Thus we have performed an additional non-trivial gauge transformation that still preserves $\gamma_{\alpha\beta} = \eta_{\alpha\beta}$.

Let's start by examining this infinitesimally. Under the infinitesimal diffeomorphisms

$$\xi'^\alpha = \xi^\alpha + \epsilon^\alpha(\xi) \qquad \Rightarrow \qquad g'^{\alpha\beta} = g^{\rho\delta} \frac{\partial \xi'^\alpha}{\partial \xi^\rho} \frac{\partial \xi'^\beta}{\partial \xi^\delta} = g^{\alpha\beta} + \partial^\alpha \epsilon^\beta + \partial^\beta \epsilon^\alpha + \mathcal{O}(\epsilon^2). \tag{4.29}$$

If $\delta g^{\alpha\beta} = g'^{\alpha\beta} - g^{\alpha\beta} = \partial^\alpha \epsilon^\beta + \partial^\beta \epsilon^\alpha$ obeys $\delta g^{\alpha\beta} = \Lambda(\xi) g^{\alpha\beta}$, then the diffeomorphism is also a Weyl rescaling such that there is some $\omega(\xi)$ such that

$$g'^{\alpha\beta} = e^{2\omega(\xi)} g^{\alpha\beta} \tag{4.30}$$

and so the effect of the diffeomorphism on the metric can be *undone* by a Weyl rescaling.

Define light cone coordinates $\xi^{\pm}$ and infinitesimal transformations $\epsilon^{\pm}$ via

$$\xi^{\pm} = \xi^0 \pm \xi^1 \qquad\qquad \epsilon^{\pm} = \epsilon^0 \pm \epsilon^1. \tag{4.31}$$

In these coordinates the derivatives (e.g. $\partial_0 = \partial/\partial\xi^0$) are given by

$$\begin{aligned} \partial_0 &= \ \ 2\,(\partial_+ + \partial_-) & \qquad \partial^0 &= \ -2\,(\partial_+ + \partial_-) \\ \partial_1 &= \ \ 2\,(\partial_+ - \partial_-) & \qquad \partial^1 &= \ +2\,(\partial_+ - \partial_-)\,. \end{aligned} \tag{4.32}$$

with similar expressions for upper-index derivatives. If we consider a very special class of infinitesimal diffeomorphisms where the $\epsilon^+ = \epsilon^+(\xi^+)$ and $\epsilon^- = \epsilon^-(\xi^-)$, then we find

$$\partial^0\epsilon^0 = -2(\partial_+ + \partial_-)(\epsilon^+ + \epsilon^-)\cdot\frac{1}{2} = -\partial_+\epsilon^+ - \partial_-\epsilon^- \tag{4.33}$$

$$\partial^1\epsilon^1 = \ \ 2(\partial_+ - \partial_-)(\epsilon^+ - \epsilon^-)\cdot\frac{1}{2} = \ \ \partial_+\epsilon^+ + \partial_-\epsilon^- \tag{4.34}$$

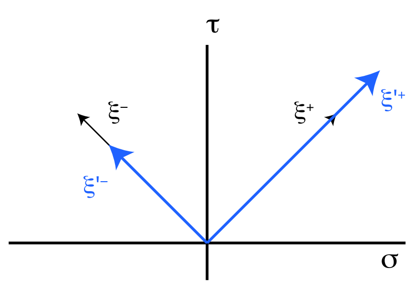
$$\partial^0\epsilon^1 = -2(\partial_+ + \partial_-)(\epsilon^+ - \epsilon^-)\cdot\frac{1}{2} = -\partial_+\epsilon^+ + \partial_-\epsilon^- \tag{4.35}$$

$$\partial^1\epsilon^0 = \ \ 2(\partial_+ - \partial_-)(\epsilon^+ + \epsilon^-)\cdot\frac{1}{2} = \ \ \partial_+\epsilon^+ - \partial_-\epsilon^-. \tag{4.36}$$

In other words,

$$\left.\begin{aligned} 0 &= \partial^0\epsilon^1 + \partial^1\epsilon^0 \\ 0 &= \partial^1\epsilon^1 + \partial^0\epsilon^0 \end{aligned}\right\} \Rightarrow \partial^\alpha\epsilon^\beta + \partial^\beta\epsilon^\alpha = \Lambda\eta^{\alpha\beta}. \tag{4.37}$$

We thus find that infinitesimal diffeomorphisms of the type $\xi^{\pm} \rightarrow \xi^{\pm} + \epsilon^{\pm}(\xi^{\pm})$ are also Weyl transformations and hence live in diff$\,\cap$Weyl. These diffeomorphisms are just reparameterizations of the light cone coordinates and this structure is almost single-handedly responsible for the appearance of holomorphy in string theory.



4.5 Complex coordinates

These sorts of transformations have some interesting properties that become very valuable if we shift to complex variables:

$$z = \xi^0 + i\xi^1 \qquad\qquad \bar{z} = \xi^0 - i\xi^1. \tag{4.38}$$

It will be convenient to treat z and $\bar{z}$ as independent variables. This leads to too many real degrees of freedom (four), so we should remember that we are 'really' only working on a real subspace of this complexified space. We may write vectors and tensors with respect to z and $\bar{z}$ using the metric (factors of 2 are to match Polchinski's notation)

$$g_{zz} = g_{\bar{z}\bar{z}} = g^{zz} = g^{\bar{z}\bar{z}} = 0 \tag{4.39}$$

$$g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2} \tag{4.40}$$

$$g^{z\bar{z}} = g^{\bar{z}z} = 2. \tag{4.41}$$

Thus we may write vectors as

$$v^z = v^0 + iv^1 \qquad\qquad v_z = \frac{1}{2}(v^0 + iv^1) \tag{4.42}$$

$$v^{\bar{z}} = v^0 - iv^1 \qquad\qquad v_{\bar{z}} = \frac{1}{2}(v^0 - iv^1). \tag{4.43}$$

Partial derivatives are

$$\partial = \partial_z = \frac{1}{2}(\partial_0 - i\partial_1) \tag{4.44}$$

$$\bar{\partial} = \partial_{\bar{z}} = \frac{1}{2}(\partial_0 + i\partial_1) \tag{4.45}$$

and the volume form is given by

$$d^2z = 2\, d\xi^0\, d\xi^1. \tag{4.46}$$

In these coordinates our light cone reparameterizations (the leftover diff $\cap$Weyl gauge transformations) are

$$z \to z + \epsilon^z(z) \tag{4.47}$$

$$\bar{z} \to \bar{z} + \epsilon^{\bar{z}}(\bar{z}). \tag{4.48}$$

In complex coordinates, our light cone reparameterizations are holomorphic! And we all know that holomorphy is always very important in physics (just ask Seiberg). Before we proceed, let's make two important generalizations. We proved the above result for infinitesimal light cone reparameterizatios, but in fact all infinitesimal diff $\cap$Weyl reparameterizations are holomorphic. Further, we can generalize this to finite as well as infinitesimal transformations. Let's prove both of these in one fell swoop using complex coordinates. Let us consider diffeomorphisms $z, \bar{z} \to w, \bar{w}$ such that $g'_{\alpha\beta} \propto g_{\alpha\beta}$ (i.e. elements of diff $\cap$Weyl). This proportionality tells us that

$$g'^{w\bar{w}} = \frac{\partial w}{\partial z}\frac{\partial \bar{w}}{\partial \bar{z}} g^{z\bar{z}} \neq 0 \tag{4.49}$$

$$g'^{ww} = \frac{\partial w}{\partial z}\frac{\partial w}{\partial \bar{z}} g^{z\bar{z}} = 0 \tag{4.50}$$

$$g'^{\bar{w}\bar{w}} = \frac{\partial \bar{w}}{\partial z}\frac{\partial \bar{w}}{\partial \bar{z}} g^{z\bar{z}} = 0. \tag{4.51}$$

The first equation tells us that

$$\frac{\partial w}{\partial z}, \frac{\partial \bar{w}}{\partial \bar{z}} \neq 0, \tag{4.52}$$

so that the second and third equations tell us

$$\frac{\partial w}{\partial \bar{z}} = \frac{\partial \bar{w}}{\partial z} = 0. \tag{4.53}$$

This, of course, is just telling us that $w = w(z)$ and $\bar{w} = \bar{w}(\bar{z})$ so that this transformation is holomorphic.

Let us remark that the space of diffeomorphisms is the space of functions from $\mathbb{R}^2 \to \mathbb{R}^2$. The intersection diff$\cap$Weyl, however, is the space of *holmorphic* functions from $\mathbb{C} \to \mathbb{C}$. This is a very small subset indeed, in fact, it is a measure-zero subset of the space of all diffeomorphisms. This little sliver of diffeomorphism space, however, is what gives us the leftover gauge freedom to go to light cone gauge and quantize our string in a quick-and-dirty manner.

4.6 Light cone gauge

More concretely, we can use diff$\cap$Weyl to allow us to fix our τ parameterization. An obvious choice might be $\tau = X^0$, but this doesn't turn out to be very useful. Instead, we will transform to

$$\tau \propto X^+ = \frac{1}{\sqrt{2}}\left(X^0 + X^1\right). \tag{4.54}$$

To motivate this, we can reparameterize τ such that it satisfies

$$\nabla^2 \tau = \partial\overline{\partial}\tau(z, \bar{z}) = 0. \tag{4.55}$$

This is precisely the same wave equation satisfied by the X^μ fields, $\nabla^2 X^\mu = 0$. In particular, this holds even in light cone coordinates *in spacetime*, i.e.

$$X^\pm = \frac{1}{\sqrt{2}}\left(X^0 \pm X^1\right); \qquad\qquad \nabla^2 X^+(\tau, \sigma) = 0. \tag{4.56}$$

Thus we can make the choice

$$X^+ = 2\alpha' P^+ \tau, \tag{4.57}$$

where the main point is that $X^+ \propto \tau$ and $2\alpha' P^+$ is just a convenient normalization. P^+ is just the light cone momentum. One can check that the dimensions are correct for a two-dimensional field theory, e.g. from (4.19) we know that $[P^+] = 0$. This is what we call **light cone gauge**. It should be clear that unlike going to light cone coordinates, we have actually done something when we fix this gauge freedom. For a more thorough proof that one can actually make this gauge choice, see Polchinski [2].

The strategy from here is to solve the Virasoro constraints (3.78) - (3.79) for the classical string and then extend this to quantize the string excitations.

Light cone coordinates. It is handy to review the salient features of light cone coordinates, (4.27). In these coordinates the metric takes the form (e.g. in 4D)

$$\eta_{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.58}$$

The inner product of two vectors can then be decomposed as

$$v^\mu w_\mu = -v^- w^+ - v^+ w^- + v^i w^i, \tag{4.59}$$

where the last term is just the usual Euclidean inner product over the remaining spacelike directions. By raising and lowering indices we find

$$v_+ = -v^- \quad ; \quad v_- = -v^+ \qquad \Rightarrow \qquad v^\mu w_\mu = v_+ w^+ + v_- w^- + v^i w^i. \tag{4.60}$$

In terms of the momentum four vector, the energy is given by

$$E = H = -p_+ = +p^- \tag{4.61}$$

4.7 Solving the Virasoro constraint in light cone gauge

Let us recall that the Virasoro constraint tells us

$$\begin{aligned} \dot{X} \cdot X' &= 0 \\ \dot{X}^2 + X'^2 &= 0 \end{aligned} \qquad \Leftrightarrow \qquad (\dot{X} \pm X')^2 = 0. \tag{4.62}$$

The straightforward thing to do is to plug in our X^μ mode expansion (4.18), but we can see that this gets rather messy very quickly. Instead, let's try to be a little more slick by working in light cone coordinates.

$$(\dot{X} \pm X')^2 = -2(\dot{X}^+ \pm X^{+\prime})(\dot{X}^- \pm X^{-\prime}) + (\dot{X}^i \pm X^{i\prime})^2 = 0. \tag{4.63}$$

Since we've just made a big hullabaloo that $X^+ = 2\alpha' P^+ \tau$, we know that $\dot{X}^+ = 2\alpha P^+$ and that $X^{+\prime} = 0$. Hence

$$(\dot{X} \pm X')^2 = -2\dot{X}^+(\dot{X}^- \pm X^{-\prime}) + (\dot{X}^i \pm X^{i\prime})^2 = 0. \tag{4.64}$$

Since we already have a solution for X^+ (by construction), it is now trivial to read off a set of solutions for the Virasoro constraints. We just have to pick

$$\dot{X}^- \pm X^{-\prime} = \frac{1}{4\alpha' P^+} \left(\dot{X}^i \pm X^{i\prime} \right)^2. \tag{4.65}$$

Easy peasy! Once we specify the transverse oscillations (X^i) and the zero modes (P^+ and X_0^-), we can just read off the Virasoro-constrained evolution of the state. Let's now go back to the mode expansion (4.18) and do this a little more explicitly. In light cone target space coordinates,

$$X^-(\tau,\sigma) = X_0^- + \sqrt{2\alpha'}\alpha_0^-\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^- e^{-in\tau}\cos n\sigma \tag{4.66}$$

$$X^i(\tau,\sigma) = X_0^i + \sqrt{2\alpha'}\alpha_0^i\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^i e^{-in\tau}\cos n\sigma \tag{4.67}$$

$$X^+(\tau,\sigma) = 2\alpha' P^+\tau \equiv \sqrt{2\alpha'}\alpha_0^+\tau \tag{4.68}$$

where in the last expression we recall (4.13). The first two equations give us explicit forms that we can differentiate,

$$\dot{X}^\beta \pm X'^\beta = \sqrt{2\alpha'}\sum_{n=-\infty}^{\infty}\alpha_n^\beta e^{-in(\tau\pm\sigma)}, \tag{4.69}$$

where $\alpha \in \{-, i\}$. Substituting into our Virasoro constraint (4.65),

$$\sqrt{2\alpha'}\sum_{n=-\infty}^{\infty}\alpha_n^- e^{-in(\tau\pm\sigma)} = \frac{1}{2P^+}\sum_n\sum_p \alpha_p^i\alpha_{n-p}^i e^{-in(\tau\pm\sigma)}. \tag{4.70}$$

We can project out each Fourier coefficient to obtain

$$\sqrt{2\alpha'}\alpha_n^- = \frac{1}{P^+}\left[\frac{1}{2}\sum_p \alpha_p^i\alpha_{n-p}^i\right] \equiv \frac{1}{P^+}L_n^\perp, \tag{4.71}$$

where we have defined L to be the modes of the Virasoro constraint, i.e. the "transverse Virasoro modes." One can check that these are just the Fourier coefficients of the stress-energy tensor, (3.75). Let's pause to emphasize what we've done: we are now able to express the oscillators of $X^-(\sigma,\tau)$ in terms of the transverse oscillators in $X^i(\sigma,\tau)$. Finally, we have

$$\dot{X}^- \pm X^{-\prime} = \frac{1}{P^+}\sum_n L_n^\perp e^{in(\tau\pm\sigma)}. \tag{4.72}$$

Let us recall our definition (4.13)

$$\sqrt{2\alpha'}\alpha_0^- = 2\alpha' P^- = \frac{1}{P^+}L_0^\perp \quad \Rightarrow \quad 2\alpha' P^+ P^- = L_0^\perp = \frac{1}{2}\sum_p \alpha_{-p}^i\alpha_p^i. \tag{4.73}$$

Now that you're probably bored of all of these redefinitions, let's squeeze something useful out of this. We will now determine the *classical* mass spectrum of the string.

4.8 The classical string spectrum

We know that the mass is given by the relation

$$P^2 = -2P^+P^- + P^iP^i = -M^2, \tag{4.74}$$

where the typist arrogantly notes unfortunate choice of metric convention. From (4.73) we note that we can write

$$L_0^\perp = 2\alpha' P^+P^- = \frac{1}{2}\alpha_0^i\alpha_0^i + \sum_{p>0}\alpha_{-p}^i\alpha_p^i. \tag{4.75}$$

Now if you try and remember way back decades ago when you first learned quantum field theory, you'll recall that this is precisely where one should start hyperventilating while trying to say something about operator ordering ambiguities associated with the raising and lowering operators, the $\alpha_{\pm p}^i$s. For now let's just reassure ourselves that we are still working with a classical theory and keep this in mind for later. Our expression for the string mass can be written in terms of $L_0^\perp$ as

$$M^2 = \frac{1}{\alpha'}L_0^\perp - P^iP^i. \tag{4.76}$$

We can now go back to our original canonically normalized a_n raising and lowering operators from equations (4.13-4.15) and write out the mass spectrum,

$$\frac{1}{\alpha'}L_0^\perp = \frac{1}{2\alpha'}(2\alpha')P^iP^i + \frac{1}{\alpha'}\sum_{n=1}^{\infty} n\ a_n^{i*}a_n^i \quad \Rightarrow \quad M^2 = \frac{1}{\alpha'}\sum_{n=1}^{\infty} n\ a_n^{i*}a_n^i. \tag{4.77}$$

This formula has an obvious and intuitive interpretation: the mass of the string is just given by counting the number of excitations in each transverse direction.

4.9 Quantizing the relativistic string in light cone coordinates

Okay! So that's the classical string with Neumann boundary conditions. So far this could have been an involved project for freshman physics majors learning analytic mechanics. Now it's time to own up and see what this all looks like when we quantize the string. We already tried this once in Section 4.3 using a covariant formulation. Our flawed attempt was to write the canonical commutation relation,

$$[\alpha_m^\mu, \alpha_n^\nu] = i\eta^{\mu\nu}\delta_{m,-n}. \tag{4.78}$$

We failed miserably due to the appearance of negative norm states. Now we're armed with the light cone gauge framework which breaks manifest target space Lorentz covariance, but which come with the Virasoro constraints built in:

$$[X^i, P^j] = i\delta^{ij}\delta(\sigma - \sigma') \tag{4.79}$$

$$[\alpha_m^i, \alpha_n^j] = m\,\delta^{ij}\delta_{m,-n}. \tag{4.80}$$

The second constraint comes from $[X^-(\tau,\sigma), P^+(\tau,\sigma')] = i\eta^{+-}\delta(\sigma-\sigma') = -i\delta(\sigma-\sigma')$ and then expressing the Fourier coefficients α_n^- of X^- in terms of the transverse coefficients α_n^i in (4.73). The X^+ is gauged away in (4.73). It is understood that all other commutation relations vanish. Note that the commutation relation for the $\alpha_n = \sqrt{m}\, a_n$ and $\alpha_{-n} = \sqrt{m}\, a_n^\dagger$ verify that the a and $a^\dagger$ are canonically normalized raising/lowering operators.

Our claim is that because we've already incorporated the Virasoro constraints our quantized theory now only deals with physical states[1]. We can write down a handy set of commuting operators: P^+, P^i, α_m^i where $m < 0$ and $i = 2, \cdots, D-1$.

Our next task is to use these to construct the spacetime Hilbert space. Yes, we do mean *Hilbert* space; we will be *first* quantizing the string wherein we consider the operators that excite modes of a string. This should be contrasted with second quantization wherein strings themselves are created (rather than just excitations on a string). This is called **string field theory** and is beyond the scope of these lectures. Waxing poetic a bit longer, let us remark that on the worldsheet side, we are properly second quantizing the 2D field theory.

To fix our terminology, we will refer to the zero-excitation string state to be the **string ground state**, $|0;k\rangle$, where k refers to the overall center-of-mass motion of the string. This should be contrasted with a notion of 'vacuum state,' $|\text{vac}\rangle$, which one ought to reserve for the 'no string' state of string field theory. The action of our set of commuting operators on the string ground state is

$$P^+|0;k\rangle = k^+|0;k\rangle \tag{4.81}$$

$$P^i|0;k\rangle = k^i|0;k\rangle \tag{4.82}$$

$$\alpha_m^i|0;k\rangle = 0 \qquad\qquad (m > 0) \tag{4.83}$$

A general string state can then be constructed by acting upon the ground state with raising operators, α_{-m}^i for $m > 0$:

$$\prod_{i=2}^{D-1}\prod_{n=1}^{\infty} \frac{1}{\sqrt{n^{N_{i,m}} N_{i,m}!}} \left(\alpha_{-n}^i\right)^{N_{i,n}} |0;k\rangle, \tag{4.84}$$

where $N_{i,n}$ is the occupation number of excitations of the n mode in the transverse i direction. One should thing of the zero mode state $|0;k\rangle$ as a small non-oscillating bit of string which has spatial extent only due to its zero point energy. The first excitation is the same picture but with a wave propagating along its length, and so forth. The energy of the string should go like the **level**,

$$N = \sum_{i=2}^{D-1}\sum_{n=1}^{\infty} nN_{i,n}. \tag{4.85}$$

Now we have to face up to the operator ordering ambiguity that we observed in the classical

[1] There is a subtlety that we're glossing over here associated with commutation relations for the α_n^- operators. This is discussed in the footnote in Section 2.2.2 of David Tong's lectures [7]

formula for the string spectrum. Recalling equations (4.61), (4.73), and (4.13) we may write

$$P^- = \frac{1}{2\alpha' P^+} L_0^\perp = \frac{1}{4\alpha' P^+} \sum_{p\in\mathbb{Z}} \alpha^i_{-p}\alpha^i_p \tag{4.86}$$

$$= \frac{1}{4\alpha' P^+}\left(\sum_{p=1}^{\infty} \alpha^i_p\alpha^i_{-p} + \sum_{p=1}^{\infty} \alpha^i_{-p}\alpha^i_p + \alpha^i_0\alpha^i_0\right) \tag{4.87}$$

$$= \frac{P^iP^i}{2P^+} + \frac{1}{2\alpha' P^+}\left(\sum_{p=1}^{\infty} \alpha^i_p\alpha^i_{-p} + A\right), \tag{4.88}$$

where we've introduced a normal ordering constant A to account for the normal ordering of the raising and lowering operators. One should feel a sense of *déjà vu* with the Casimir energy from quantum field theory. Plugging into our mass formula, we get the expression

$$M^2 = \frac{1}{\alpha'}\left(\sum_{p=1}^{\infty} \alpha^i_{-p}\alpha^i_p + A\right) = \frac{1}{\alpha'}\left(N + A\right). \tag{4.89}$$

Instead of specifying some prescription for the normal ordering constant, we can impose the consistency of our theory to fix A. (Later we'll derive A from a more formal argument using conformal field theory.) Reading off our low-lying mass spectrum, we see that the ground state and first excited states have masses

$$(M_0)^2 = \frac{1}{\alpha'}A \qquad \text{and} \qquad (M_1)^2 = \frac{1}{\alpha'}(1+A). \tag{4.90}$$

The lazy (i.e. clever) way to determine A is to consider what is really going into the equation for M_1. This formula accounts for $D-2$ states, as is manifest from the sum over $i = 2, \cdots, D-1$. In fact, the index structure in (4.84) tells us that this is a vector particle excitation, where light cone gauge has butchered manifest Lorentz invariance in exchange for working with only physical states. We know from working with 4D gauge theories that massive vector bosons have an additional degree of freedom over their massless cousins associated with the ability to boost to the rest frame of a massive particle. In particular, in D spacetime dimensions a massive vector has $D-1$ states while a massless vector has $D-2$ states. Clearly our first excited states, with its sum over $D-2$ indices, must be a massless. This then tells us that $M_1 = 0$, from which we can fix

$$A = -1. \tag{4.91}$$

That's not bad for a quick-and-dirty derivation. Unfortunately, purists will note that we've been a little bit *too* quick and *too* dirty. When proving $A = -1$ we took the non-covariant gauge choice $\tau \propto X^+$ (light cone gauge). In a different target space Lorentz frame this will look different. We need to ask ourselves if our gauge choice actually does respect Lorentz invariance 'under the hood.' In other words, we need to determine whether or not Lorentz invariance holds in the quantum theory or whether it is anomalous. Hence we've found that $A = -1$ is a *necessary* condition, but is is not sufficient. It turns out that a sufficient condition is $A = -1$ and $D = 26$. This is straightforwardly, but very tediously, derived by writing out the Lorentz generators and then demanding that the correct Lorentz algebra is satisfied in light cone gauge. This is one way of observing the 'critical dimension' $D = 26$ appearing in bosonic string theory.

Happily taking that at face value, we return to the first equation in (4.90): it appears that our zero mode has become tachyonic! In fact, when α' is at the Planck scale this state has a very large negative mass indeed. *Geez Louise!* It makes you feel like the little Dutch boy: you plug one hole and another leak springs right out. At some level, though, we should appreciate that at least this is 'only' a tachyon. This is very different from the non-unitary states that we ran into in Section 4.3 that were an inconsistency of the theory. The reason why I'm trying so hard to convince you that this is not such a bad problem is that our bag of tricks is now empty and we'll have to live with this tachyonic state for the meanwhile. In fact the tachyon is a general problem of bosonic string theory. We will see, however, that in *super*string theory the zero mode is cancelled and the tachyon goes away.

The Virasoro algebra as a projection to physical states. In light cone gauge our Virasoro constraints read $T_{++} = T_{--} = 0$. This motivates us to Fourier transform the energy-momentum tensor,

$$L_m = \frac{1}{4\pi\alpha'} \int_0^\pi d\sigma\, e^{-2im\sigma} T_{--} = \frac{1}{2} \sum_p \alpha_p^\mu \alpha_{(m-p)\mu}, \tag{4.92}$$

where we note that we are considering all oscillators, not only the perpendicular modes ($L_m^\perp$). This is the origin of the definition of the Ls in (4.71). The L_0 component is special. For one, it is equivalent to the Hamiltonian, $H = L_0$. However, in light of the commutation relations (4.80) we can see that L_0 is special in a different way: it is the only operator which picks up a normal ordering constant because the αs don't commute.

We can write our Virasoro constraint as a set of equations where the Ls act upon physical states as operators:

$$(L_0 - A)|\text{phys.}\rangle = 0 \qquad\qquad L_m|\text{phys.}\rangle = 0 \qquad (m > 0). \tag{4.93}$$

In other words, the Ls can be understood as projection operators onto the space of physical states. We can remark that the first equation has the same content as the second equation of (4.90); it sets the normal ordering coefficient to make the lowest physical state massless. One might ask why we restrict to the case $m > 0$. The commutation relations of the αs imposes an algebra for the Ls called the **Virasoro algebra**,

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{D}{12}(m^3 - m)\delta_{m+n}. \tag{4.94}$$

There is a non-zero central term when $n = -m$ so that the projection onto physical states would be inconsistent if we included negative values of m.

4.10 Open string: Dirichlet boundary conditions

We will postpone our discussion of the open string with Dirichlet boundary conditions since this case is best discussed in the context of D-branes, which exist as the Dirichlet boundary conditions of precisely these kinds of strings. The reader is encouraged to perform the above analysis for the open string as homework.

4.11 Closed string: Periodic boundary conditions

We now make some remarks about the closed string in light cone gauge. We claim that we end up with what amounts to two copies of the open string. Details can be found in your favorite string theory text. We still have the same equation of motion (3.73), Virasoro constraint (3.74), and general solution of left and right movers (4.3). The big difference are the boundary conditions which are now periodic,

$$X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+2\pi). \tag{4.95}$$

We start by expanding the general solution for independent X_L and X_R,

$$X_L^\mu(\tau+\sigma) = \frac{1}{2}X_0^{L\mu} + \frac{\sqrt{\alpha'}}{2}\widetilde{\alpha}_0^\mu(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\widetilde{\alpha}_n^\mu e^{-in(\tau+\sigma)} \tag{4.96}$$

$$X_R^\mu(\tau-\sigma) = \frac{1}{2}X_0^{R\mu} + \frac{\sqrt{\alpha'}}{2}\alpha_0^\mu(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in(\tau-\sigma)}, \tag{4.97}$$

where we emphasize that α and $\widetilde{\alpha}$ are independent Fourier coefficients and are not related by any kind of complex conjugation. Imposing the periodic boundary conditions we obtain

$$X_L^\mu(\tau+\sigma) + X_R^\mu(\tau-\sigma) = X_L^\mu(\tau+\sigma+2\pi) + X_R^\mu(\tau-\sigma-2\pi) \tag{4.98}$$

rearranging this, we have

$$X_L^\mu(\tau+\sigma) - X_L^\mu(\tau+\sigma+2\pi) = X_R^\mu(\tau-\sigma-2\pi) - X_R^\mu(\tau-\sigma). \tag{4.99}$$

We know, however, that $(\tau+\sigma)$ and $(\tau-\sigma)$ are completely independent variables. Thus we can take derivatives with each separately to obtain

$$X_L'^\mu(\tau+\sigma+2\pi) - X_L'^\mu(\tau+\sigma) = \frac{\partial}{\partial(\tau+\sigma)}\text{RHS} = 0, \tag{4.100}$$

where $' = d/d(\tau+\sigma)$. Simillarly for the right-moving modes,

$$X_R'^\mu(\tau-\sigma+2\pi) - X_R'^\mu(\tau-\sigma) = 0, \tag{4.101}$$

where now $' = d/d(\tau-\sigma)$. From this one may write

$$X_L'^\mu(\tau+\sigma) = \sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}}\widetilde{\alpha}_n^\mu e^{-in(\tau+\sigma)} \tag{4.102}$$

$$X_R'^\mu(\tau-\sigma) = \sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}}\alpha_n^\mu e^{-in(\tau-\sigma)}. \tag{4.103}$$

Integrating these equations we get

$$X_L^\mu = \frac{1}{2}X_0^{L\mu} + \sqrt{\frac{\alpha'}{2}}\widetilde{\alpha}_0^\mu(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\widetilde{\alpha}_n^\mu}{n}e^{-in(\tau+\sigma)} \tag{4.104}$$

$$X_R^\mu = \frac{1}{2}X_0^{R\mu} + \sqrt{\frac{\alpha'}{2}}\alpha_0^\mu(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}e^{-in(\tau-\sigma)}. \tag{4.105}$$

Now one can take the difference after a shift $\sigma \to \sigma + 2\pi$,

$$X_L^\mu(\tau+\sigma) - X_L^\mu(\tau+\sigma+2\pi) = \sqrt{\frac{\alpha'}{2}}\widetilde{\alpha}_0^\mu(-2\pi) \tag{4.106}$$

$$X_R^\mu(\tau+\sigma) - X_R^\mu(\tau+\sigma+2\pi) = \sqrt{\frac{\alpha'}{2}}\alpha_0^\mu(2\pi). \tag{4.107}$$

And thus we find that the periodicity condition gives us an additional constraint on the otherwise independent α_n^μ and $\widetilde{\alpha}_n^\mu$,

$$\alpha_0^\mu = \widetilde{\alpha}_0^\mu. \tag{4.108}$$

This has a strightforward interpretation: while the left- and right-moving excitations are independent, the center of the string only moves in one way. One can derive this using a more sophisticated Noether current approach, for which the reader is referred to Zwiebach exercise 13.4 [1].

Let as now impose light cone gauge. This allows us to set

$$X^+ = \alpha' P^+ \tau, \tag{4.109}$$

note that there is a factor of two with respect to our normalization of the open string. Solving the Virasoro constraint using (4.65) with the appropriate modification to overall normalization,

$$\dot{X}^- \pm X^{-'} = \frac{1}{2\alpha' P^+}\left(\dot{X}^i \pm X'^i\right)^2. \tag{4.110}$$

Then we can write

$$(\dot{X}^i + X'^i)^2 = 4\alpha' \sum_{n\in\mathbb{Z}}\left(\frac{1}{2}\sum_{p\in\mathbb{Z}}\widetilde{\alpha}_p^i\widetilde{\alpha}_{n-p}^i\right)e^{-in(\tau+\sigma)} \equiv 4\alpha'\sum_{n\in Z}\widetilde{L}_n^\perp e^{-in(\tau+\sigma)} \tag{4.111}$$

and, analogously for the lower sign,

$$(\dot{X}^i - X'^i)^2 = 4\alpha' \sum_{n\in\mathbb{Z}}\left(\frac{1}{2}\sum_{p\in\mathbb{Z}}\alpha_p^i\alpha_{n-p}^i\right)e^{-in(\tau-\sigma)} \equiv 4\alpha'\sum_{n\in Z}L_n^\perp e^{-in(\tau+\sigma)}. \tag{4.112}$$

Using the mode expansion and following the same steps that we did in Section 4.7,

$$\dot{X}^- + X'^- = \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\widetilde{\alpha}_n^- e^{-in(\tau+\sigma)} \tag{4.113}$$

$$\dot{X}^- - X'^- = \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\alpha_n^- e^{-in(\tau-\sigma)}. \tag{4.114}$$

Using this we can once again relate the oscillator modes in the minus light cone direction to those in in the transverse direction,

$$\sqrt{2\alpha'}\widetilde{\alpha}_n^- = \frac{2}{P^+}\widetilde{L}_n^\perp \tag{4.115}$$

$$\sqrt{2\alpha'}\alpha_n^- = \frac{2}{P^+}L_n^\perp, \tag{4.116}$$

where we have written our Ls as

$$L_n^\perp = \frac{1}{2}\sum_{p\in\mathbb{Z}}\alpha_p^i\alpha_{n-p}^i \tag{4.117}$$

$$\widetilde{L}_n^\perp = \frac{1}{2}\sum_{p\in\mathbb{Z}}\widetilde{\alpha}_p^i\widetilde{\alpha}_{n-p}^i. \tag{4.118}$$

We also note that our additional constraint (4.108) gives us

$$L_0^\perp = \widetilde{L}_0^\perp. \tag{4.119}$$

Plugging in these modes we get

$$\frac{1}{2}\alpha_0^i\alpha_0^i + \sum_{p=1}^{\infty}\alpha_{-p}^i\alpha_p^i = \frac{1}{2}\widetilde{\alpha}_0^i\widetilde{\alpha}_0^i + \sum_{p=1}^{\infty}\widetilde{\alpha}_{-p}^i\widetilde{\alpha}_p^i. \tag{4.120}$$

Defining the excited state sums

$$N = \sum_{p=1}^{\infty}\alpha_{-p}^i\alpha_p^i \qquad\qquad \widetilde{N} = \sum_{p=1}^{\infty}\widetilde{\alpha}_{-p}^i\widetilde{\alpha}_p^i, \tag{4.121}$$

we get from (4.120) the **level matching** condition

$$N = \widetilde{N}. \tag{4.122}$$

This tells us that the left-moving and right-moving levels (N and $\widetilde{N}$) are equal. Note that this does *not* mean that each mode number matches individually ($N_{i,n} \neq \widetilde{N}_{i,n}$), but rather that the total number matches. Again, the physical origin of htis level matching condition can be traced back to σ-translation invariance.

Now is is easy to quantize our closed string. The set of commuting operators (degrees of freedom) are as before but with the addition of separate left- and right-moving operators,

$$X^i, P^i, X^-, P^+, \alpha_n^i, \widetilde{\alpha}_n^i. \tag{4.123}$$

Their commutation relations are as before,

$$[\widetilde{\alpha}_m^i, \widetilde{\alpha}_n^j] = m\,\delta^{ij}\,\delta_{m,-n} \qquad\qquad [\alpha_m^i, \widetilde{\alpha}_n^j] = 0, \tag{4.124}$$

along with the usual center of mass commutation relations

$$[X^-, P^+] = -i \qquad\qquad [X^i, P^j] = i\delta^{ij}. \tag{4.125}$$

All other commutators vanish.

Indeed, up to the requirement of level matching, this is the algebra of two non-interacting copies of the open string. We can now write out a general state in terms of raising operators acting on the ground state $|0,0;k\rangle$,

$$|N, \widetilde{N}; k\rangle = \prod_{i=2}^{D-1}\prod_{n=1}^{\infty} \frac{(\alpha^i_{-n})^{N_{i,n}}(\widetilde{\alpha}^i_n)^{\widetilde{N}_{i,n}}}{\sqrt{n^{N_{i,n}}N_{i,n}!\, n^{\widetilde{N}_{i,n}}\widetilde{N}_{i,n}!}}|0,0;k\rangle. \tag{4.126}$$

Let us consider the spectrum of the closed bosonic string:

$$\begin{aligned} M^2 &= 2P^+P^- - P^iP^i & (4.127)\\ &= \frac{2}{\alpha'}\left[\sum_{n=1}^{\infty}\left(\alpha^i_{-n}\alpha^i_n + \widetilde{\alpha}^i_{-n}\widetilde{\alpha}^i_n\right) + A + \widetilde{A}\right] & (4.128)\\ &= \frac{2}{\alpha'}\left(N + \widetilde{N} + A + \widetilde{A}\right) = \frac{2}{\alpha'}\left(2N-2\right) & (4.129)\\ &= \frac{4}{\alpha'}(N-1). & (4.130)\end{aligned}$$

We've used the fact $A = \widetilde{A} = -1$, which we will not prove in detail. And, again as before, we still have a tachyonic ground state, $(M_0)^2 = -4/\alpha'$. We make the same remark that this state is removed in the superstring. Okay, so thus far this should all be a mind-numbing review of quantizing in light cone gauge. The reason that we've gone through this exercise, however, is that the closed string does buy us one very interesting new feature: the mode

$$\alpha^i_{-1}\widetilde{\alpha}^j_{-1}|0,0;k\rangle \qquad \text{with} \qquad M^2 = 0. \tag{4.131}$$

This is a massless two-index tensor. We know that this can be decomposed into a symmetric, antisymmetric, and trace components. In particular, the moment we hear 'antisymmetric 2-index tensor,' we know we're talking about **gravitons**.

To be more precise, (4.131) is a basis of general massless two-index states. We can write a general state as

$$\sum_{ij} e_{ij}\alpha^i_{-1}\widetilde{\alpha}^j_{-1}|0,0;k\rangle \tag{4.132}$$

where e_{ij} is a general tensor. It is useful to decompose e_{ij} into irreducible representations of $\mathrm{SO}(D-2)$,

$$e_{ij} = \frac{1}{2}\left(e + ij + e_{ji} - \frac{D}{D-2}\delta_{ij}e^k{}_k\right) + \frac{1}{2}\left(e_{ij} - e_{ji}\right) + \frac{1}{D-2}\delta_{ij}e^k{}_k, \tag{4.133}$$

where we can write these three terms as $e_{ij} = g_{ij} + b_{ij} + \Phi$. These will be our field names for the graviton, the antisymmetric tensor, and the scalar (trace) of our decomposition.

4.12 Remarks: consistency and the zero-point energy

The A (and $\widetilde{A}$) factors in the open (and closed) string spectrum came from the reordering of the Fourier coefficients as operators with non-trivial commutation relations, i.e. taking

$$\frac{1}{2}\sum_{n=1}^{\infty}\alpha_{-n}^{i}\alpha_{n}^{i}+\alpha_{n}^{i}\alpha_{-n}^{i}\to\sum_{n=1}^{\infty}\alpha_{-n}^{i}\alpha_{n}^{i}+A. \tag{4.134}$$

A has the interpretation of being a sum of zero-point energies for all oscillator modes, i.e.

$$A=\sum\frac{1}{2}\omega=\sum_{i=2}^{D-1}\sum_{n=1}^{\infty}\frac{1}{2}n=\frac{D-2}{2}\sum_{n=1}^{\infty}n. \tag{4.135}$$

Those of you who have dabbled in string theory before already know what the very non-intuitive result is: this sum, which looks 'obviously' divergent, can be taken to converge to a value of

$$\sum_{n=1}^{\infty}n=-\frac{1}{12}. \tag{4.136}$$

This is absurd on many levels, but it turns out to be a sensible thing to do. One way to motivate this is to appeal to the Zeta function,

$$\zeta(s)\equiv\sum_{n=1}^{\infty}n^{-s} \tag{4.137}$$

so that the sum of all positive integers is just $\zeta(-1)$. By performing an analytic continuation of ζ we obtain $\zeta(-1)=-1/12$. This 'explanation' should be completely unsatisfactory. It can be understood more rigorously using conformal methods, though we will not dwell on this issue. For further reading see Tong's notes [7] or a lucid account by Lubos Motl at `http://motls.blogspot.com/2007/09/zeta-function-regularization.html`.

The result from all this is that

$$A=\frac{D-2}{2}\cdot\left(-\frac{1}{12}\right)=\frac{2-D}{24.} \tag{4.138}$$

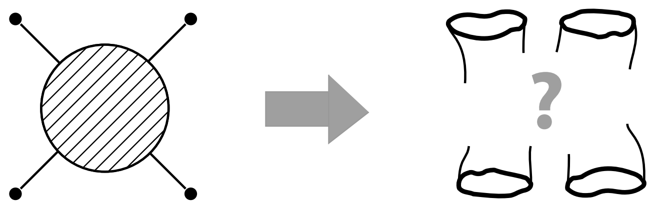
We already know that Lorentz invariance requires $A=-1$, from which we obtain the (in)famous result from bosonic string theory, $D=26$. This matches the claim earlier that demanding the Lorentz algebra gives us $D=26$.

5 The Polyakov path integral

So far we've developed a theory of a free string with a flat target space. One generalization we'd like to make is to develop the theory of an interacting string. Along the way we'll develop a handy path integral formalism for the quantized string.

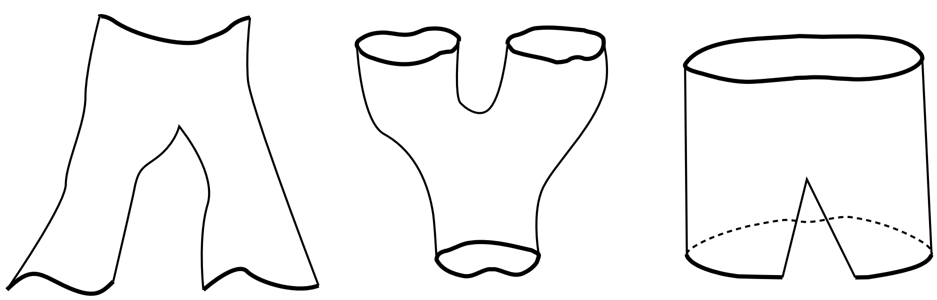
5.1 Cartoon picture of string scattering

There are two ways to think about string interactions. The first way is to appeal to what we are used to from quantum field theory. We have the usual LSZ formalism for determining interactions by calculating correlation functions and amputating the external legs:



We would like to develop a similar picture for the worldsheet, where instead of external point particles we have initial and final string states. We thus propose that between these external string states we draw any connected worldsheet surface and count this as a contribution towards the string amplitude. This is a vague statement; in fact, the vague part is the word 'any.' What we mean is that we consider any embedding of the worldsheet into the target space ($X^\mu(\sigma,\tau)$) and any intrinsic metric on the worldsheet ($\gamma_{\alpha\beta(\sigma,\tau)}$). In particular, we will consider any *topology* of this worldsheet connecting the final states. We would like to perform a path integral over all such worldsheets.

A second picture is to actually consider open and closed strings in spacetime merging and splitting off one another. This is most simply seen in pictures:



Here we depict two open strings combining into a single open string, a closed string splitting into two closed strings, and two open strings combining into a closed string.

5.2 A general Polyakov action

Thus far we've studied the Polyakov action S_P and studied its symmetries. Going to Euclidean complex coordinates and unit gauge, S_P takes the form

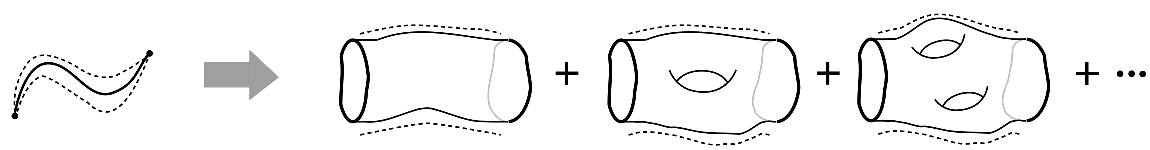
$$S_\text{P} = -\frac{1}{4\pi\alpha'}\int d^2\sigma\,\sqrt{\gamma}\gamma^{\alpha\beta}\,\partial_\alpha X^\mu \partial_\beta X_\mu \quad \longrightarrow \quad \frac{1}{2\pi\alpha'}\int d^2z\,\partial X^\mu \bar{\partial} X_\mu. \tag{5.1}$$

This, however, is *not* the most general action that we could have written that is invariant under the gauged diff $\times$ Weyl and global Poincaré symmetry that we made such a big deal about. There is an additional *topological* term which we could have introduced,

$$\Delta S = \frac{\lambda}{4\pi} \int d^2\sigma \sqrt{\gamma}\, R. \tag{5.2}$$

We say that this is "topological" in the sense that it is equivalent to the **Euler number**, χ, so that $\Delta S = \lambda\chi$. You're probably starting to feel queasy: we just went through a lot of work (albeit abridged) in Section 4 to quantize the Polyakov action. Now we have a new action, and nobody wants to have to go through all that again. Fortunately, this topological term does *not* contribute to the equation of motion. One nice way to see this is to note that ΔS is just the 2D Einstein-Hilbert term (3.20) and recall that general relativity in 1+1 dimensions is, as we noted earlier, non-dynamical. The lesson for 2D general relativity is that the Einstein-Hilbert action is independent of the metric and only depends on the worldsheet topology.

Thus trying to construct a path integral formalism for strings, we should properly weight our path integral by the combined action $S_{\text{P}} + \Delta S = S_{\text{P}} + \lambda\chi$. We would like to relate this to our [worldsheet] Fock space (i.e. the spacetime Hilbert space). For point particles the path integral sums over all paths connecting the initial and final states. We generalize this to all worldsheets connecting initial and final strings. The cartoon picture of this generalization is:

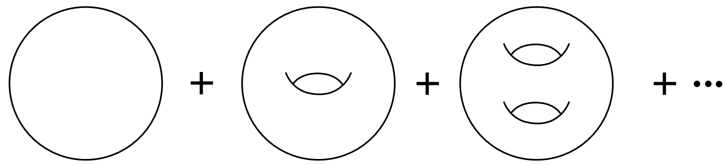


A subtle assumption is that we know how express the external state data. We'll get to this in due course. We can write out the partition function,

$$Z = \sum_{\text{topologies}} \int \frac{[\mathcal{D}X]\,[\mathcal{D}g]}{(\text{overcounting})}\, e^{-(S_{\text{P}}+\lambda\chi)}. \tag{5.3}$$

We've shifted notation for future convenience by writing $g_{\alpha\beta}$ as the worldsheet metric, i.e. $\gamma_{\alpha\beta} \to g_{\alpha\beta}$. (We don't want to get confused from having too many γs floating around.) Here the sum is over all suitable compact, connected topologies. The overcounting is just the volume of the gauge group, $V_{\text{diff}\times\text{Weyl}}$. To account for this gauge redundancy one must apply the usual Fadeev-Popov procedure that one is familiar with from field theory. For brevity we'll not review the details but will assume that it's clear what we mean when we mod out by $V_{\text{diff}\times\text{Weyl}}$. Finally, one might worry about the $\mathcal{D}g$ integral. Is $g_{\alpha\beta}$ really a degree of freedom? In the preceding derivation of the string spectrum we gauged fixed to $g_{\alpha\beta} = \eta_{\alpha\beta}$ at the cost of imposing the Virasoro constraints. Certainly we should keep track of these constraints, especially when considering extrema of the path integral, but the $\mathcal{D}g$ integral reflects the integration over off-shell paths.

We can understand the partition (in particular, the $\lambda\chi$ term) as an expansion in topologies of the vacuum-to-vacuum amplitude $Z = \langle 1 \rangle$:



Ultimately we would like correlation functions of operators, $\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle$. To do this, we need to understand how to properly encode the initial and final state data into our path integral.

5.3 Conformal invariance

Our primary tool to answer this question is **conformal invariance**. This is a powerful symmetry of the Polyakov action S_{P} that is *not* present in the Nambu-Goto action S_{NG}. In fact, conformal invariance is the true *raison d'être* of using the Polyakov action over the Nambu-Goto action, much more so than any of the square root business we used to originally motivated S_{P}. However, just like the details of the Fadeev-Popov procedure, we're going to skip a lot of the formalism of 2D conformal field theories (CFTs).

Wait a second! Are you being short-changed? Maybe the only reason you're reading this is because you want to learn the AdS/CFT correspondence. Isn't the CFT part half of the deal? Fortunately, you can relax. The formalism for two-dimensional conformal field theory is rather special compared to a general $D > 2$, e.g. there are an infinite number of generators which can be used to strongly constrain the form of correlation functions. In string theory $D = 2$ case is used mainly for developing a way to calculate string amplitudes. This will not be of particular importance for the 'modern developments' that we will focus on in these lectures, in particular the 'CFT' in the AdS/CFT correspondence is decoupled from the details of $D = 2$ conformal field theory. For those with a burning desire to learn more (en route to string scattering), we particularly recommend David Tong's lectures [7], the large text by Di Francesco *et al.* [11], the reviews by Ginsparg [12] and Schellekens [13], Freddy Cachazo's 5th lecture on string theory at the 2009-10 Perimeter Scholars International school [14], as well as the usual classic string texts [2, 4].

We've already seen a glimpse of how all of this works, so let's review the 'leftover' gauge freedom diff$\times$Weyl that played such a key role in our lightcone gauge quantization of the string. This time we will work in unit gauge complex coordinates,

$$S_{\mathrm{P}} = \frac{1}{2\pi\alpha'} \int d^2z\, \partial X^\mu \bar{\partial} X_\mu. \tag{5.4}$$

The 2D complex line element is

$$ds^2 = g_{z\bar{z}}\, dz\, d\bar{z}. \tag{5.5}$$

We claim that there is a leftover gauge freedom after fixing to unit gauge; from our earlier quantization in light cone gauge complex coordinates we know this has to do with holomorphic diffeomorphisms $z \to z'(z)$. These are, we recall, a very (infinitesimally) small subset of the set of all possible diffeomorphisms. Under such a holomorphic change of coordinates, the line element transforms as

$$ds^2 \to ds'^2 = \left(\left| \frac{\partial z}{\partial z'} \right|^2 g_{z\bar{z}} \right) \left(\left| \frac{\partial z'}{\partial z} \right|^2 dz\, d\bar{z} \right) = ds^2, \tag{5.6}$$

that is to say that the line element does not transform. Of course, we expected this since *lengths* are unchanged by a a change in coordinates even though the *metric* itself ($g_{z\bar{z}}$) is transformed, as seen by the first term in parentheses. The transformation of the metric is an annoyance since we are interested in fixing the metric to unit gauge. Following our observations from Section 4.4, we know that we can get back to unit gauge via a Weyl transformation. In particular, if we chose

$$\omega = \ln \left| \frac{\partial z'}{\partial z} \right|, \tag{5.7}$$

then our line element would then transform as

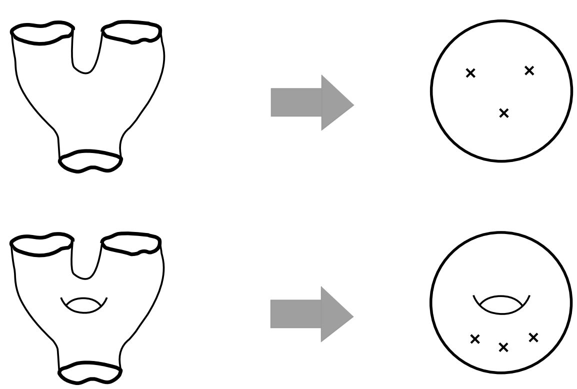
$$ds''^2 = e^{2\omega} ds'^2 = g_{z\bar{z}} \left| \frac{\partial z'}{\partial z} \right|^2 dz\, d\bar{z} = g_{z\bar{z}}\, dz'\, d\bar{z}' \neq ds^2. \tag{5.8}$$

Thus the actual line element has changed, though our metric has not. It is easy to miss the point of what we have done here. We have not just simply 'undone' the holomorphic diffeomorphism. A Weyl transformation is a very different beast than a diffeomorphism. While we have indeed reverted our metric back into its unit gauge form so that the action is unchanged, this has come at a cost: actual lengths (as determined by the line element) in the system have changed.

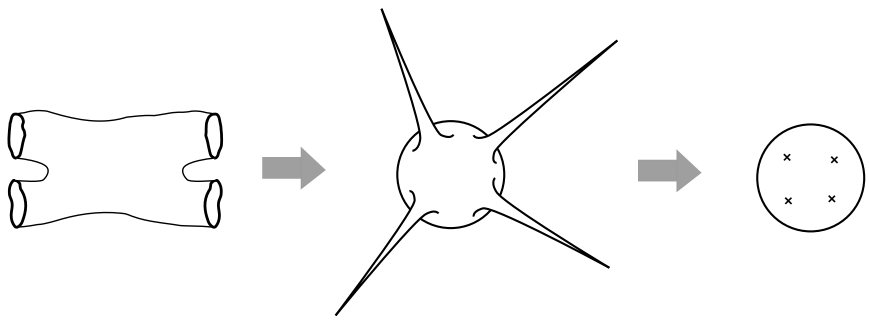
We will now use this to our advantage. The sequence of transformations we've just done are just **conformal maps**. We don't mean this in any fancy AdS/CFT sense, but rather the conformal maps that people studied in the 1900s to do complicated electrostatics problems before the days of *Mathematica*.

5.4 Conformal Maps: some pictures

We will use conformal invariance to map our complicated interacting string worldsheets to simpler objects, namlely spheres and torii with punctures. This is best described by the following cartoon:



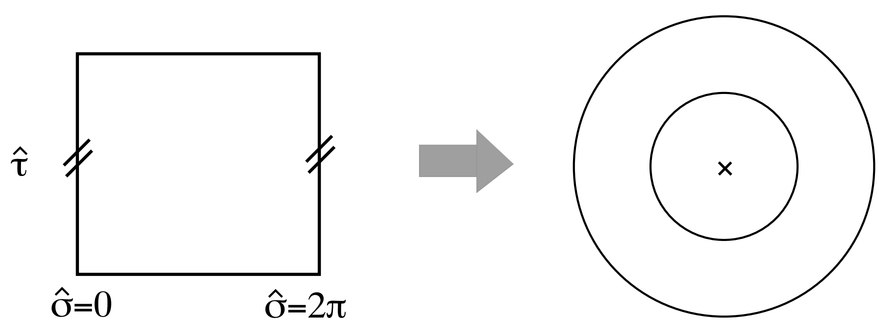
The external string states, which were holes on the worldsheet have been mapped to punctures nice-looking spacetime embeddings. More precisely, we would like to consider the scattering of states that are asymptotically far away from each other before and after the interaction, so we can imagine this as



A somewhat subtle point that is that internal states can have different numbers of strings; consider e.g. constant time slices of complicated worldsheet topologies. It thus seems like we are in fact creating and annihilating strings themselves, not just excitations of strings. In this sense it looks like we are second quantizing the string (i.e. doing string field theory). This, however, is still not quite what we're doing. The correct analog is to recall the way that physicists of yesteryear worked with Dirac theory before quantum field theory was developed. With those tools they were able to add particles via interactions to existing external lines, but there was no true second quantization of particle fields. In the same way, we can consider worldsheets with funny topologies that are interpreted as interactions that add additional string states to existing external strings, but we really have not second quantized anything.

5.5 A hint of radial quantization

Let's now consider a much simpler map:

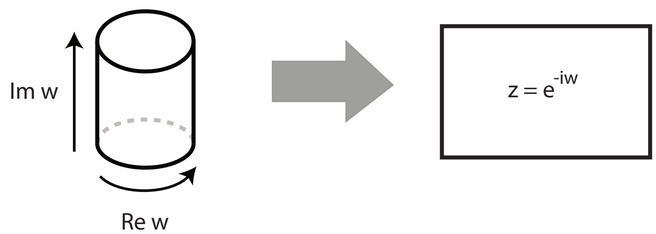


Here we map the string worldsheet into polar coordinates. The hats refer to Euclidean worldsheet coordinates. We've identified the $\hat{\sigma} = 0$ and $\hat{\sigma} = 2\pi$ coordinates so that the object on the left is really a cylinder representing the worldsheet of a closed string. If we complexify our worldsheet coordinates via

$$w = \hat{\sigma} + i\hat{\tau}, \tag{5.9}$$

then we consider the conformal map from $w \to z$ coordinates

$$z = e^{-iw} = e^{-i\hat{\sigma}+\hat{\tau}}. \tag{5.10}$$



These 'radial' coordinates have some nice features. We can see that the origin of this space $z = 0$ corresponds to the asymptotic past $\hat{\tau} = -\infty$. Circles about the origin correspond to equal time contours. In fact, this is the first step to **radial quantization**. In quantum field theory we care about time-ordered operators. In these radial coordinates, time ordering corresponds to radial ordering. Let us remark that, as one would imagine, the open string can be mapped conveniently to the half plane.

Now let's write our our mode expansion. In Minkowski space, this was

$$X_L^\mu = \frac{1}{2}X_0^{L\mu} + \sqrt{\frac{\alpha'}{2}}\alpha_0^\mu(\tau - \sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}e^{-in(\tau-\sigma)} \tag{5.11}$$

with a similar expression for $L \to R$, $\alpha \to \widetilde{\alpha}$, and $\tau - \sigma \to \tau + \sigma$. Going from Minkowski to Euclidean worldsheet coordinates $ds^2 = -d\tau^2 + d\sigma^2 \to d\hat{\tau}^2 + d\hat{\sigma}^2$ with $\tau = -i\hat{\tau}$ and radial coordinates $\ln z = -i(\sigma - \tau)$, we find

$$X_L^\mu = \frac{1}{2}X_0^{L\mu} + \sqrt{\frac{\alpha'}{2}}\alpha_0^\mu(-i\ln z) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}z^{-n}. \tag{5.12}$$

One can check that the right-handed modes are the same with the antiholomorphic variable, i.e. $L \to R \Rightarrow z \to \bar{z}$. This is a nice result, we've found that our Fourier expansion of modes has turned into a Laurent-like expansion (up to the logarithmic term) in the radial coordinates.

We can thus write out our D-plet of 2D Euclidean CFT scalar fields as

$$X^\mu(z,\bar{z}) = X_L^\mu(z) + X_R^\mu(\bar{z}). \tag{5.13}$$

Furthermore, taking the appropriate derivatives,

$$\partial_z X_L^\mu(z) = -i\sqrt{\frac{\alpha'}{2}}\sum_n \alpha_n^\mu z^{-n-1} \qquad \text{and} \qquad \partial_z X_R^\mu(\bar{z}) = -i\sqrt{\frac{\alpha'}{2}}\sum_n \widetilde{\alpha}_n^\mu \bar{z}^{-n-1}. \tag{5.14}$$

Now this really *is* a Laurent expansion and we can perform contour integrals. In radial quantization a circular contour integral about the origin is just an integration over constant time. Since contour integrals just pick up simple poles, we can use this to determine the coefficients rather simply,

$$\alpha^\mu_{-n} = i\sqrt{\frac{2}{\alpha'}}\oint \frac{dz}{2\pi i} z^{-(n-1)}\partial_z X^\mu(z) \tag{5.15}$$

$$= i\sqrt{\frac{2}{\alpha'}}\oint \frac{dz}{2\pi i} z^{-(n-1)}\left[\cdots + \frac{1}{(n-1)!} z^{n-1}\partial_z^n X^\mu(0) + \cdots\right] \tag{5.16}$$

$$= \sqrt{\frac{2}{\alpha'}}\oint \frac{dz}{2\pi} z^{-n}\partial_z X^\mu(z). \tag{5.17}$$

5.6 The state-operator map

The beauty of working in these radial complex coordinates is that the integral over a circle about the origin is related to an integral over a spacelike surface,

$$\oint \quad \underset{\text{radial quantization}}{\Longleftrightarrow} \quad \int_{\text{spacelike surface}}. \tag{5.18}$$

Integrals over spacelike surfaces are usually how we define things like (*ta-dah*) conserved charges in the Noether procedure. This is actually a rather nice statement with more significance than it lets on. On the left-hand side we have the Fourier coefficient α^μ_{-n}, which in our quantum theory is promoted to an *operator*. However, on the right-hand side, the integrand contains no operator-like objects. In fact, the integrand is made up of the derivative of a *local field*, which is a completely different kind of object in quantum field theory.

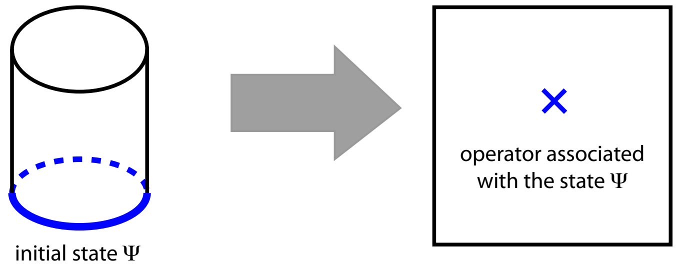
It turns out that this is a rather 'deep' relation between operators and local fields, i.e. states in 2D conformal field theory. We now claim, albeit heuristically at this level, that this relation can be made more rigorous into an isomorphism between operators and states. A handy analogy is the charge associated with a current j^μ in ordinary QFT:

$$Q \sim \int_{t=t_0} j^0. \tag{5.19}$$

The left-hand side is a conserved charge which, in QFT, is the generator of the symmetry associated with the current. The right-hand side is again a construct of the local fields of the theory integrated over a spacelike surface. Evaluating equation (5.17),

$$\alpha^\mu_n = \sqrt{\frac{2}{\alpha'}}\frac{i}{(n-1)!}\partial^n X^\mu(0), \tag{5.20}$$

where we've identified the simple pole term from expanding $\partial_z X^\mu(z)$ about the origin. As the case of the Noether charge, the left-hand side is an operator (of the creation/annihilation variety) and the right-hand side is some construct of fields in the QFT. We would like to [somewhat] formalize the relation between the creation/annihilation operators associated with the external states of a string scattering diagram and the local operators of the worldsheet QFT. A heuristic picture is:



For a state $|\Psi\rangle = \alpha^{\mu}_{-m}\widetilde{\alpha}^{\nu}_{-m}|0,0\rangle$, we can construct the associated operator

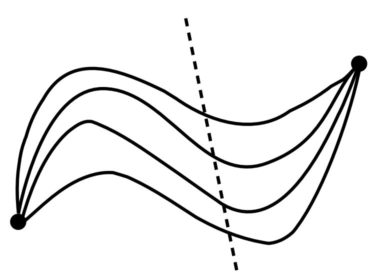
$$\mathcal{O}_\Psi = \left[\sqrt{\frac{2}{\alpha'}}\frac{i}{(m-1)!}\partial^m X^\mu(0)\right] \times \left[\sqrt{\frac{2}{\alpha'}}\frac{i}{(m-1)!}\bar{\partial}^m X^\nu(0)\right]. \tag{5.21}$$

To check this properly, one should check that this transformation maps states faithfully. See chapter 2 of Polchinski [2] or Tong's lecture notes [7] for details. Here we'll be content to motivate this correspondence from elementary quantum mechanics.

Consider the path integral for a point particle propagating from state $|q_i\rangle$ at time $t=0$ to $|q_f\rangle$ at time $t=T$. The path integral associated with this evolution is given by

$$\langle q_f, T|q_i, 0\rangle = \int [\mathcal{D}q]\, e^{\frac{i}{\hbar}\int_0^T dt\, L(q,\dot{q})}. \tag{5.22}$$

We can "cut open" the path integral at some intermediate time $t = t^*$. In pictures, we would slice the set of paths from $|q_i\rangle$ to $|q_f\rangle$:



This is equivalent to inserting $\mathbb{1} = \int dq(t^*)|q,t^*\rangle\langle q,t^*|$, where we emphasize that this is an *ordinary* integral, not a path integral:

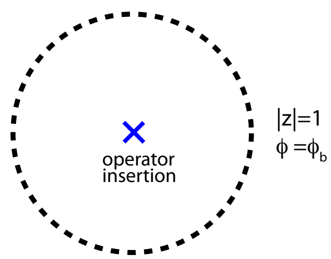
$$\begin{aligned}\langle q_f, T|q_i, 0\rangle &= \int dq(t^*)\int_{q_0,t^*}^{q_f,T}[dq]\, e^{\frac{i}{\hbar}\int_{t_*}^{T} dt\,\mathcal{L}}\int_{q_i,0}^{q_i,t^*}[dq]\, e^{\frac{i}{\hbar}\int_0^{t_*} dt\,\mathcal{L}} &(5.23)\\ &= \int dq(t^*)\,\langle q_f,T|q,t^*\rangle\langle q,t^*|q_i,0\rangle. &(5.24)\end{aligned}$$

What we've inserted here is just a complete set of states. We cold have replaced $\int dq\,|q,t\rangle\langle q,t|$ with *any* complete set of states. The lesson here is that the path integral up to a fixed time,

$$\int_{q_i,0}^{q,t^*} [\mathcal{D}q]\, e^{\frac{i}{\hbar}\int_0^{t^*} L(q,\dot{q})}, \tag{5.25}$$

is actually a state. As a sanity check, we note that the path integral up to a particular final state is (of course) the amplitude to propagate to that state. What we are saying is that the sum (ordinary integral) of the path integral up to all such final states is a state.

We can do this slightly more formally from the point of view of radial quantization. Suppose we have an operator insertion $\mathcal{O}(0)$ at the origin and fix the value of the field $\phi = \phi_b$ on some constant radius surface on the complex worldsheet, e.g. $|z| = 1$.



We can now do the path integral on the internal disc up to this boundary. We claim that this gives us a state, $\Psi_{\mathcal{O}}$ isomorphic to the operator. In particular,

$$\int [\mathcal{D}\phi_i]_{\phi_b}\, e^{-S[\phi_i]}\mathcal{O}(0) \equiv \Psi_{\mathcal{O}}[\phi_b]. \tag{5.26}$$

This is a functional of boundary field configurations ϕ_b, i.e. for any given ϕ_b it spits out a number. This, however, is precisely what we call a state.

This can be a bit nebulous because in QFT one doesn't often make a distinction between states and operators. Operators create and annihilate states, and that's that. It is useful to appeal to quantum mechanics. In quantum mechanics we have a state which is the wavefunction $\Psi(q)$. This is an object which exists over all space for a fixed time t_0. Its argument is a quantum configuration q and it tells us the probability amplitude associated with that configuration at t_0. (The time evolution is, of course, governed by the Schrödinger equation.) In quantum field theory (i.e. after second quantization) the quantum configuration is promoted to a quantum field $q \to \phi(x)$ and the wavefunction is promoted to a wave*functional* $\Psi[\phi(x)]$. The meaning of this wavefunctional is the again the probability amplitude for field configuration $\phi(x)$ for each point x in space at a fixed time t_0. This is exactly what we claimed above in (5.26). Let us further remark that operators are a completely different object than wavefunction(al)s (states). Operators exist at a fixed point in spacetime and are associated with exciting the first-quantized oscillators of the quantum field. Cast this way, it is indeed rather surprising that an isomorphism between states and operators should exist in CFTs. For more details, see Tong's lecture notes [7].

One straightforward example of the state-operator map is the zero-excitation state with momentum k, $|0,0;k\rangle$, which is mapped to a plane wave,

$$|0,0;k\rangle \quad \Leftrightarrow \quad e^{ik_\mu X^\mu(0)}. \tag{5.27}$$

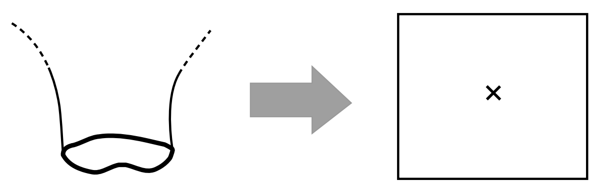
Similarly, we can map the one- and two-tensor excitations

$$\alpha^\mu_{-m}|0,0;k\rangle \quad \Leftrightarrow \quad \sqrt{\frac{2}{\alpha'}}\frac{i}{(m-1)!}\partial^m X^\mu(0)e^{ik\cdot X(0)} \tag{5.28}$$

$$\alpha^\mu_{-m}\widetilde{\alpha}^\nu_{-m}|0,0;k\rangle \quad \Leftrightarrow \quad \left(\sqrt{\frac{2}{\alpha'}}\frac{i}{(m-1)!}\right)^2 \partial^m X^\mu(0)\bar{\partial}^m X^\nu(0)e^{ik\cdot X(0)}. \tag{5.29}$$

One remark we can make is that we're working with a conformal field theory of fields X^μ. Our local operators are spanned by polynomials in $\partial X^\mu, \partial^2 X^\mu, \cdots, \bar{\partial}X^\mu, \bar{\partial}^2 X^\mu, \cdots$ multiplied by the plane wave $e^{ik_\mu X^\mu(0)}$. In this sense the state-operator correspondence is rather straightforward. This is just the heuristic association between creation operators and quantum fields that one could imagine from canonical quantization and the path integral formalism. Conformal field theories are somewhat special in that one can actually formalize this heuristic relation.

Thus far we've learned how to relate external states in spacetime to local operators at the origin of the worldsheet corresponding to a particular on-shell state at $\tau = -\infty$.



One should be concerned that we chose to put our operators at the origin since this is manifestly not diffeomorphism invariant. In order to restore diffeomorphism invariance, we must perform the d^2z integral over the worlsheet. For example, the vertex operator for the string ground state with momentum k, $|0,0;k\rangle$ (or $|0;k\rangle$ for the open string) is

$$\mathcal{V}_0 = g_c \int d^2z\, e^{ik_\mu X^\mu(z,\bar{z})}, \tag{5.30}$$

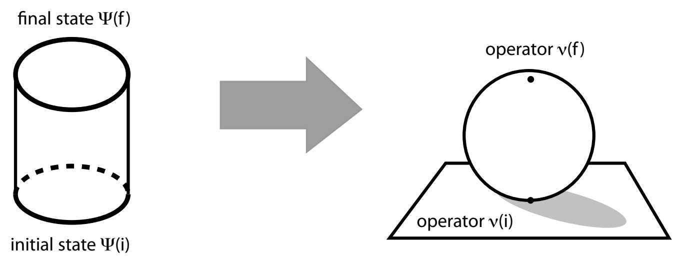
where g_c is a normalizing constant. More interesting is the vertex operator for the first excited state, $\alpha^\mu_{-1}\widetilde{\alpha}^\nu_{-1}|0,0;k\rangle$,

$$\mathcal{V}_{-1,-1} = g_c \int d^2z \left(\frac{2}{\alpha'}\right) \partial X^\mu(z)\bar{\partial}X_\mu(\bar{z})e^{ik_\mu X^\mu(z,\bar{z})} \tag{5.31}$$

$$= \frac{2}{\alpha'}g_c \int d^2z\, \partial X^\mu \bar{\partial} X^\nu e^{ik\cdot X}. \tag{5.32}$$

This maps, e.g., a graviton on the cylindrical worldsheet to a local operator on the complex plane.

Inserting a final state is analogous, except that now one would like to put the final state at the asymptotic future, i.e. at the complex infinity of the Riemann sphere,

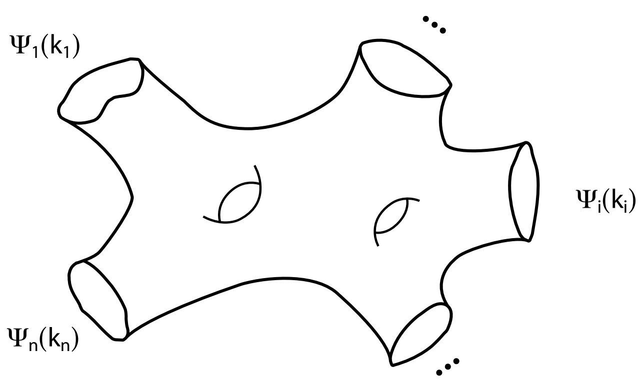


[**Question:** Are these positions just for illustrative purposes? The integral over the complex plane seems to violate any sense of positioning of the operators.] We conclude that the leading-order propagation amplitude between given initial and final states is the correlation function on the sphere of

$$\langle \mathcal{V}_i \mathcal{V}_f \rangle \quad \Leftrightarrow \quad \int \frac{[\mathcal{D}X][\mathcal{D}g]}{V_{\text{diff}\times\text{Weyl}}} e^{-S_{\text{P}}-\lambda\chi_{\text{sphere}}} \int d^2\sigma_i \sqrt{g}\, \mathcal{V}_i \int d^2\sigma_f, \sqrt{g}\, \mathcal{V}_f. \tag{5.33}$$

5.7 Scattering amplitudes

Let's now press on and perform the path integral. We would like to compute the scattering amplitude (or correlation function) of spacetime diagrams like



The n-string correlator can be computed through the path integral using the appropriate insertions of the vertex operators coming from the state-operator map,

$$\int \frac{[\mathcal{D}X][\mathcal{D}g]}{V_{\text{diff}\times\text{Weyl}}} e^{-S_{\text{P}}-\lambda\chi} \prod_{i=1}^{n} \nu_i(k_i) \equiv S^{(c)}_{\Psi_1\cdots\Psi_n}(k_1, \cdots, k_n). \tag{5.34}$$

$S^{(c)}_{\Psi\cdots}$ is the scattering amplitude for external states Ψ_i and a particular topology c. We can think of this as the $\langle V_1(k_1)\cdots V_n(k_n)\rangle_c$ correlation function. The full amplitude is given by a sum over topologies (i.e. arbitrary number of intermediate holes),

$$S_{\Psi_1\cdots\Psi_n} = \sum_{\text{topologies}} e^{-\lambda\chi}\langle\prod_{i=1}^{n}\mathcal{V}_i\rangle_c = e^{-2\lambda}\langle\prod\mathcal{V}\rangle_{S^2} + e^{-0}\langle\prod\mathcal{V}\rangle_{T^2} + \cdots , \tag{5.35}$$

where we noted the formula for the Euler characteristic

$$\chi = 2 - 2g - b, \tag{5.36}$$

where g is the genus and b is the number of boundaries of the topology. In particular, $\chi(S^2) = 2$ and $\chi(T^2) = 0$. The higher topology terms drop off very quickly for small λ.

5.8 The Weyl Anomaly

Before we can say anything meaningful, we have to suitably normalize our results by dividing by the partition function Z. This corresponds to calculating $\langle 1\rangle$,

$$Z = \int \frac{[\mathcal{D}X][\mathcal{D}g]}{V_{\text{diff}\times\text{Weyl}}}\, e^{-S_{\text{P}}-\lambda\chi}. \tag{5.37}$$

We remind ourselves that the point of the measure is that it an be decomposed into a parts that are gauge-equivalent and those that are not (i.e. unique orbits within the gauge group),

$$[\mathcal{D}X][\mathcal{D}g] = [\mathcal{D}(\text{orbit})]\cdot[\mathcal{D}(\text{gauge equivalent})]. \tag{5.38}$$

The second factor is just $V_{\text{diff}\times\text{Weyl}}$ and cancels the denominator. The Fadeev-Popov method tells us that this decomposition is always allowed and also buys us an additional consistency condition.

We already know that the system is classically invariant under the local diffeomorphism $\times$ Weyl symmetry and the global Poincaré symmetry. We need to check if these symmetries turn out to be anomalous. (If the gauged symmetries are anomalous then our theory dies.) Since it's not necessarily obvious that anomalies could appear, let's motivate the possibility that such an inconsistency could appear. Consider, for example, introducing a regulator into our theory. One method that rarely failed us in field theory was the introduction of a Pauli-Villars field,

$$\Delta S = \int d^2\sigma\,\sqrt{\gamma}\,\left(\mu^2 Y_\mu Y^\mu\right). \tag{5.39}$$

This is manifestly Poincaré invariant (from the contraction of Lorentz indices), it is diffeomorphism invariant (since we know we can have massive fields in general relativity), but it is *not* Weyl invariant. This is easy to see since we already know that $\sqrt{\gamma}\gamma^{\alpha\beta}\cdots$ *is* Weyl invariant, therefore $\sqrt{\gamma}$ by itself cannot be. We thus have reason to ask whether or not Weyl symmetry is anomalous, i.e. does there exist a Weyl invariant regulator? A more sophisticated way of saying this is to take the Fujikawa perspective and ask whether or not both the action and the measure are invariant under Weyl transformations. (This way one doesn't have to ask about regulating a divergence which doesn't necessarily exist.) We know, of course, that an anomaly in a gauge symmetry such as Weyl invariance would be an automatic game-over: we would lose covariance and could even lose unitarity as the theory becomes inconsistent.

Let's sketch out the derivation of the Weyl anomaly cancellation condition. A correlation function for "$\cdots$" (i.e. for any operator) takes the general form

$$\langle \cdots \rangle = \int \frac{[\mathcal{D}X][\mathcal{D}g]}{V_{\text{diff}\times\text{Weyl}}}\, e^{-S[X,g]} \cdots . \tag{5.40}$$

Let us introduce a **fiducial metric** $\hat{g}$ that we introduce as part of the Fadeev-Popov procedure so that we can write correlation functions with respect to this gauge-fixed metric,

$$\langle \cdots \rangle_{\hat{g}} = \int [\mathcal{D}X]\, e^{-S[X,\hat{g}]} \cdots . \tag{5.41}$$

We've cheated a bit since the action should properly contain a term for the Fadeev-Popov ghosts which are introduced to cancel shifts in the Jacobian. We'll ignore all this and leave the reader to work out the details or follow them in the usual references. What we're looking for when we say that there should be no Weyl anomaly is that the correlation functions should not change if the metric undergoes a Weyl transform, $\hat{g} \to \hat{g}^{\xi}$ where ξ is some Weyl transformation parameter, $\hat{g}^{\xi}_{\alpha\beta} = e^{2\omega_\xi}\hat{g}_{\alpha\beta}$. Thus for Weyl invariance to be anomaly-free, it is necessary and sufficient that

$$\langle \cdots \rangle_{\hat{g}} = \langle \cdots \rangle_{\hat{g}^{\xi}}. \tag{5.42}$$

Let's see a correlator changes under a variation of the metric,

$$\delta\langle \cdots \rangle_{\hat{g}} = \int d^2\sigma\, \sqrt{g} \frac{\delta}{\delta g_{\alpha\beta}(\sigma)} \langle \cdots \rangle_{\hat{g}}. \tag{5.43}$$

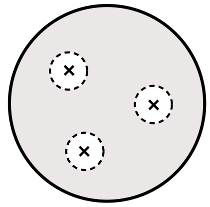
Recall, however, that

$$\frac{\delta S}{\delta g_{\alpha\beta}(\sigma)} = \frac{\sqrt{g}}{4\pi} T^{\alpha\beta}(\sigma) \tag{5.44}$$

from the definition of the energy-momentum tensor in general relativity. Let us stress that $T^{\alpha\beta}$ is the energy-momentum tensor of the worldsheet quantum field theory, not the target spacetime. Now we make a big simplification. Thus we can write

$$\delta\langle \cdots \rangle_{\hat{g}} = -\frac{1}{4\pi} \int d^2\sigma\, \delta g_{\alpha\beta}(\sigma)\, \langle T^{\alpha\beta}(\sigma) \cdots \rangle_{\hat{g}}. \tag{5.45}$$

We want this to hold as an operator equation. That is to say that we would like this to hold 'off shell' and for any operators "$\cdots$". Further, we would like this to hold even when the operators inside the "$\cdots$" become coincident. Recall that when different operators in a correlation function become local, i.e. $\langle \cdots \mathcal{O}_i(z_i)\mathcal{O}_j(z_j) \cdots \rangle$ with $z_i \to z_j$, one often finds singularities. We shall assume that $\delta g_{\alpha\beta}(\sigma) = 0$ in some neighborhood of each of the operator insertions. In other words, the support of our variation of the metric looks like



This is a very dramatic simplification that borders on dishonesty. For our 'quick-and-dirty' purposes, however, it will be sufficient. The proper way to treat this is to do an operator product expansion (OPE) which turns out to be not-so-painful thanks to the simplifications of working with holomorphic functions. A good review of this can be found in Tong's lectures [7].

For a variation of the metric that is a Weyl rescaling, δ_ξ, we find that

$$\delta_\xi g_{\alpha\beta} T^{\alpha\beta} = T^\alpha_{\ \alpha}\left(e^{2\omega_\xi} - 1\right) \tag{5.46}$$

since $\delta_\xi g_{\alpha\beta} = (e^{2\omega_\xi} - 1)g_{\alpha\beta}$. Now we see that we want to demand that

$$\langle T^\alpha_{\ \alpha} \cdots \rangle = 0, \tag{5.47}$$

i.e. that $T^\alpha_{\ \alpha} = 0$ as an operator equation. This is equivalent to the Weyl invariance being non-anomalous. Recall, however, that the tracelessness of the energy-momentum tensor is just the statement that the theory is scale invariant.

In unit gauge on a flat worldsheet embedding, we already knew that the energy-momentum tensor is traceless. This was part of our Virasoro constraints. For a more general embedding, then, we know that the trace of the energy-momentum tensor must be proportional to the curvature,

$$T^\alpha_{\ \alpha} = a_1 R^{(2)}, \tag{5.48}$$

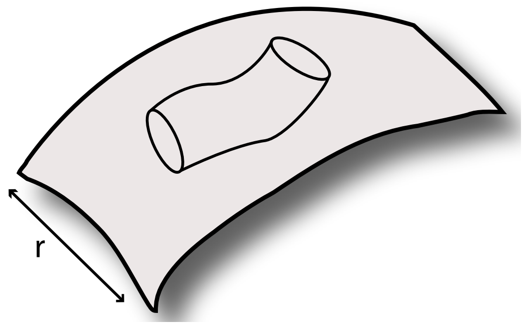
in order to ensure the correct limit in the flat case. (We write $R^{(2)}$ to mean the Ricci scalar on the 2D worldsheet theory.) One can then spend eight hours learning about conformal field theories to derive that the coefficient a_1 depends on the dimension of the spacetime as

$$a_1 = \frac{26 - D}{12}. \tag{5.49}$$

We see that in order for the Weyl anomaly to vanish on a general target space, we must have a target spacetime of dimension $D = 26$.

6 The stringy nonlinear sigma model

Now let's build on all that we've developed and write out a low-energy effective action for the light modes of our theory. Recall when we have a quantum field theory with a spontaneously broken global symmetry we can write out an effective theory of light states, i.e. (pseudo-)Nambu-Goldstone bosons. Recall, for example, chiral perturbation theory for pions in QCD. We call these effective theories nonlinear sigma models (NLΣM), and now we'd like to work out a stringy version. In particular, we will be interested in the theory of the graviton and its massless friends. This means we'd like to study the low-energy string states on a general target space metric. Here's a cartoon picture. We shall write r to be some characteristic curvature scale of the target space whose precise definition isn't something we'll lose sleep over.



We began with the Polyakov action (written here with Euclidean signature) for a Minkowski target space metric,

$$S_{\mathrm{P}} = \frac{1}{4\pi\alpha'} \int d^2\sigma \, \sqrt{g}\, g^{\alpha\beta}\, \partial_\alpha X^\mu \partial_\beta X^\nu \, \eta_{\mu\nu}. \tag{6.1}$$

We can generalize this by replacing $\eta_{\mu\nu}$ with $G_{\mu\nu}(X)$,

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \, \sqrt{g}\, g^{\alpha\beta}\, \partial_\alpha X^\mu \partial_\beta X^\nu \, G_{\mu\nu}(X). \tag{6.2}$$

One can see that we're starting to connect our string theory to something resembling quantum excitations of spacetime general relativity. With this generalization we can expand our action in terms of the flat spacetime metric, $G_{\mu\nu}(X) = \eta_{\mu\nu} + \chi_{\mu\nu}(X)$,

$$e^{-S} = e^{-S_{\mathrm{P}}} \left(1 - \frac{1}{4\pi\alpha'} \int d^2\sigma \, \sqrt{g}\, g^{\alpha\beta}\, \partial_\alpha X^\mu \partial_\beta X^\nu \chi_{\mu\nu}(X) + \cdots \right). \tag{6.3}$$

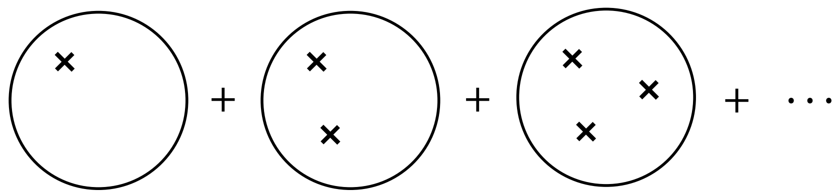
Recall our state-operator correspondence (where the subscript 'graviton' refers to any of the massless closed string excitations, not just the symmetric one)

$$\mathcal{V}_{\text{graviton}}(s_{\mu\nu}, k) \quad \Leftrightarrow \quad \frac{g_c}{\alpha'} \int d^2\sigma \, \sqrt{g}\, g^{\alpha\beta}\, \partial_\alpha X^\mu \partial_\beta X^\nu \, s_{\mu\nu} e^{ik\cdot X}, \tag{6.4}$$

where $s_{\mu\nu}$ is a polarization vector ($s_{\mu\nu}k^\mu = 0$ and $s^\mu{}_\mu$). Comparing to our expansion of the target space metric about $\eta_{\mu\nu}$. Thus our expansion (6.3) can be understood in terms of an expansion of the correlation functions of arbitrary operators ("$\cdots$"),

$$\langle \cdots \rangle_G = \langle \cdots \rangle_{G=\eta} + \sum_{k=1}^{\infty} \langle \cdots \prod_{i=1}^{j} \mathcal{V}_i(s^i_{\mu\nu}, k_i) \rangle_{G=\eta}. \tag{6.5}$$

In other words, the correlator with respect to the general [target] spacetime metric is equal to an expansion of correlators with respect to the 'graviton' (and friends) vertex operators in flat space. In pictures,



where each puncture is a graviton (or antisymmetric tensor, or dilaton).

We could have picked any background metric to expand about. The Minkowski metric is the obvious choice since we know how to calculate with respect to it, but we'll soon be interested in more general backgrounds (e.g. black holes). Let's remind ourselves that our general background of massless closed string states can be decomposed into a graviton $G_{\mu\nu}(X)$, an antisymmetric tensor $B_{\mu\nu}(X)$, and a dilaton $\Phi(X)$. Thus we may write our nonlinear sigma model action as

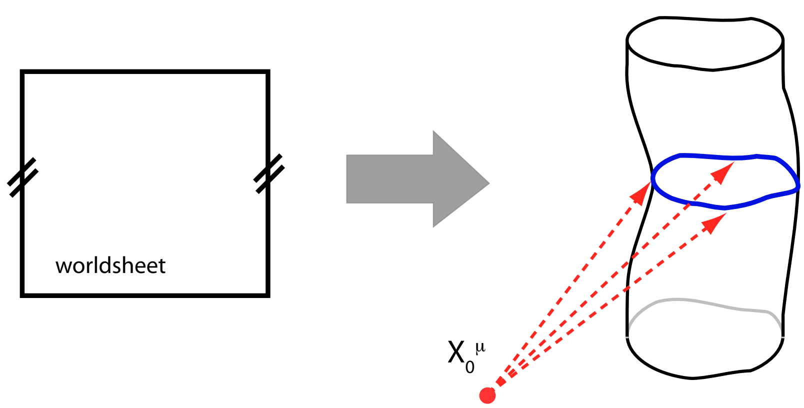
$$S_{\mathrm{NL\Sigma M}} = \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{g}\left\{\left[g^{\alpha\beta}G_{\mu\nu}(X) + i\epsilon^{\alpha\beta}B_{\mu\nu}(X)\right]\partial_\alpha X^\mu\partial_\beta X^\nu + \alpha' R^{(2)}\Phi(X)\right\}. \tag{6.6}$$

We've written $R^{(2)}$ to mean the worldsheet 2D Ricci scalar to distinguish it from the geometric quantities of the target spacetime. This NLΣM action corresponds to **coherent states** of the massless fields 'generated' by exponentiating the massless vertex operators, $\mathcal{V}$.

6.1 The sigma model expansion

This is all a nice story, but how do we get a handle for working about this general target space geometry with a non-trivial $G_{\mu\nu}(X)$, $B_{\mu nu}(X)$, and $\Phi(X)$.

We will start by expanding our worldsheet-to-target space map X^μ about a point in the target space X_0^μ,



i.e. we define an origin in the target space via

$$X^\mu(\tau,\sigma) = X_0^\mu + Y^\mu(\tau,\sigma). \tag{6.7}$$

Then the NLΣM action can be Taylor expanded about this reference piont,

$$S_{\mathrm{NL}\Sigma\mathrm{M}} = \frac{1}{4\pi\alpha'}\int d^2\sigma\,\sqrt{g}\,\Big\{g^{\alpha\beta}\partial_\alpha Y^\mu\partial_\beta Y^\nu\big[G_{\mu\nu}(X_0) + G_{\mu\nu,\omega}(X_0)Y^\omega + \cdots\big] \\ + i\epsilon^{\alpha\beta}\partial_\alpha Y^\mu\partial_\beta Y^\nu\big[B_{\mu\nu}(X_0) + B_{\mu\nu,\omega}(X_0)Y^\omega + \cdots\big] \\ + \alpha' R^{(2)}\big[\Phi(X_0) + \Phi_{,\omega}Y^\omega + \cdots\big]\Big\} \tag{6.8}$$

In principle there should also be a tachyon term, but it carries its own set of issues and we'll assume that it is projected out. We should interpret this action as a 2D interacting quantum field theory. One can see that it has the usual kinetic term for the fields Y^μ. The couplings are given by spacetime derivatives of the metric and its friends (G, B, and Φ). There are infinitely many couplings. In fact, to be more precise, we have whole functions worth of couplings: the interactions depend on where one is in field space. This is indeed an expansion, the so-called **sigma model expansion**. To make the expansion scheme more plain, we can introduce dimensionless coordinates

$$\phi^\mu = \frac{Y^\mu}{\sqrt{\alpha'}}. \tag{6.9}$$

It then becomes more clear that the derivative expansion about the reference point X_0^μ can really be taken as an expansion in powers of (α'/r^2), where r is the characteristic curvature scale of the target space (the radius of curvature). Recall further that the string tension is given by $T = (2\pi\alpha')^{-1}$ so that $\sqrt{\alpha'}$ is the length scale of the string. Thus the sigma model expansion is valid when the string too small to probe the curvature of the target space, i.e. when the length scale for change in the geometry of spacetime is large in string units.

6.2 Consistency of the sigma model expansion

We should now pause and consider consistency. We emphasized that we are now working in a general target space background. But we also know that when we quantized our string theory we were very particular about our set of consistency conditions. We now ought to check when, in terms of our general target space, these consistency conditions are still satisfied.

Recall that varying the action with respect to the metric gives us the stress-energy (or "energy-momentum") tensor,

$$\frac{\delta}{\delta g^{\alpha\beta}}S \quad\leftrightarrow\quad T_{\alpha\beta}. \tag{6.10}$$

Thus it is straightforward that a Weyl rescaling $\delta_{\mathrm{Weyl}}\,g_{\alpha\beta} = f(x)g_{\alpha\beta}$ gives the trace of $T_{\alpha\beta}$. Indeed, we recall that we had to require that the Weyl anomaly vanishes,

$$\langle T^\alpha{}_\alpha\rangle = 0. \tag{6.11}$$

Weyl rescaling is just a dilatation so that in particular we have to require (at least) invariance with respect to **rigid dilatations**. Actually, the vanishing of the Weyl anomaly is a stronger statement than this, but for our present purposes we will work with this weaker statement. Invariance under rigid dilatations tells us that the theory must be scale invariant, i.e. the couplings (heuristically λ) must have vanishing β functions,

$$\beta_\lambda(M_0) = M\frac{\partial}{\partial M}\lambda(M)\bigg|_{M=0} = 0. \tag{6.12}$$

In our 2D QFT we may compute $\langle T^\alpha{}_\alpha\rangle$ to any order we desire in the sigma model expansion parameter (α'/r^2). From (5.48) we already know that at $\mathcal{O}[\alpha'/r^2)^0]$, that is even in infinite flat space, we have

$$T^\alpha{}_\alpha = -\frac{1}{12}(D-26)R^{(2)}. \tag{6.13}$$

There are, however, corrections at higher orders in (α/r^2).

$$T^\alpha{}_\alpha = -\frac{1}{2\alpha'}\beta^G_{\mu\nu}g^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu \tag{6.14}$$
$$= -\frac{1}{2\alpha'}\beta^B_{\mu\nu}\epsilon^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu - \frac{1}{2}\beta^\Phi R^{(2)}. \tag{6.15}$$

Where, after some hard work, one finds that the β functions are

$$\beta^G_{\mu\nu} = \alpha' R_{\mu\nu} + 2\alpha'\nabla_\mu\nabla_\nu\Phi - \frac{\alpha'}{4}H_{\mu\lambda\omega}H_\nu{}^{\lambda\omega} + \mathcal{O}(\alpha'^2) \tag{6.16}$$
$$\beta^B_{\mu\nu} = \frac{\alpha'}{2}\nabla^\omega H_{\omega\mu\nu} + \alpha'\nabla^\omega\Phi H_{\omega\mu\nu} + \mathcal{O}(\alpha') \tag{6.17}$$
$$\beta^\Phi = \frac{D-26}{6} - \frac{\alpha'}{2}\nabla^2\Phi + \alpha'\nabla_\omega\Phi\nabla^\omega\Phi - \frac{\alpha'}{24}H_{\mu\nu\lambda}H^{\mu\nu\lambda} + \mathcal{O}(\alpha'^2) \tag{6.18}$$

and the 3-form $H_{\mu\nu\rho}$ is the exterior derivative of the $B_{\mu\nu}$ tensor, $H = dB$,

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \{\text{antisymmmetric permutations}\}. \tag{6.19}$$

We can think of these as loop corrections (in the sigma model) to the β functions. Our condition of Weyl invariance requires that these loop corrections vanish,

$$\beta_G(X) = \beta_B(X) = \beta_\Phi(X) = 0. \tag{6.20}$$

To zoom out to the big picture once again, we started with a worldsheet theory of fields $X^\mu(\tau,\sigma)$ with a set of equations of motion to impose. Now the vanishing of the beta functions is another set of equations that have to be imposed. [**Check:**] We understand that these constraints come from Weyl invariance in the full string theory, but how should we understand this condition from the point of view of the nonlinear sigma model?

It turns out that these equations can be understood as coming from come from extremizing the 26-dimensional action

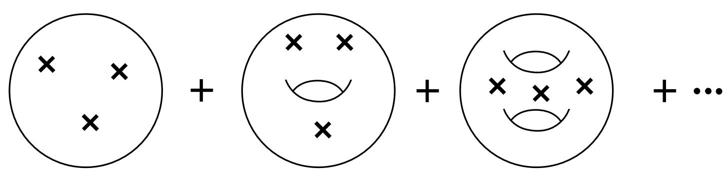
$$S_{26} = \frac{1}{2\kappa_0^2}\int d^2X\,\sqrt{G}e^{2\Phi}\left\{R - \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda} + 4\partial_\mu\Phi\partial^\mu\Phi + \mathcal{O}(\alpha')\right\}. \tag{6.21}$$

Here κ_0 is some constant that is not yet determined and all geometric quantities (e.g. $\sqrt{G}$, R) are evaluated on the target space. We see that up to a Φ rescaling S_{26} looks just like the Einstein-Hilbert action plus 'stuff' (matter). The H^2 term should be understood as a 3-form generalization of the usual fieldstrength and we can see the usual scalar kinetic term. We know exactly what we're going to get from varying this action: the Einstein equation. One can perform a metric transformation to the Einstein frame, see e.g. Polchinski chapter 3, in which case one would see that, that $\kappa_0 \exp(\Phi_0)$ should be $(8\pi G_N)^{1/2} = (8\pi)^{1/2}/M_{\rm P}$ as required to match gravity. Moreover, going to a higher order in α' one finds (through more calculation)

$$\beta^G_{\mu\nu} = \alpha' R_{\mu\nu} + \frac{\alpha'^2}{2} R_{\mu\kappa\lambda\tau} R_\nu{}^{\kappa\lambda\tau} + \mathcal{O}(\alpha'^3), \tag{6.22}$$

the second term is the stringy correction to GR. It seems weird that we're imposing these extra spacetime equations of motion on top of the X equations of motion. Further, it seems disconcerting that these appear to come only from a special subset of theories, i.e. those which look like S_{26}. Indeed, this would seem to say that only a subset of string theories are self consistent.

It turns out that there is yet another way still to understand where these 'additional' constraints come from. In particular we will see that S_{26} comes from somewhere else still, i.e. we will find a different way to get the low-energy EFT of massless modes. Let's start by computing S-matrix elements of massless states. We already have a nice picture of this:

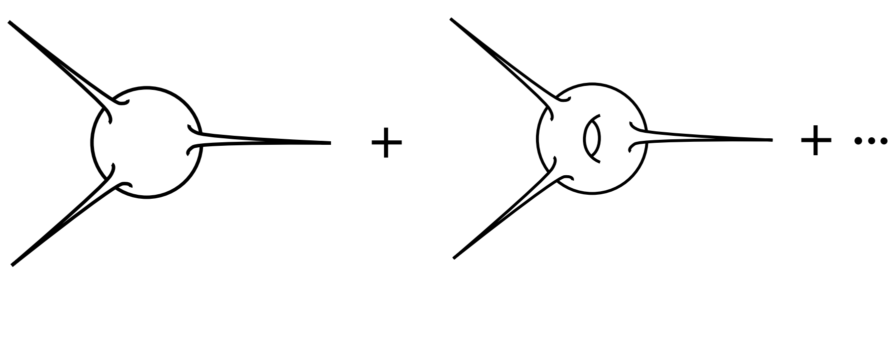


where the crosses represent insertions of the vertex operators $\mathcal{V}_{\rm massless}$ for massless modes ($G_{\mu\nu}$, $B_{\mu\nu}$, Φ). This is the picture coming from the path integral,

$$\int \mathcal{D}X\, \mathcal{V}_1 \cdots \mathcal{V}_n e^{-S_{\rm P} - \lambda\chi}. \tag{6.23}$$

We would have to do this for all numbers of insertions and for all Riemann surfaces. Note that this is really *really* hard for topologies beyond the torus. Fortunately, there are aspects that survive to any order.

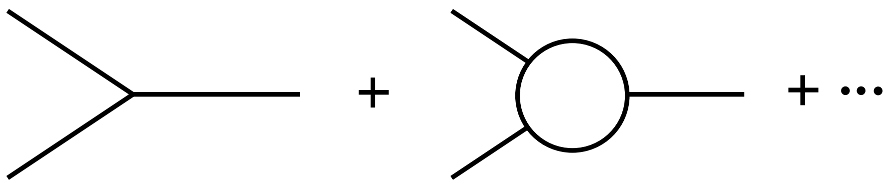
Assume that we compute S-matrices that look like:



(We don't care about the specifics of the external states other than that they are massless.) We can ask ourselves the following question: Does there exist a 26-dimensional QFT coupled to gravity (i.e. a long-wavelength low-energy EFT) that has S-matrix elements that match our string theory up to $\mathcal{O}(\alpha')$:

$$\mathcal{A}_{\text{EFT}} = \mathcal{A}_{\text{string up to }\mathcal{O}(\alpha')}. \tag{6.24}$$

Thus we wonder if there exist a QFT in which the above stringy diagrams can be calculated via normal Feynman diagrams,



So does such a theory exist? It turns out that the answer is *yes!* and that the theory is precisely S_{26} in (6.21). The effective action that 'reproduces' the string S-matrix is precisely the effective action that gives the β function equations that we wanted to impose for Weyl invariance.

We can now rejoice and jump up and down emphatically since we now see that Weyl consistency is imposed *automatically* in string theory. On the one hand, it is imposed (1) by the cancellation of the Weyl anomaly and (2) by looking for a QFT that gives the same S-matrix elements. We find that the bulk (target space) Einstein equation appears both from Weyl consistency in string theory and the scattering of gravitons in the low-energy effective theory. Thus just by starting with the Polyakov action S_{P} we find that we force the background of our target space to obey the Einstein equations. Cool.

6.3 Two expansion schemes: genus vs. sigma model

Now we arrive at an often-confusing point. There are *two* expnsions in string theory. We've already familiarized ourselves with the sigma model expansion. Let us introduce the **string loop (genus) expansion** as an expansion in the number of donut holes of our S-matrix element. Recall that in a non-trivial Φ background and in the limit where $\sqrt{\alpha'}\frac{\partial}{\partial X}\langle\Phi\rangle$ is small, our action looks like

$$S = S_{\text{P}} + \frac{1}{4\pi\alpha'}\int d^2\sigma\,\sqrt{g}\,R^{(2)}\alpha'\Phi \tag{6.25}$$

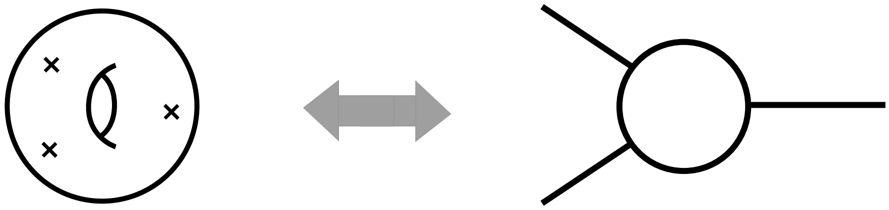
where we can readily identify the second term as $\Phi\chi = \Phi\cdot$(Euler number). We thus have $\lambda = \langle\Phi\rangle$. In other words, we can define a string coupling

$$g_{\text{S}} = e^{\Phi} \tag{6.26}$$

so that the expansion in genus g (g = number of holes) is

$$\sum_{g=0}^{\infty} = \mathcal{A}_g e^{-\lambda\chi} = \sum_{g=0}^{\infty}\mathcal{A}_g e^{-2\Phi+2g\Phi} = \frac{1}{g_{\text{S}}^2}\sum_{g=0}^{\infty} g_S^{2g}\mathcal{A}_g. \tag{6.27}$$

Since the powers of g_S in the genus expansion count the number of holes in the scattering amplitude for the worldsheet, the expansion in g_S is indeed a loop expansion in the 26-dimensional spacetime QFT. This is somewhat obvious is we just draw some pictures, e.g.,



Classical effects in spacetime (tree-level Feynman diagrams) correspond to vertex insertions on the sphere while quantum effects correspond to more complicated non-spherical (hole-y) worldsheets. Note that g_S plays a role that is very much analogous to $\hbar$ in that it counts the loop level of spacetime quantum effects.

On the other hand, for a *fixed genus* we can also consider the 2D QFT expanded about flat space, i.e. the sigma model expansion (α' expansion) that we've already familiarized ourselves with. This is an expansion in powers of (α'/r^2) via derivatives of the massless fields (e.g. $G_{\mu\nu,\omega,\rho}$). We haven't been rigorous in our definition of r mostly because we don't need to be. But we can see where that it comes from the term in the effective action that looks like $\alpha' H^4$ from which we can very crudely say that $H \sim 1/r$. The sigma model expansion corresponds to loops in the two-dimensional quantum field theory (this is not *necessarily* a CFT, though usually we will find that this is necessary).

We thus have two expansions schemes. If we wanted to use string theory to study, for example, four-graviton scattering, then we can write out our amplitude with respect to each of these expansions. Let us write $\mathcal{A}^i_j$ to mean the amplitude contribution from the i^{th} order in the sigma model expansion and j^{th} order in the loop expansion.

$$\begin{array}{ccccccccc} \mathcal{A} = & \mathcal{A}^0_0 & + & \mathcal{A}^0_1 g_S & + & \mathcal{A}^0_2 g_S^2 & + \cdots \\ & + & & + & & + & \\ & \mathcal{A}^1_0\left(\frac{\alpha'}{r^2}\right) & + & \mathcal{A}^1_1\left(\frac{\alpha'}{r^2}\right) g_S & + & \mathcal{A}^1_2\left(\frac{\alpha'}{r^2}\right) g_S^2 & + \cdots \\ & \vdots & & \vdots & & \vdots & \end{array} \tag{6.28}$$

Brute force *never* gets further than these terms. However, clever use of symmetries and nonrenormalizations theorems will allows us to get results to all orders in g_S and α'.

We could ask ourselves where the higher-order terms come from, e.g. the α'^2 term in (6.22) appears to be an expansion in $\alpha'\times$curvature. The true quantum gravity effects are g_S effects since quantum gravity in the target spacetime requires $\hbar \neq 0$. However, string theory provides *other* high-scale effects from 'stringy' α' physics. For example, we know that the Einstein-Hilbert action for general relativity is $\mathcal{L} = \sqrt{g}R$. The correction to this at string tree-level (i.e. $\mathcal{O}(g_S^2) \leftrightarrow S^2$) is given by

$$\Delta\mathcal{L} = \sqrt{g}\,\alpha'^3\zeta(3)\left(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} + \cdots\right). \tag{6.29}$$

Note that we are at $\mathcal{O}(\alpha'^3)$, but still at string tree-level. Including string loops would introduce $\mathcal{O}(g_3)$ terms which come from corrections like $\langle h \cdots h\rangle_{T^2}$. The lesson is this:

1. g_S terms represents a genuine loop expansion ("string loop") capturing effects from string theory as a quantum gravity

2. α' represents different "stringy" corrections coming from string theory as a theory of extended objects.

Now a comment: we started with a long-wavelength effective field theory. It was certainly not finite; you knew well before you started with these lectures that QFT and GR do not typically combine to give a finite theory. What repairs this theory are α' and g_S corrections. We *cannot* compute the α' corrections within the string sigma model. This will be very important for quantum black holes where the two expansions will give important new structure. Finally, in this course we will spent a lot of time on topics where we can avoid these expansions and instead focus on non-perturbative effects that hold to all orders.

6.4 String instantons

The vertex operator for our massless antisymmetric tensor $B_{\mu\nu}$ is

$$\mathcal{V} = \frac{g_c}{\alpha'} \int d^2\sigma \sqrt{g}\, \epsilon^{\alpha\beta} \partial_a X^\mu \partial_\beta X^\nu a_{[\mu\nu]} e^{ik\cdot X}. \tag{6.30}$$

For zero momentum $k = 0$ this simplifies (integrating by parts and using $\epsilon^{\alpha\beta}\partial_\alpha\partial_\beta X^\nu = 0$)

$$\mathcal{V} = \frac{g_c}{\alpha'} \int d^2\sigma \sqrt{g}\, \partial_\alpha \left(\epsilon^{\alpha\beta} X^\mu \partial_\beta X^\nu g_{\mu\nu}\right). \tag{6.31}$$

This is a total derivative so that if the worldsheet has no boundary $\mathcal{V}$ vanishes. This is true to all orders in α'. In fact, this is true to all orders in g_S since we never invoked the genus of the target space embedding of worldsheet. Thus the zero-momentum coupling of $B_{\mu\nu}$ vanishes at zero momentum, i.e. the zero momentum mode of this plane wave *decouples*. From this we conclude that the effective spacetime action cannot have any terms involving undifferentiated $B_{\mu\nu}(X)$ fields; i.e. there are no non-derivative couplings of $B_{\mu\nu}$ to any order in g_S and α'. Thus the terms involving this field in the spacetime action can look like

$$S_{26}^{\text{Eff}} \supset H_{\alpha\beta\gamma} H^{\alpha\beta\gamma} + \cdots \tag{6.32}$$

but it can*not* have terms like, e.g., $B_{\mu\nu}B^{\mu\nu}$.

We now ask ourselves: When can this fail?

1. If the worldsheet contains a boundary, then certainly the total derivative term in (6.31) can contribute.

2. Alternately, we can consider the term in the worldsheet action

$$\Delta S = \int_{\Sigma_2} d^2\sigma \sqrt{g}\, \epsilon^{\alpha\beta} B_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu = \int_{\Sigma_2} B_2. \tag{6.33}$$

   In sigma model perturbation theory Σ_2 is contractible (we're just calculating about some fixed X_0^μ) and so this vanishes. However, we could consider cases where Σ_2 is nontrivial and $\Delta S \neq 0$.

The second case is what we'll now briefly consider. If we suppose that the target space has a big sphere, i.e. a "nonvanishing 2-cycle," then we know from gauge theory in QFT that we can have some non-vanishing winding about this topology. This is what we call a **worldsheet instanton**. It is manifestly a non-perturbative effect. The lesson is that we can write an instanton action

$$S_{\text{inst.}} = \frac{1}{2\pi\alpha'}\left(\int\sqrt{g} + i\int B\right). \tag{6.34}$$

Thus the 26-dimensional spacetime action can depend on $H = dB$ at $\mathcal{O}\left(\alpha'^k g_S^j\right)$ for any finite k and j, but it can depend on B itself only as

$$\exp\left(\frac{n}{2\pi\alpha'}\int\sqrt{g} - \frac{in}{2\pi\alpha'}\int B\right) = \exp\left[-\frac{n}{2\pi\alpha'}\left(\text{Vol}(\Sigma_2) + i\int_{\Sigma_2} B\right)\right] = e^{-n(t+ib)}. \tag{6.35}$$

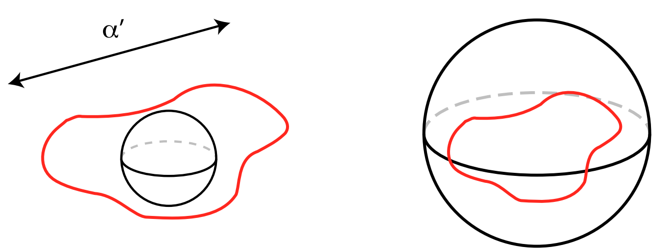
Schematically we may write

$$\mathcal{L}_{\text{Pert.}} \supset c_2 H_{\mu\nu\rho}H^{\mu\nu\rho} + c_4\alpha'\left(H_{\mu\nu\rho}H^{\mu\nu\rho}\right)^2 + \cdots \tag{6.36}$$

where $c_a = \sum_{j=0}^{\infty} c_a^{(j)} g_S^j$. To this perturbative piece we add the non-perturbative instanton contribution,

$$\mathcal{L}_{\text{NP}} \supset \sum_{\Sigma_2}\sum_j d_j e^{-j\left[\frac{\text{Vol}\Sigma_2}{2\pi\alpha'} + i\int_{\Sigma_2} B\right]}. \tag{6.37}$$

We can close with a cartoon picture of what's going on. Suppose we have a string probe (of characteristic scale $\sqrt{\alpha'}$) probing the non-trivial cycles of the bulk [target] spacetime.



When the string wraps a small 2-cycle we get $\mathcal{O}(1)$ contributions. However, when the 2-cycle is larger than the string scale we get contributions which are exponentially suppressed. Thus non-perturbatively we can have zero-momentum contributions from $B_{\mu\nu}$ (that are not pre-packaged in $H = dB$). These are pseudo-Nambu-Goldstone bosons due to their shift symmetry. (One could call these 'axions,' but for technical reasons this terminology would be too strong.)

7 Superstrings

We have already noted that bosonic string theory is not suitable to describe our physical reality. In addition to the instability from the tachyon modes, the most obvious deficiency is the lack of fermions. We know of an easy way to get fermions out of a theory: imposing supersymmetry. This leads us naturally to the **superstring**. Two of the key results from superstring theory should already be well-known from popular expositions on string theory: (1) the tachyonic mode is projected out and (2) the spacetime dimension is reduced to $D = 10$. In this section we'll build up the structure of superstring theory to see where we get these results. Then, in the remainder of these lectures, we'll really put our superstring theory to work. We shall assume that the reader has a strong background in supersymmetry so that details regarding superspace an the SUSY algebra can be omitted.

7.1 Fermions in 2D

Consider the action

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \left(\partial_\alpha X^\mu \partial_\beta X_\mu \, \eta^{\alpha\beta} + \bar{\Psi}^\mu \rho^\alpha \partial_\alpha \Psi_\mu \right), \tag{7.1}$$

where we are working in unit gauge. We note that for now we shall work with a flat Minkowski metric in both the target space and worldsheet. It is worth noting that the Minkowski signature is important on the worldsheet since we will want the Minkowski signature Clifford algebra. We're already familiar with our friends the X^μ fields. Now joining the party are **worldsheet fermions** Ψ^μ. Again μ is a target spacetime Lorentz index (i.e. a worldsheet 'flavor') and for the moment we have suppressed spinor indices. The ρ^α are two-dimensional Dirac matrices which obey the 2D Clifford algebra

$$\left\{ \rho^\alpha, \rho^\beta \right\} = 2\eta^{\alpha\beta}. \tag{7.2}$$

This admits a representation of 2×2 *real* matrices

$$\rho^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad\qquad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{7.3}$$

Armed with ρ^0 we can define the 'conjugate' spinor field as usual,

$$\bar{\Psi} = \Psi^\dagger i \rho^0. \tag{7.4}$$

Now the fact that the ρ matrices are real tells us that we can take Ψ^μ to be real, i.e. Majorana, two-component spinors. We shall label these two components as ψ_-^μ and ψ_+^μ so that $\Psi = \psi_- \oplus \psi_+$. We can also define a chirality operator

$$\Gamma = \rho^0 \rho^1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{7.5}$$

this is a 2D version of the 4D chirality operator $-i\gamma^5 = \mathrm{diag}(\mathbb{1}_{2\times 2}, -\mathbb{1}_{2\times 2})$. These obey the usual relations,

$$\Gamma^2 = 1 \qquad\qquad \{\Gamma, \rho^\alpha\} = 0. \tag{7.6}$$

The existence of such an operator that commutes with the Hamiltonian tells us that Γ eigenstates do not mix and we can decompose the reducible Ψ representation into irreducible representations according to chirality: $\Psi = \Psi_- \oplus \Psi_+$. Thus the irreps are simply the states Ψ_- and Ψ_+ taken independently, or $(\Psi^\mu_-, 0)^T$ and $(0, \Psi^\mu_+)^T$ if one wants to be picky.

To recap we started with 2ℂD (two complex dimensional) Dirac spinors. The projection operator Γ allows us to construct chiral irreducible representations $\Psi_\pm$ are which are 2ℂD Weyl spinors since a Weyl spinor is one that is projected onto a chirality. Separate from any chiral considerations, we also noted that our 2D space furnishes a representation of the Clifford algebra that is purely real. Thus we learned that our Dirac spinors could in fact be written as $2\mathbb{R}D$ (two real dimensional) Majorana spinors. Combining this with the chiral projections we find that in 2D we have irreducible representations which are 1ℝD **Majorana-Weyl** spinors.

This can be somewhat counterintuitive to those who are used to working with spinors only in four dimensions where there are no Majorana-Weyl representations. One can only simultaneously impose the Majorana and Weyl conditions when the chiral projector is real. It turns out that this is only allowed in spaces with dimension $D = 2 \mod 8$. Now is a good time to mention that that for the superstring, $D = 10$. More details can be found in the appendix of Polchinski's second volume [3].

The Dirac equation in 2D is

$$\rho^\alpha \partial_\alpha \Psi^\mu = 0 \qquad \text{i.e.} \qquad \begin{pmatrix} 0 & -\partial_0 + \partial_1 \\ \partial_0 + \partial_1 & 0 \end{pmatrix} \begin{pmatrix} \Psi^\mu_- \\ \Psi^\mu_+ \end{pmatrix}. \tag{7.7}$$

We can define

$$\partial_\pm = \frac{1}{2}\left(\partial_0 \pm \partial_1\right) \tag{7.8}$$

so that the Dirac equation reduces to the relations

$$\partial_\mp \Psi^\mu_\pm = 0. \tag{7.9}$$

Now we can work with the irreducible, single real dimensional Majorana-Weyl spinors. This means we choose our fermions to be a single chirality and decompose into left- and right-movers. (For this chiral worldsheet theory, choice of Ψ_+ or Ψ_- chirality is irrelevant, so pick one.) We will write the left-movers as ψ^μ and the right-movers as $\widetilde{\psi}^\mu$ so that the fermion Lagrangian takes the form

$$\mathcal{L}_F = \psi^\mu \bar{\partial} \psi_\mu + \widetilde{\psi}^\mu \partial \widetilde{\psi}_\mu. \tag{7.10}$$

Now we cut to the chase. What special property does the action (7.1) have? Supersymmetry. In particular, if we take our supersymmetry transformation parameter to be $\varepsilon = (\varepsilon_-, \varepsilon_+)^T$, a Majorana spinor, then the action obeys a symmetry

$$\delta X^\mu = \bar{\varepsilon} \Psi^\mu \tag{7.11}$$

$$\delta \Psi^\mu = \rho^\alpha \partial_\alpha X^\mu \varepsilon. \tag{7.12}$$

In fact, our action obeys more than just supersymmetry. It obeys an $\mathcal{N} = (1,1)$ **superconformal** symmetry. What we mean by this is that we have two $\mathcal{N} = 1$ sectors in our theory: the independent

Q_α and $\bar{Q}_{\dot{\alpha}}$ Majorana-Weyl generators of SUSY (the indices are just meant to be suggestive and only take one value).

That's silly notation. Indeed, the notation $\mathcal{N} = (1,1)$ might be unfamiliar to those used to four-dimensional SUSY. In particular, we could recall 'vanilla' 4D SUSY and ask why we write $\mathcal{N} = 1$ when this theory also has left and right chiral sectors, i.e. the Q_α and $\bar{Q}_{\dot{\alpha}}$. The key is that in 4D the Weyl representation is complex so that the representation of $\bar{Q}$ is fixed to be the conjugate of the Q representation. In 2D, on the other hand, our Weyl representation is real (*Majorana*-Weyl) and the $\bar{Q}$ representation is completely independent of the Q representation. For more on this, see the relevant chapters of West [15].

7.2 A motivation for light cone gauge

We will not discuss the details of superconformal field theory on the worldsheet, but let's imagine what would happen if we wanted to plow ahead and quantize the theory.

First of all, we would start with the canonical (anti-)commutation relations

$$[\alpha^\mu_m, \alpha^\nu_n] = m\,\delta_{m+n}\eta^{\mu\nu} \tag{7.13}$$

$$\{\Psi^\mu_m, \Psi^\nu_n\} = \delta_{m+n}\eta^{\mu\nu}. \tag{7.14}$$

Both of these relations are bad. We know from our experience in Section 4.3 that these relations can give us negative norm states that invalidate our theory (much worse than a tachyon).

Let's recall how we dealt with this problem in the bosonic string. We introduced the Polyakov action which has an additional degree of freedom $g_{\alpha\beta}$ (previously called $\gamma_{\alpha\beta}$) whose equations of motion gave the Virasoro constraints, $T_{\alpha\beta} = 0$. We saw that these constraints are an avatar of the conformal invariance of the Polyakov action (associated with the leftover diff$\cap$Weyl gauge freedom) and furnished a constraint algebra (the Virasoro algebra) to identify physically allowed states.

Now for the superstring we could follow the analogous procedure to formulate a theory with a *local worldsheet* supersymmetry and derive the resulting constraint algebra. The equations of motion for the Ψ^μ will give us a condition for the worldsheet fermionic energy-momentum tensor $T_F = 0$. This, in turn, will be our signal of **superconformal symmetry**, whose algebra will act as our generalization of the Virasoro algebra acting on states. As before, anomaly cancellation will give us the critical dimension; the result of SCFT calculations is that $D = 10$.

In the bosonic string, conformal symmetry (diff$\cap$Weyl) allowed us to set $X^+(\sigma,\tau) = X^+ + P^-\tau$ and thus remove the $\alpha^\dagger$ oscillators, i.e. we can go to light cone gauge. In this gauge we were able to solve for $X^-(\sigma,\tau)$ in terms of the transverse modes $X^i(\sigma,\tau)$ with

$$[\alpha^i_m, \alpha^j_n] = m\,\delta_{m+n}\delta^{ij}. \tag{7.15}$$

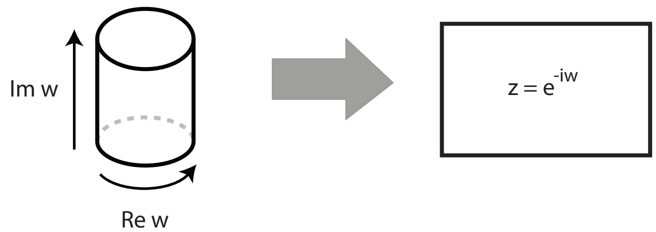
Similarly, superconformal symmetry allows us to set $\psi^+(\sigma,\tau) = 0$ and solve for $\psi^-(\sigma,\tau)$ in terms of the transverse $\psi^i(\sigma,\tau)$ with

$$\{\psi^i_m, \psi^j_n\} = \delta_{m+n}\delta^{ij}. \tag{7.16}$$

Since we already know the punchline ($D = 10$), we'll skip the long story. It's covered thoroughly in the usual references.

7.3 Boundary conditions: Ramond and Neveu-Schwarz

Now let us consider the boundary conditions of the closed superstring. We shall use the complex worldsheet coordinate w that we introduced in (5.9). These were just a straightforward complexification of the τ and σ coordinates including the periodicity in σ. As a reminder, let's draw the picture again:



As a remark, most textbooks (i.e. all except Polchinski) discuss this topic using (τ, σ) coordinates. We will follow Polchinski's notation, but it may be helpful to mentally keep track of w and $\bar{w}$ in terms of the original worldsheet coordinates. Recall our fermionic action,

$$S_\psi = \frac{1}{4\pi\alpha'} \int d^2w \; \psi^\mu \bar{\partial} \psi_\mu + \widetilde{\psi}^\mu \partial \widetilde{\psi}_\mu. \tag{7.17}$$

Note the invariance of S_ψ under $w \to w + 2\pi$. The variation of the fermion action allows more choices for the open ψ and $\widetilde{\psi}$ strings: each mode can *independently* be periodic or anti-periodic,

$$\psi^\mu(w + 2\pi) = \pm \psi^\mu(w) \tag{7.18}$$

$$\widetilde{\psi}^\mu(w + 2\pi) = \pm \widetilde{\psi}^\mu(\bar{w}). \tag{7.19}$$

These fermion boundary conditions have names: **Ramond** (R) for the periodic condition and **Neveu-Schwarz** (NS) for the anti-periodic condition. Thus for the open string there are four distinct sectors corresponding to the choice of boundary conditions on each end of the string: R-R, R-NS, NS-R, and NS-NS.

Two boundary conditions. Why did we not also consider anti-periodic boundary conditions for our bosonic closed strings? Certainly the bosonic action is also invariant under $X(\sigma+2\pi) \equiv -X(\sigma)$ in the exact same way that the fermionic action is; they are both quadratic in the field. This is actually a rather subtle topic and is related to the Lorentz symmetry of the target spacetime. Bosonic fields with antiperiodic boundary conditions, i.e. 'twisted sector' fields, break spacetime translation invariance. This is straightforward to see because $X^\mu \to -X^\mu$ does not commute with $X^\mu \to X^\mu + A^\mu$. Fermionic directions (target superspace coordinates) are inherently quantum and so we never have to worry about breaking symmetry in the 'classical' limit. This should become more clear with we discuss D-branes shortly.

A handy notation in the w coordinates is to write

$$\psi^\mu(w + 2\pi) = e^{2\pi i \nu} \psi^\mu(w) \qquad \widetilde{\psi}^\mu(\bar{w} + 2\pi) = e^{2\pi i \widetilde{\nu}} \widetilde{\psi}^\mu(\bar{w}) \tag{7.20}$$

where $\nu = 0, \frac{1}{2}$ according to Ramond or Neveu-Schwarz boundary conditions respectively. We can expand in Fourier modes,

$$\psi(w) = i^{-\frac{1}{2}} \sum_{r\in\mathbb{Z}+\nu} \psi_r e^{irw} \qquad \widetilde{\psi}(\bar{w}) = i^{-\frac{1}{2}} \sum_{r\in\mathbb{Z}+\widetilde{\nu}} \widetilde{\psi}_r e^{ir\bar{w}}. \tag{7.21}$$

We now go back to the complex z worldsheet coordinates where these Fourier expansions map to

$$\psi(z) = \sum_{r\in\mathbb{Z}+\nu} \frac{\psi_r}{z^{r+1/2}} \qquad \widetilde{\psi}(\bar{z}) = \sum_{r\in\mathbb{Z}+\nu} \frac{\psi_r}{\bar{z}^{r+1/2}}. \tag{7.22}$$

We remark that the factor of 1/2 in the power of z introduces a branch cut associated with the half-integer moding of the R and NS sectors.

7.4 Open strings and the doubling trick

When we first introduced the closed bosonic string we remarked that it is essentially 'two copies' of the open string. This is a general feature that one should keep in mind when relating open and closed strings. Without going into details, one may describe the open superstring using a 'doubling trick' to relate it to the closed string.

The boundary terms in the variation of the action for the open string in light cone gauge are

$$\delta S_\psi = \frac{1}{2\pi} \int d\tau \, \left[\psi_+^i \delta\psi_+^i - \psi_-^i \delta\psi_-^i\right]_{\sigma=0.}^{\sigma=\pi} \tag{7.23}$$

In order to have a non-trivial solution (imposing, e.g. $\psi_\pm(\tau, 0) = 0$ forces the entire field to vanish by the solution of the wave equation) one has to relate the ψ_+ and ψ_- chiralities so that

$$\psi_-^i(\tau, \sigma = 0, \pi) = \pm\psi_+^i(\tau, \sigma = 0, \pi), \tag{7.24}$$

Because we can choose the overall signs of ψ_+ and ψ_- arbitrarily, we may chose—without loss of generality—to set

$$\psi_-(\tau, 0) = \psi_+(\tau, 0). \tag{7.25}$$

On the other end of the string, we have a choice to make:

$$\psi_-(\tau, \pi) = \pm\psi_+(\tau, \pi). \tag{7.26}$$

where the top sign corresponds to the Ramond sector while the bottom sign corresponds to the Neveu-Schwarz sector. Now for the trick: we can combine these boundary conditions into a *closed* string over the interval $\sigma \in [-\pi, \pi]$ (one can then shift to the usual interval $\sigma \in [0, \pi]$):

$$\psi^i(\tau, \sigma) \begin{cases} \psi_-^i(\tau, \sigma) & \sigma \in [0, \pi] \\ \psi_+^i(\tau, -\sigma) & \sigma \in [-\pi, 0] \end{cases} . \tag{7.27}$$

This closed string has Ramond or Neveu-Schwarz boundary conditions according to (7.26). The payoff is that we can work with an open string of either chirality by looking at a closed string formed by gluing together the chiral open strings.

7.5 The superstring spectrum

Now we've established boundary conditions, mode expansions, and commutation relations. Let's quickly go over the low energy spectrum. We can take the Neveu-Schwarz vacuum $|0\rangle_{\rm NS}$ to obey

$$\psi_r^\mu|0\rangle_{\rm NS} = 0 \qquad \text{for } r > 0 \quad \left(r = \frac{1}{2}, \frac{3}{2}, \cdots\right). \tag{7.28}$$

As a sanity check recall that NS boundary conditions correspond to r taking half-integer values. We can build states in the usual way,

$$\psi^\mu_{-1/2}|0\rangle_{\rm NS}. \tag{7.29}$$

Because this is a *fermionic* raising operator, we can only add one at a time. Do not be confused, however: this is a *worldsheet* fermion. As far as the target spacetime is concerned, this is a vector boson. This is indeed a bit weird. More generally, the first excited state in the NS sector is given by $\psi^\mu_{-r}|0\rangle_{\rm NS}$ for $r = 1/2, 3/2, \cdots$. Following the same arguments as the bosonic string we find a general mass formula

$$\alpha' M^2 = \sum_{n=1}^{\infty} \alpha^i_{-n}\alpha_{ni} + \sum_{r=1/2}^{\infty} r\,\psi^i_{-r}\psi_{ri} - \frac{1}{2}. \tag{7.30}$$

The first term is the usual bosonic contribution, the second term is its fermionic analog, and the final factor of $-1/2$ is a zero point energy contribution that comes from normal ordering. We can find it from the usual slick argument: we count the degrees of freedom and we argue that they are only consistent with a massless state (given spacetime dimensionality $D = 10$, which came from anomaly cancellation). In fact, we will find that the spacetime graviton will come from the NS sector worldsheet fermion. (This is indeed surprising since in the bosonic theory the graviton came from the X^μ fields.)

Let's move on to the Ramond sector, which turns out to be much more interesting. We start by looking at the low-lying anticommutation relation

$$\{\psi_0^\mu, \psi_0^\nu\} = \eta^{\mu\nu}. \tag{7.31}$$

Hmm! Do you see it? Let's define a suggestively-named rescaled operator, $\Gamma^\mu = \sqrt{2}\psi_0^\mu$. Then our above anticommutation relation reads

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}. \tag{7.32}$$

This is just the D-dimensional Clifford algebra! (Where we already argued that superconformal anomalies require $D = 10$.) We can impose the usual condition that the Ramond ground state, $|0\rangle_{\rm R}$, is annihilated by lowering operators,

$$\psi_r^\mu|0\rangle_R = 0 \qquad \text{for } r > 0 \quad (r \in \mathbb{Z}), \tag{7.33}$$

but what about the ψ_0^μ operator? One might also want to impose $\psi_0^\mu|0\rangle_R = 0$, but it turns out that this would be inconsistent. From the anti-commutation relations, we can read off

$$\psi_0^\nu\,\psi_0^\mu|0\rangle_R + \psi_0^\mu\,\psi_0^\nu|0\rangle_R = \eta^{\mu\nu}|0\rangle_R \neq 0. \tag{7.34}$$

Thus we learn that the Ramond ground state is degenerate and furnishes a representation of the 10D Clifford algebra. By the spin-statistics theorem in spacetime, we learn that $|0\rangle_R$ is a spacetime fermion. Fermions in the worldsheet have now given us fermions in spacetime. These fermions will turn out to be massless, but one would have to compute this 'honestly' since it's hard to apply our 'slick' argument based on degrees of freedom on spacetime spinors.

Let's go to light cone gauge where we have some familiarity with identifying physical states. In this gauge, the relevant anti-commutation relation is

$$\{\sqrt{2}\psi_0^i, \sqrt{2}\psi_0^j\} = 2\delta^{ij}. \tag{7.35}$$

This is the eight dimensional Euclidean Clifford algebra, Spin(8), which is represented by eight 16×16 Γ matrices, $\Gamma^1, \cdots, \Gamma^8$. We can again define a chirality operator, $\Gamma^9 \equiv \Gamma^1$ with the usual properties

$$\left(\Gamma^9\right)^2 = 1 \qquad\qquad \left\{\Gamma^9, \Gamma^i\right\} = 0. \tag{7.36}$$

To repeat the usual *spiel*, the existence of the chirality operator tells us that our representation can be decomposed into $\pm$ eigenvalues of Γ^9. In other words, we are decomposing $\mathbf{16} \to \mathbf{8}_+ \oplus \mathbf{8}_-$. (The $\mathbf{16}$ refers to SO(9,1), which goes to SO(8) in light cone gauge.) To match Polchinski's notation, we'll write this as $\mathbf{8}\oplus\mathbf{8}'$. So SO(8) has three eight-dimensional representations: two spinors ($\mathbf{8}, \mathbf{8}'$) and a vector ($\mathbf{8}_V$) [the obvious rep]; these are related by the so-called 'triality' property of the SO(8) Dykin diagram. The reader is directed to Appendix B of Polchinski [3] for a comprehensive treatment of spinors in various dimensions.

Dirac in 10D. See Liam old notes page 21. Or Simon Ross notes 12/3/08. [**Do this:** Fill this in.]

We now know that the Ramond vacuum has to be a spinor representation of SO(8). Which one? We are free to choose either the $\mathbf{8}$ ($|0\rangle_R^+$) or the $\mathbf{8}'$ ($|0\rangle_R^-$) since these are related by parity.

Analogous to (7.30), our general mass formula for the Ramond sector is

$$\alpha' M^2 = \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_{ni} + \sum_{r=1}^{\infty} r\, \psi_{-r}^i \psi_r + 0, \tag{7.37}$$

where we have explicitly written that the zero point energy vanishes. The easiest proof for this is to dive in and use the properties of the superconformal field theory. We'll content ourselves with more indirect proofs down the road. Now we claim that the ground states $|0\rangle_R^\pm$ are massless. Unlike the case with bosonic target spacetime representations, we are now dealing with target spacetime fermions and this is no longer a straightforward exercise in counting degrees of freedom. [**Question:** why?]

Let's pause to summarize the upshot of everything we've done so far. Our low-lying spectrum now looks like

$$\begin{array}{rcl} |0\rangle_{\text{NS}} & \alpha' M^2 = -\frac{1}{2} & \text{tachyon} \\ \psi^i_{-\frac{1}{2}}|0\rangle_{\text{NS}} & \alpha' M^2 = 0 & \mathbf{8}_V \\ |0\rangle_{\text{R}}^+ & \alpha' M^2 = 0 & \mathbf{8} \\ |0\rangle_{\text{R}}^- & \alpha' M^2 = 0 & \mathbf{8}' \end{array}$$

We've only writte the left-movers (holomorphic functions of z); the transition to right-movers is as simple as adding tildes and bars. A full state written as a tensor product of a left-moving and a right-moving state, e.g. $|0\rangle_R^+ \otimes |0\rangle_R^-$. As we can plainly see, our theory still has a problem: the tachyon. In fact, this problem is now slightly worse: not only do we have a negative-norm state, but the tachyon breaks spacetime supersymmetry.

On the topic of supersymmetry, we remark that the naïve unprojected spectrum contains a massless spin 3/2 field, a gravitino. Such a theory is *only* consistent if it is supersymmetric.

7.6 A proposed solution: GSO projection

We've dealt with unwanted states a couple times already. A tool which has come in handy is the idea of some kind of operator to project out unwanted states. (Recall, for example, the Virasoro algebra and chirality operators for spinors). We propose, with this admittedly scant motivation, a solution: perhaps there is an operator $\mathcal{O}$ that squares to one, $\mathcal{O}^2 = 1$ and that can be used to project out the tachyon in the same way that we use a chirality operator to project a Dirac spinor representation to a Weyl representation. Since our tachyon is just the NS vacuum, we would require $\mathcal{O}|0\rangle_{\text{NS}} = -|0\rangle_{\text{NS}}$.

To be clear, at this point in our discussion there's no *a priori* reason to believe that such an operator should exist. (This is partially because of the fast track we took through the material.) Don't worry about this for now. We'll retroactively motivate this later on.

Gliozzi, Sherk, and Olive (GSO) proposed one such operator, $(-)^F$, by

$$(-)^F = \begin{cases} (-1)^{\sum_{r=1/2}^{\infty} \psi^i_{-r}\psi_{ri}+1} & \text{(NS)} \\ \Gamma^9(-1)^{\sum_{r=1}^{\infty} \psi^i_{-r}\psi_{ri}} & \text{(R)} \end{cases} . \tag{7.38}$$

F is an operator that counts the worldsheet fermion number. The Γ^9 will turn out to be be important for projecting out extra fermions to get a supersymmetric spectrum of spacetime fields. With this operator we find the following parities, (recalling once again that we are only explicitly writing one sector, e.g. left-movers, of the theory)

$$(-)^F|0\rangle_{\text{NS}} = -|0\rangle_{\text{NS}} \tag{7.39}$$

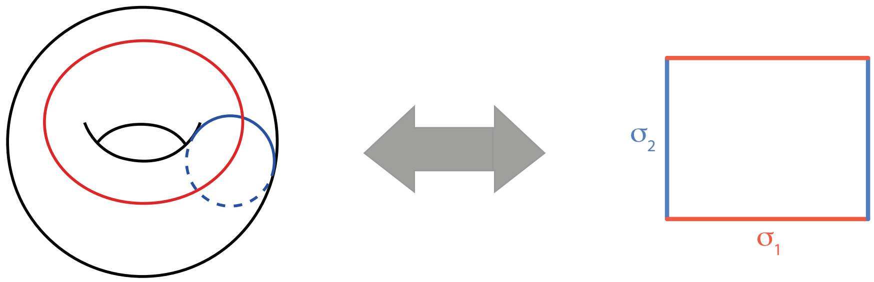
$$(-)^F\psi^i_{-1/2}|0\rangle_{\text{NS}} = +\psi^i_{-1/2}|0\rangle_{\text{NS}} \tag{7.40}$$

$$(-)^F|0\rangle^{\pm}_{\text{R}} = \pm|0\rangle^{\pm}_{\text{R}}. \tag{7.41}$$

This looks pretty good; it gives us a way to segregate our tachyon and project it out, taking one of the Ramond vacuum states with it. In fact, this seems too good to be true, a little *deus ex machina*. Where exactly did this operator come from? It turns out that we can properly motivate the GSO projection by looking at the quantum behavior of our theory.

7.7 Modular invariance and boundary conditions

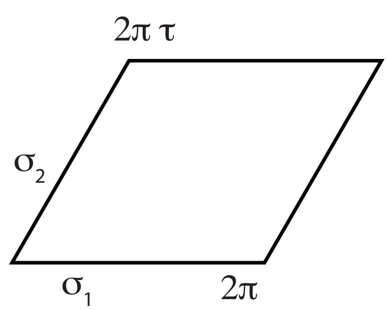
Fujikawa taught us that the quantum consistency of our theory (i.e. that gauge symmetries be anomaly-free) requires an invariance of our path integral measure under diffeomorphisms. For infinitesimal transformations this is clear, but there is one place where this is tricky. Consider the computation of vacuum amplitudes on T^2, the torus, the analog to one-loop bubble diagrams in quantum field theory.



The torus is periodic in two directions, for example, $0 \leq \sigma_1, \sigma_2 \leq 2\pi$. Alternately, the torus can be defined with a complex structure τ and a flat metric via $w = \sigma_1 + \tau\sigma_2$ with the identifications

$$w \sim w + 2\pi n \qquad\qquad w \sim w + 2\pi m\tau \tag{7.42}$$

and the metric $ds^2 = dw\, d\bar{w}$. [**Check:** understand this better? See D-branes, p. 88.] The complex number τ determines the shape of the metric: i.e. it determines whether the donut is fat or skinny. The cartoon picture for this is



Now we can consider a special set of transformations

$$\begin{pmatrix} \sigma_1' \\ \sigma_2' \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} \qquad \left.\begin{aligned} ad - bc &= 1 \\ a, b, c, d &\in \mathbb{Z} \end{aligned}\right\} \mathrm{PSL}(2\mathbb{Z}). \tag{7.43}$$

These transformations are those of the group SL(2,$\mathbb{Z}$); the P in PSL(2,$\mathbb{Z}$) refers to reflections $a, b, c, d \to -a, -b, -c, -d$. SL(2,$\mathbb{Z}$) is generated by the **modular transformations**:

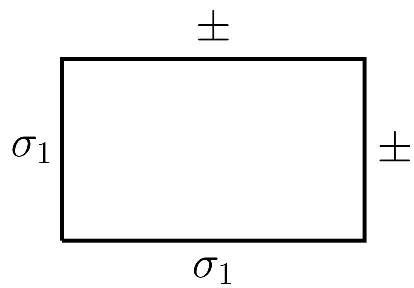
$$\begin{aligned} a &= d = 0 \\ b &= 1 \\ c &= -1 \end{aligned} \qquad\qquad \begin{aligned} a &= b = d = 1 \\ c &= 0 \end{aligned} \tag{7.44}$$

This group is turns out to be rather special. Under the transformations (7.43), the metric has the same form but with a modified modular parameter

$$\tau \to \tau' = \frac{a\tau + b}{c\tau + d}. \tag{7.45}$$

Thus under the combined transformations (7.43) and (7.45) the torus is mapped to an *equivalent* torus under the identifications (7.42).

Now let us discuss the boundary conditions in σ_1 and σ_2. We now work in Euclidean time (σ_2). Recall (for example, from statistical mechanics) that the path integral for fermions have anti-periodic boundary conditions in time, i.e. under $\sigma_2 \to \sigma_2 + 2\pi$ in the torus. (Bosons have the usual periodic identification). The derivation of this is a long exercise in quantum mechanics, so we'll leave it to the reader to review elsewhere. For σ_1, we already know that there is a choice of boundary conditions: + for Ramond superstrings and - for Neveu-Schwarz superstrings. We will write our boundary conditions as $(\sigma_1, \sigma_2) = (+, -)$ for Ramond strings and $(-, -)$ for Neveu-Schwarz strings.



Now for the critical question: how do modular transformations affect our choice of boundary conditions? Under our first modular transformation (associated with $\tau \to -1/\tau$),

$$\begin{pmatrix} \sigma_1' \\ \sigma_2' \end{pmatrix} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} -\sigma_2 \\ \sigma_1 \end{pmatrix}. \tag{7.46}$$

The minus sign doesn't matter, but the thing to notice is that we have swapped $\sigma_1 \leftrightarrow \sigma_2$, i.e. we've swapped which circle is which. For boundary conditions (α, β), this modular transformation produces boundary conditions (β, α). For example, Ramond boundary conditions (+,$-$) get mapped to ($-$,+). Now we pause awkwardly: what the heck is ($-$,+)? This is a boundary condition that is anti-periodic in σ_1 but periodic in σ_2 (i.e. bosonic). This is manifestly a new kind of boundary condition!

This is the lesson that we wanted to get to. We started by emphasizing that diffeomorphism invariance should be anomaly free. We then considered a special subset of diffeormorphisms, the modular transformations. Starting with our theory of Ramond and Neveu-Schwarz superstrings, invariance under the first modular transformation now tells us that in order to be consistent we must consider other boundary conditions. A similar thing happens for the second modular transformation.

Under the transformation associated with $\tau \to \tau + 1$, we have

$$\begin{pmatrix} \sigma_1' \\ \sigma_2' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}. \tag{7.47}$$

One can check that this sends the boundary condition (α, β) to $(\alpha\beta, \beta)$. In particular, for the Neveu-Schwarz boundary condition, we map $(-,-) \to (+,-)$. This is just the Ramond boundary condition, so that modular invariance tells us that we *must* have a Ramond state. Thus we see that modular invariance relates the

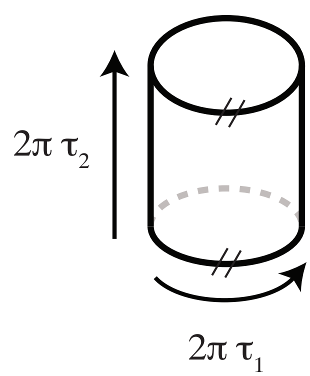
$$(-,-) \qquad (+,-) \qquad (-,+) \tag{7.48}$$

boundary conditions, but not (+,+). If the reader will now permit a statement without proof, the (+,+) condition joins its friends at the two-loop level so that one should consider the entire set of four boundary conditions. While this is already quite a development that we should try to grok, we haven't yet understood how to interpret the + boundary condition in the Euclidean time direction.

7.8 The twisted partition function

Recall that the unit operator corresponds to the partition function itself,

$$\langle 1 \rangle_{T^2(\tau)} = Z(\tau). \tag{7.49}$$



We can recall from quantum mechanics that the partition function can be written as

$$Z = \mathrm{Tr}\ e^{-\beta H} = \sum_\alpha \langle \alpha | e^{-\beta H} | \alpha \rangle. \tag{7.50}$$

We can read $e^{-\beta H}|\alpha\rangle$ as "$|\alpha\rangle$ at $t_E = 2\pi\tau_2$," so that

$$e^{-\beta H}|\phi, t_E = 0\rangle = |\phi, t_E = 2\pi\tau_2\rangle. \tag{7.51}$$

In other words, $e^{-\beta H}$ time translates by $2\pi\tau_2$, using $\beta = 2\pi\tau_2$. In order to interpret the $(\pm, +)$ boundary conditions, we want a sign flip upon a similar time translation. How do we introduce such a sign flip? We can appeal to our new friend, $(-)^F$. Let us compute a 'twisted partition function,'

$$Z_F = \text{Tr}\ (-)^F e^{-\beta H} = \sum_\alpha \langle\alpha|(-)^F e^{-\beta H}|\alpha\rangle. \tag{7.52}$$

This gives us precisely the 'time translation plus sign flip' that we wanted, at least for states with an odd fermion number. In other words, Tr $(-)^F e^{-\beta H}$ is the partition function 'twisted' by $(-)^F$. This object is properly an index: it gives us

$$\text{Tr}\ (-)^F e^{-\beta H}|\phi, t_E = 0\rangle = [\dim(+) - \dim(-)]\, e^{\beta H}|\text{eigenspace}\rangle, \tag{7.53}$$

which gives the number of + chirality zero modes minus the number of − chirality zero modes (since the excited states are always paired).

Let's recap the story to this point. We started with boundary conditions that we understood: Ramond (+,−) Neveu-Schwarz (−,−) corresponding to periodic and anti-periodic boundary conditions in the σ direction. Both of these had fermionic (anti-periodic) boundary conditions in Euclidean time, τ_E. We then looked at modular transformations, a particular set of diffeomorphisms that take the torus to itself. Quantum consistency then requires that these transformations don't change our theory. However, one particular aspect of our theory – the boundary conditions – are very clearly changed under modular transformations: it takes our Ramond and Neveu-Schwarz boundary conditions to weird periodic-in-time boundary conditions that we didn't understand: (−,+) and, at two-loop, (+,+). We decided that this is the theory telling us that we should have included these odd boundary conditions to our theory. We wanted to do what the theory is telling us, but we were thoroughly confused about how to actually do this. Now, with the introduction of a twisted partition function, we have found a way to implement the $(\pm, +)$ boundary conditions.

The remaining step is to combine this twisted partition function with the untwisted partition function so that we have a theory that includes all of our boundary conditions. The correct way to do this is to do the 'obvious' thing and to sum the two partition functions with no further weighting. This is because our modular transformations maps single states to [other] single states. Thus the map $(-,-) \to (-,+)$ also maps $\text{Tr}_{\text{NS}} e^{-\beta H} \to \text{Tr}_{\text{NS}} e^{-\beta H}(-)^F$. We are then led to consider the summed partition function,

$$\frac{1}{2}\text{Tr}\left(e^{-\beta H} + (-)^F e^{-\beta H}\right) = \text{Tr}\left(\frac{1}{2}\left[1 + (-)^F\right] e^{-\beta H}\right). \tag{7.54}$$

The object $\frac{1}{2}\left(1 + (-)^F\right)$ is just a projection onto the + eigenspace of the $(-)^F$ operator. *Voilà!* This is precisely the GSO projection acting on the partition function. We now come back to the question raised at the end of Section 7.6: how do we properly motivate the GSO projection? What we've found here is that we don't have to: the GSO projection is *required* by the quantum consistency of our theory as one can see by just looking at the boundary conditions and the symmetries of a special set of vacuum donut diagrams. We have a choice of how to choose the overall charge of $(-)^F$ which translates to choosing between projecting out the NS tachyon or the first and third excited states; it is obvious which choice we make.

If we start with the Neveu-Schwarz sector, we saw that modular invariance implies the existence of both twisted $(-,-)$ and untwisted $(-,+)$ NS sectors as well as the untwisted R sector $(+,-)$. We learned that this means we have to include the GSO projection $[1+(-)^F]_{\text{NS}}$ on the NS sector.

We could have also started with the Ramond sector. We know from above that at one-loop level modular invriance tells us that the $(+,-)$ state hangs out with the NS sector. The twisted R sector $(+,+)$, however, has its own distinct orbit under SL(2, $\mathbb{Z}$) and is not related to anything. One might then wonder if we could just start with the twisted R sector and never worry about NS states. However, one would find techncial evidence that this is not correct since scattering amplitudes would have NS states as poles and the operator product expansion would not close. More heuristically, two-loop modular invariance turns out to mix the $(+,+)$ boundary condition to the other three. One can check (see Polchinski) that this allows the OPE to close.

Thus we can never have a theory of only Ramond states and even in the Ramond sector one must also compute the GSO projection $[1+(-)]^F_{\text{R}}$. The correct perscription, then, is to include *all* boundary conditions and project with the GSO operator in each sector.

7.9 Super-spectrum

Let us write NS$\pm$ for the $\pm$ eigenstates of the $(-)^F$ operator in the Neveu-Schwarz sector. Similarly, let us write R$\pm$ for the corresponding eigenstates in the Ramond sector. Let us be clear that we have a choice in which sector we project onto. Or, in other words, we can say that we project onto the $+$ eigenspace but we have a choice in how to define the overall sign of $(-)^F$. We will choose the following definitions:

$$(-)^F = \begin{cases} +(-1)^{\sum_{r=1/2}^{\infty}\psi^i_{-r}\psi_{ri}+1} & \text{(NS)} \\ +\Gamma^9(-1)^{\sum_{r=1}^{\infty}\psi^i_{-r}\psi_{ri}} & \text{(R)} \end{cases}, \tag{7.55}$$

where we have explicitly written out the leading $+$ as our choice of sign. This corresponds a definition of F. More explicitly, we may write a general NS state as

$$\left[\prod_{i=2}^{D-1}\prod_{n=1}^{\infty}(\alpha^i_{-n})^{N_{i,n}}\right]\prod_{j=1}^{m}\psi^{i_j}_{-r_j}|0\rangle_{\text{NS}}. \tag{7.56}$$

The NS+ states correspond to m odd while NS$-$ states correspond to m even. Writing the exact same equation for the general R state,

$$\left[\prod_{i=2}^{D-1}\prod_{n=1}^{\infty}(\alpha^i_{-n})^{N_{i,n}}\right]\prod_{j=1}^{m'}\psi^{i_j}_{-r_j}|0\rangle^{\pm}_{\text{R}}. \tag{7.57}$$

The R+ states correspond to m' even if we choose the $|0\rangle^+_{\text{R}}$ vacuum or otherwise m' odd if we choose $|0\rangle^-_{\text{R}}$. Similarly, the R$-$ states correspond to m' odd if we choose the $|0\rangle^+_{\text{R}}$ vacuum or otherwise m' even if we choose $|0\rangle_{\rm R}^-$. As a reminder, the lowest states in each sector are:

$$\begin{aligned}
\text{NS+}: &\qquad \psi^i_{-1/2}|0\rangle_{\rm NS} &(7.58)\\
\text{NS}-: &\qquad |0\rangle_{\rm NS} &(7.59)\\
\text{R}\pm: &\qquad |0\rangle_{\rm R}^{\pm}. &(7.60)
\end{aligned}$$

Thus far this discussion has only focused on the left-movers. Do not forget that we also have a right-moving sector which is related to the left-movers by level matching. We would like to contruct the space of states built upon $|0\rangle_L \otimes |0\rangle_R$. Because modular transformations act in the same way on the $\psi(z)$ and $\widetilde{\psi}(\bar{z})$ boundary conditions, the story is exactly the same for the right-movers.

We now have to choose our projection. Indeed, we have a few choices to make: for the left- and the right-moving sectors, we *independently* must choose which $(-)^F$ eigenstate to projet out of *both* the NS and the R sectors. We've chosen our signs such that $(-)^F|0\rangle_{\rm NS} = -|0\rangle_{\rm NS}$, i.e. tachyon is projected out – both for the left- and right-movers. This corresponds to choosing the NS+ sector for both the left- and right-movers. Now we still have some choices:

	Left	**Right**
	NS+	NS+
IIB	R+	R+
IIB'	R−	R−
IIA	R+	R−
IIA'	R−	R+

To the left we've written some characters which should look familiar to those who peek at `hep-th`. These are names for different kinds of superstring theories. The primed and unprimed theories are physically equivalent because they are related by chirality and if we say our theory has some fixed chirality, we cannot distinguish between + chirality and − chirality without additional structure. Thus, for example, in type IIB theories we can have vacuum states like $|0\rangle_R^+ \otimes |0\rangle_R^+$ while IIB' theories can have $|0\rangle_R^- \otimes |0\rangle_R^-$ vacuum states. These are equivalent up to chirality.

Other superstring theories. IIA and IIB theories are not the only superstring theories. Let's mention some of the others briefly. Heterotic theories are another type of superstring theory that will be of interest to us later in these notes. Alternately, we could consider type 0A and 0B theories. We shall define these using another notation. Let us write our thories in terms of the possible pairs of left and right boundary conditions (L,R). In this notation

$$\text{IIA} = (\text{NS+},\text{NS+}),\ (\text{NS+},\text{R}-),\ (\text{R+},\text{NS+}),\ (\text{R+},\text{R}-) \qquad (7.61)$$

$$\text{IIB} = (\text{NS+},\text{NS+}),\ (\text{NS+},\text{R+}),\ (\text{R+},\text{NS+}),\ (\text{R+},\text{R+}). \qquad (7.62)$$

One should note that IIA and IIB are related by $(..., R-) \to (..., R+)$. These boundary conditions were chosen to eliminate the tachyon. Alternately we could have chosen different boundary conditions, the 0A and 0B theories,

$$0\mathrm{A} = (\mathrm{NS}+,\mathrm{NS}+),\ (\mathrm{NS}-,\mathrm{NS}-),\ (\mathrm{R}+,\mathrm{R}-),\ (\mathrm{R}-,\mathrm{R}+) \tag{7.63}$$
$$0\mathrm{B} = (\mathrm{NS}+,\mathrm{NS}+),\ (\mathrm{NS}-,\mathrm{NS}-),\ (\mathrm{R}+,\mathrm{R}+),\ (\mathrm{R}-,\mathrm{R}-). \tag{7.64}$$

These theories are modular invariant but they do not project out the tachyon and do not have target spacetime supersymmetry. For this reason we will ignore them for the remainder of these lectures. For now we shall choose to ignore the gaping hole in our list, the type I theories (which we haven't defined), since they will show up when we discuss open strings and D-branes.

Let's now flesh out the low-energy states of our type II theories. We can write this out in terms of the choices we can make for the (left-mover)⊗(right-mover) ground states. For IIA theories, we have

$$\begin{pmatrix} \psi^i_{-1/2}|0\rangle_{\mathrm{NS}} \\ \text{or} \\ |0\rangle^+_{\mathrm{R}} \end{pmatrix} \otimes \begin{pmatrix} \widetilde{\psi}^j_{-1/2}|0\rangle_{\mathrm{NS}} \\ \text{or} \\ |0\rangle^-_{\mathrm{R}} \end{pmatrix}. \tag{7.65}$$

Similarly, for IIB theories,

$$\begin{pmatrix} \psi^i_{-1/2}|0\rangle_{\mathrm{NS}} \\ \text{or} \\ |0\rangle^+_{\mathrm{R}} \end{pmatrix} \otimes \begin{pmatrix} \widetilde{\psi}^j_{-1/2}|0\rangle_{\mathrm{NS}} \\ \text{or} \\ |0\rangle^+_{\mathrm{R}} \end{pmatrix}. \tag{7.66}$$

We can work out SO(8) representations of our fields, recalling that SO(9,1)→SO(8) when we look at physical states, e.g. if we go to light cone gauge. As mentioned in Section 7.5, our states arrange themselves into tensors of three types of eight-dimensional representations of SO(8): the vector $\mathbf{8}_V$, and two spinors $\mathbf{8}$ and $\mathbf{8}$' according to

$\mathbf{8}_V$	$\psi^i_{1/2}\vert 0\rangle_{\mathrm{NS}}$
$\mathbf{8}$	$\vert 0\rangle_{\mathrm{R}^+}$
$\mathbf{8}'$	$\vert 0\rangle_{\mathrm{R}^-}$

Now we just have to compute the relevant tensor products of these representations in terms of irreducible representations. This is the usual "representation theory for physicists" game that we play.

$\mathbf{8}_V \otimes \mathbf{8}_V$	NS-NS	IIA,B
$\mathbf{8} \otimes \mathbf{8}$	R-R	IIB
$\mathbf{8} \otimes \mathbf{8}'$	R-R	IIA
$\mathbf{8}_V \otimes \mathbf{8}$	NS-R	IIA
	R-NS	IIB
$\mathbf{8}_V \otimes \mathbf{8}'$	NS-R	IIA

This corresponds to doing the usual tensor product decompositions for group representations. The $\mathbf{8}_V \otimes \mathbf{8}_V$ representation is particularly straightforward since this is the product of vector representations. We can write this two-tensor as c^{ij} and decompose it into a symmetric and antisymmetric piece: $c^{ij} \to s^{ij} + a^{ij}$. The symmetric piece can, in turn, be decomposed into a scalar trace and a traceless symmetric tensor,

$$c^{ij} \to \left(s^{ij} - \frac{1}{8}s^i{}_i\right) + a^{ij} + \frac{1}{8}s^i{}_i. \tag{7.67}$$

This is now written in terms of irreducible representations. Counting dimensions, we know that c contains $8 \times 8 = 64$ degrees of freedom, while the symmetric (antisymmetric) tensors contain $8(8 \pm 1)/2$ degrees of freedom. Separating the trace, we get

$$\mathbf{8}_V \otimes \mathbf{8}_V = 35 \oplus 28 \oplus 1. \tag{7.68}$$

Another handy way of writing this is $\mathbf{8}_V \otimes \mathbf{8}_V = (2) + [2] + [0]$, where we've written the parenthesis for symmetric indices and square brackets for antisymmetric indices. Now one should stop and think: *ah, a symmetric traceless tensor, an antisymmetric tensor, and a scalar. Where have I heard that before?* That's right: these are just our old friends: the graviton g_{ij}, the antisymmetric B_{ij}, and the dilaton Φ. This is just like the bosonic string, except now these states come from $D = 10$ fermion operators acting on the NS left- and right-moving ground states.

Neat. That was just the appetizer. Let's get to the real work and consider the states with spinor indices. Given our previous revelation, we can already expect that one of these ought to correspond to the gravitino. Let's start with the $\mathbf{8}_V \otimes \mathbf{8}$. We write a state as $|\zeta\rangle_i$ where the i indexes the vector $\mathbf{8}_V$ index while the α indexes the spinor $\mathbf{8}$. We can construct an irreducible representation by constructing

$$|\zeta\rangle_i{}^\alpha \Gamma^i_{\alpha\beta'}, \tag{7.69}$$

where we recall that the Γ converts indices of one chirality $(\mathbf{8}, \alpha)$ to the other $(\mathbf{8}', \beta')$. Thus the state above is in the $\mathbf{8}'$ representation. We now state without proof that this is as far as we can go, so that our decomposition is

$$\mathbf{8}_V \otimes \mathbf{8} : \quad \mathbf{64} \to \mathbf{8}' \oplus \mathbf{56}. \tag{7.70}$$

The $\mathbf{56}$ is an irreducible vector-spinor irreducible representation which we write as $\Psi_i{}^\alpha$. By parity the $\mathbf{8}_V \otimes \mathbf{8}'$ has the same structure,

$$\mathbf{8}_V \otimes \mathbf{8}' : \quad \mathbf{64} \to \mathbf{8} \oplus \mathbf{56}', \tag{7.71}$$

where write the $\mathbf{56}'$ as $\widetilde{\Psi}_i{}^{\beta'}$.

The next object to calculate is $\mathbf{8} \otimes \mathbf{8}$. Fortunately, we already know how to tensor together spinors in quantum field theory. For example, we know from 4D QFT that we can write a fermion bilinear as a linear combination of Γ matrices sandwiched by fermion bilinears; these relations are called Fierz identities. Thus for spinors ζ and χ, we can write something like

$$\zeta \otimes \chi \sim \sum \bar{\zeta}\Gamma^{[\mu_1 \cdots}\Gamma^{\mu_m]}\chi, \tag{7.72}$$

where the sum is over suitable values of m so that each term is an m-form. These are irreducible fields, so what we find is

$$\mathbf{8} \otimes \mathbf{8} = [0] + [2] + [4]_+ = \mathbf{1} + \mathbf{28} + \mathbf{35}_+, \tag{7.73}$$

where we make use of the handy notation for antisymmetrized indices that we introduced above. For example, the [2] refers to an antisymmetric two-index tensor, $a^{[ij]}$, while the $[4]_+$ refers to an antisymmetric four-index tensor, $c^{[ijk\ell]}$. The subscript + refers to a self dual representation, which turns out to be the correct irreducible representation. Explicitly, the $\mathbf{35}_+$ is a field C_4 such that $*dC_4 = dC_4$. This should sound plausible since d(4-form) is a 5-form and in $D = 10$ a 5-form can be self-dual. In the same way, we can write

$$\mathbf{8}' \otimes \mathbf{8}' = [0] + [2] + [4]_- = \mathbf{1} + \mathbf{28} + \mathbf{35}_-, \tag{7.74}$$

where now the $-$ subscript refers to the anti-self dual. Finally, the tensor product of the different-chirality representations gives

$$\mathbf{8} \otimes \mathbf{8}' = [1] + [3] = \mathbf{8}_V + \mathbf{56}_T, \tag{7.75}$$

where the subscript T is a reminder that this is a tensor representation, not one of the vector-spinors above.

Let's review what we've done. We started with the $\mathbf{8}_V, \mathbf{8}$ and $\mathbf{8}'$ reps. We ended up decomposing the relevant tensor products into the irreducible representations

$$\begin{array}{cccc} \mathbf{1}, & \mathbf{8}_V, & \mathbf{35}, & \mathbf{56}, \\ & & \mathbf{35}_+, & \mathbf{56}', \\ & & \mathbf{35}_-, & \mathbf{56}_T \end{array}$$

We can write this out to more clearly compare the IIA and IIB low-energy spectra:

	NS-NS bosons		R-NS and NS-R fermions				R-R bosons
IIA:	$(\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35})$	$\oplus$	$(\mathbf{8} \oplus \mathbf{56}')$	$\oplus$	$(\mathbf{8}' \oplus \mathbf{56})$	$\oplus$	$(\mathbf{8}_V \oplus \mathbf{56}_+)$
	Φ, B_{ij}, g_{ij}		$\lambda_\beta, \Psi_i^{\beta'}$		$\widetilde{\lambda}_{\beta'}, \widetilde{\Psi}_i^{\beta}$		C_1, C_3
IIB:	$(\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35})$	$\oplus$	$(\mathbf{8}' \oplus \mathbf{56})$	$\oplus$	$(\mathbf{8}' \oplus \mathbf{56})$	$\oplus$	$(\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_+)$
	Φ, B_{ij}, g_{ij}		$\lambda_{\beta'}, \Psi_i^{\beta}$		$\widetilde{\lambda}_{\beta'}, \widetilde{\Psi}_i^{\beta}$		C_0, C_2, C_4

We remark that the NS-NS sector is universal; it is the same for the IIA and IIB theories. This makes senes because the only difference between IIA and IIB is the right-moving groundstate: $|0\rangle_R^-$ for IIA and $|0\rangle_R^+$ for IIB. The λs are dilatinos while the Ψs are gravitinos (as promised); each comes in pairs. Note that the IIA theory has fermion pairs of opposite chiralities while the IIB theories have fermion pairs that come in the same chirality. The important feature of this is that IIB theories are chiral. One final observation is that IIA theories have odd-number forms (C_1, C_3) while the IIB theories have even-number forms (C_0, C_2, C_4).

Let's make two more important points: first, the only assumption that we made to get to our IIA and IIB theories was that we wanted to project out the tachyon. This was a choice but, as we saw from our argument of modular invariance, it was a choice that we had to make one way or another. Thus what we have found the ground state spectrum of the most general one-loop consistent theory. Next, note that we never considered the states coming form the bosonic X fields. We saw that the graviton (and friends) came from the NS-NS sector with worldsheet fermion operators. The massless X fields are projected out by the GSO projection so that the lightest X excitations live at the string-scale and do not participate in our low-energy action.

So many fields! Now a 'big picture' remark. Our ground state spectrum has a disturbingly large number of fields. We understand some of them, but what's going on with these p-forms? One might question what role these fields play in the low energy effective theory of a realistic string set up. The big picture is that we'll have to compactify our target spacetime into something called a **Calabi-Yau manifold**. If the Calabi-Yau is 'empty' then there are typically hundreds of massless fields with only gravitational coupling. These fields are called **moduli** and are very strongly constrained experimentally. To avoid these constraints, we can turn on **fluxes**, i.e. vacuum expecation values of the p-forms. This tends to make the moduli go away. This leads us to the current topic of **flux compactifications**. One open question is whether or not a generic string compactification can have no moduli.

8 Superstring actions

Now we've met and acquainted ourselves with IIA and IIB superstrings. What shall we do with this? All we've done in the previous section is reproduce the ground state spectrum. This is nice, but what we *really* want are the interactions of these states. Recall the bosonic case: all the neat stuff happened when we looked at the string sigma model. In fact, this something we picked up when we learned 4D rigid SUSY: it was easy to construct the supersymmetric spectrum by just using the SUSY algebra. The interactions were the non-trivial part: these are what required a Lagrangian.

Now we'd like to build a superstring action, at least for the low-energy states. In fact, what we really want is a supersymmetric action that is the low-energy effective theory associated with our string theory. In fact, since we know that the metric is one of our fields, what we *really* really want is a supergravity action. Supergravity predates string theory by about a decade and many of the subject's heroes worked out several classic results in the field which we will use here. The detailed derivation of these results are generally unenlightening for our present discussion of SUGRA as the low-energy theory of strings so we will not provide rigorous proofs (if any at all).

8.1 Counting supercharges

One such classic result is that in $D = 4$ the largest supersymmetry algebra is $\mathcal{N} = 8$. This is because larger algebras would gives us massless fields with spin greater than 2; starting with a base a state of spin s, acting with $\mathcal{N}$ supercharges gives a state with spin $|s - \mathcal{N}/2|$ since each supercharge reduces the spin by 1/2. The highest spin state is minimized by taking $s = \mathcal{N}/2$ so that if we only restrict to fields of spin no more than 2, we must choose $\mathcal{N} = 8$.

Why do we restrict to fields with spin ≤ 2? For rather general reasons, it is very difficult to construct a consistent theory of interacting spin > 2 fields. The heuristic reason is that such fields do not have a conserved current to which they can couple. For example, the graviton can couple to the stress-energy tensor (a symmetric two-index tensor), but no higher-index current naturally occurs in our theories. A more thorough discussion can be found in Weinberg, volume 1 [16]. For our purposes we shall assume that no consistent interacting theory of spin > 2 exists so that we only need to consider theories with fields of spin no more than 2. This, in turn, restricts us to theories which, in 4D, have no more than $\mathcal{N} = 8$ supersymmetry.

This $D = 4$, $\mathcal{N} = 8$ SUSY algebra contains $8 \times 4 = 32$ supercharges. Thus any higher dimensional supersymmetric theory which is supposed to give a realistic 4D theory *must* live in a dimension which admits spinors with at most 32 elements. (Note the difference between the total number of components in a spinor versus the number of spinors.) For example, one might consider a supersymmetric theory in $D = 12$. The minimal representation of the Dirac algebra in 12 dimensions is a 64 component spinor. This means that the SUSY generators must have at least 64 components so that there is at least 64 supercharges. This gives us more than $\mathcal{N} = 8$ in 4D and so $D = 12$ theories cannot give us consistent 4D theories by assumption.

8.2 The $D = 11$ Cremmer-Julia-Scherk action

We thus are led to theories with $D \leq 11$. In the 80s and early 90s people studied supersymmetry in $D \leq 11$ as a mathematical curiosity. The results below about $D = 10, 11$ have thus been known for some time. It wasn't until much more recently, however, that we have learned that $D = 11$ SUGRA is in fact rather special: it corresponds to the low energy theory of M-theory.

The $D = 11$ spinor has 32 components and has a *unique* supersymmetric action, the Cremmer-Julia-Scherk action. For simplicity we'll write out only the bosonic terms. The fermionic terms follow from supersymmetry. In fact, since we already know that the fermions do not give contributions to the low-energy theory (they are inherently quantum and do not obtain vevs in the classical limit), then the bosonic terms are really all that we care about about. The action is

$$S_{11} = \frac{1}{2\kappa_{11}^2} \int d^{11}x \sqrt{-G} \left(R - \frac{1}{2} |F_4|^2 \right) - \frac{1}{2\kappa^2} \int d^{11}x \, A_3 \wedge F_4 \wedge F_4 + \text{fermions}. \tag{8.1}$$

κ_{11}^2 is the 11D gravitational coupling and sets the overall length scale and does not represent some additional one-parameter freedom in choosing the theory. In other words, this action really is the unique $D = 11$ SUGRA action. (We say this with conviction but without proof.) If we stopped at the R term, then we would have an action for 11-dimensional gravity. $F_4 = dA_3$ is a 4-form so that the $|F_4|^2$ term is the generalization of the usual gauge field strength. The $A_3 \wedge F_4 \wedge F_4$ term is a Chern-Simons term.

On the origin of p-forms. We've seen from the string picture that antisymmetric p-form fields come from the decomposition of tensor products of spinors into irreducible representations. Given that we propose the Cremmer-Julia-Scherk action is the most general $D = 11$ SUGRA action to give sensible 4D physics, one might ask about the origin of the A_3 3-form in this field theory. This again comes from a tensor product of spinors, but this time coming from products of supercharges acting on a state. See Weinberg Vol. III for more details [17].

8.3 The $D = 10$ Type IIA and IIB SUGRA actions

We can take (8.1) to be a fact. What can we do with it? Well, we could repeat the analysis for $D = 10$ via a dimensional reduction procedure. In ten dimensions, it turns out that instead of a single unique action, there are two possible SUGRA actions with the necessary 32 supercharges (as befitting a theory that came from a $D = 11$ theory with 32 supercharges). Perhaps unsurprisingly, they are called **type IIA** and **type IIB** *supergravity* and are the low-energy limits of type IIA and IIB string theory. (Historically the SUGRA theories were known first and lent their names to the string theories.)

We start with our eleven-dimensional theory with a Majorana generator Q_α, a real 32-component spinor. We can reduce[2] this to IIA via

$$Q_\alpha \to Q_\alpha^{(1)} + Q_{\alpha'}^{(2)} \qquad \text{(IIA)}. \tag{8.2}$$

In terms of representations, $Q^{(1)}$ is a $\mathbf{16}$ and $Q^{(2)}$ is a $\mathbf{16'}$ of SO(9,1). These go to the $\mathbf{8}$ and $\mathbf{8'}$ of SO(8) in light-cone gauge. Each of these are 10D Majorana-Weyl spinors. As we've noted before, these are spinors of opposite chirality. Alternately, we could reduce to IIB via

$$Q_\alpha \to Q_\alpha^{(1)} + Q_{\alpha}^{(2)} \qquad \text{(IIB)}, \tag{8.3}$$

where now the $Q^{(2)}$ is also a $\mathbf{16}$ and the model is chiral (note that both indices are now unprimed).

Let us review our field content. Note, in particular, the primes on the IIA fermion indices which denote a different chirality.

IIA	g_{ij}	B_{ij}	Φ	$C_1\ C_3$	$\Psi^i_s\ \widetilde{\Psi}^i_{s'}$	$\lambda_s\ \widetilde{\lambda}_{s'}$
IIB				$C_0\ C_2,\ C_4$	$\Psi^i_s\ \widetilde{\Psi}^i_s$	$\lambda_s\ \widetilde{\lambda}_s$
	graviton	2-form	dilaton	p-forms	gravitinos	dilatinos

The action for the IIA theory is

$$\begin{aligned} S_{\text{IIA}} = \; & \frac{1}{2\kappa_{10}^2}\int d^{10}x\,\sqrt{-G}\,e^{-2\Phi}\left(R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}|H_3|^2\right) \\ & - \frac{1}{4\kappa_{10}^2}\int d^{10}x\,\sqrt{-G}\,e^{-2\Phi}\left(|F_2|^2 + |\widetilde{F}_4|^2\right) \\ & - \frac{1}{4\kappa_{10}^2}\int B_2\wedge F_4\wedge F_4 + \text{fermions}. \end{aligned} \tag{8.4}$$

The three lines correspond to the NS-NS sector, the R-R sector, and the Chern-Simons term which couples the two. The field strengths are

$$F_2 = dC_1 \qquad \widetilde{F}_4 = dC_3 - C_1\wedge H_3 \qquad H_3 = dB_2. \tag{8.5}$$

It is typical to write F_{p+1} to mean the field strength coming from the exterior derivative of an a-form C_a and $\widetilde{F}_{p+1}$ to mean the same field strength with an additional term of the form $C_a \wedge H_3$. One shouldn't confuse this with a Hodge dual.

[2] Here we perform a **dimensional reduction** where we compactify the extra dimension into a circle and take the radius to zero so that only the zero-mode survives.

The IIB action is analogous with the replacements

$$\begin{array}{ccc} C_1,\ C_3 & \longrightarrow & C_0,\ C_2,\ C_4 \\ \text{opposite chirality} & \longrightarrow & \text{same chirality.} \end{array}$$

We find

$$\begin{aligned} S_{\text{IIB}} = \; & \frac{1}{2\kappa_{10}^2}\int d^{10}x\,\sqrt{-G}\,e^{-2\Phi}\left(R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}|H_3|^2\right) \\ & - \frac{1}{4\kappa_{10}^2}\int d^{10}x\,\sqrt{-G}\,e^{-2\Phi}\left(|F_1|^2 + |\widetilde{F}_3|^2 + \frac{1}{2}|\widetilde{F}_5|^2\right) \\ & - \frac{1}{4\kappa_{10}^2}\int C_4\wedge H_3\wedge F_3 + \text{fermions}. \end{aligned} \tag{8.6}$$

The new tilde-fields are

$$\widetilde{F}_3 = F_3 - C_0\wedge H_3 \tag{8.7}$$

$$\widetilde{F}_5 = F_5 - \frac{1}{2}C_2\wedge H_3 + \frac{1}{2}B_2\wedge F_3. \tag{8.8}$$

We note that there's one subtlety: we have to impose $*\widetilde{F}_5 = \widetilde{F}_5$, i.e. that the $\widetilde{F}_5$ field is self-dual. The other field strengths are defined following the convention above.

Let us make a few remarks. The IIA/B super*gravity* action above has a UV divergence. The IIA/B super*string* theory which has IIA/B SUGRA as its IR limit, is well-behaved in the UV. We say that IIA is the UV completion of IIA/B SUGRA. This action gives the same S-matrix elements as IIA string amplitudes with massless modes. The quantum (i.e. loop) corrections to the SUGRA theory should be interpreted as g_S corrections, as we discussed in Section 6.3. This theory, however, will never give α' corrections. This will become interesting when we study black hole microstates which *do* get α' corrections.

In the back of our minds we imagine that if we can relate F_4 and $\widetilde{F}_4$, then we can understand the relation between $D = 10$ and $D = 11$. For example, we could take the S_{11} action and dimensionally reduce on a circle to get $S_{\text{IIA/B}}$. Witten showed that S_{11} is a 'real theory' *on its own*. This set up is extremely suggestive of the existence of some string-like theory which gives S_{11} at low energies. In other words our 11D action is the low-energy effective action of M-theory without sources. We'll see this soon.

In terms of the big picture, there is a lot of technical work that goes into an honest derivation of the results above. One can spend a lot of time—as they did in the 80s and early 90s—doing more with these theories, but the point is that all of the relevant results for us can be distilled into relatively simple classical actions. In practice, most people just take these actions and solve the equations of motion.

9 Calabi-Yau compactifications

The next step is to search for supersymmetric string vacua with four uncompact dimensions. We need more than just supersymmetric *actions*, we need to find supersymmetric *configurations* (i.e. states).

10 Dp-branes and T-duality

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A Notation and Conventions

There's a joke that I used to tell in this section. I would say that particle physicists use the metric $(+,-,-,-)$, general relativists use the metric $(-,+,+,+)$, while string theorists use the metric $(+,+,+,+,+,+,\cdots)$. Well, I guess it's not a joke anymore. Our conventions will try to follow those in Polchinski [2, 3]. Worldsheet (two-dimensional) indices will typically be written with lower-case Greek letters from the beginning of the alphabet $(\alpha,\beta,\cdots)$ while target spacetime indices will typically be written with lower-case Greek letters from the middle of the alphabet $(\mu,\nu,\cdots)$. Occasionally we will write a Kronecker delta with only one argument, e.g. δ_{m+n}; in this case the other argument is understood to be zero.

B Conformal field theory basics

> This section is based on notes from a previous version of the course given in 2008 which focused on a more traditional introduction to bosonic string theory. It also borrows heavily from the text by DiFrancesco, et al. [11] and lecture notes by Tong [7], Ginsparg [12], and Schellekens [13].

Conformal invariance is a symmetry (of an action) under $g_{\mu\nu}(x) \to \Lambda(x)g_{\mu\nu}(x)$, i.e. it is an invertible transformation of the coordinates $x^\mu \to x'^\mu$ that preserves angles (because the metric is preserved up to a scaling). In basic string theory our primary interest is the conformal group in two dimensions since we are interested in the conformal field theory (CFT) on the worldsheet. We will see that 2D CFTs are rather special in ways that become very useful if one were interested in calculating string scattering amplitudes. However, for a general background and because we will eventually become interested in the AdS/CFT correspondence and higher dimensional conformal field theories, we will first review general properties of CFTs before specializing to the case $D = 2$.

B.1 The conformal group in D dimensions

To better understand the elements of the conformal group, let us study how the metric changes under a general coordinate transformation $x^\mu \to x^\mu + \epsilon^\mu(x)$. To leading order in ϵ, the metric changes as

$$\delta g_{\mu\nu} = g'_{\mu\nu} - g_{\mu\nu} = -\partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu. \tag{B.1}$$

Imposing that this variation must generate an infinitesimal conformal transformation we obtain

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = f(x) g_{\mu\nu} = \frac{2}{D} \partial \cdot \epsilon \, g_{\mu\nu}, \tag{B.2}$$

where we've fixed the prefactor of $g_{\mu\nu}$ by requiring the traces of each side are equal. For simplicity let us now assume a flat metric, $g_{\mu\nu} = \eta_{\mu\nu}$. One may now take an additional derivative ∂_ρ of this equation and permute indices to obtain (see chapter 4.1 of [11])

$$\left(\eta_{\mu\nu}\partial^2 + (D-2)\partial_\mu \partial_\nu\right) \partial \cdot \epsilon = 0. \tag{B.3}$$

Restriction to flat space. In two dimensions, our choice $g_{\mu\nu} = \eta_{\mu\nu}$ is not actually a restriction. A general two dimensional metric is given by three functions (each of its diagonal elements, and one off-diagonal which is related to the other by symmetry). We can use diffeomorphism invariance, which gives us two 'functions-worth' of freedom, to go to conformal gauge where $g_{12} = 0$ and $g_{11}(x) = g_{22}(x)$ so that the metric is $g_{\mu\nu} \propto \eta_{\mu\nu}$. Then it is easy to use a Weyl transformation to bring us back to the flat space metric everywhere. In the case of higher dimensions, our analysis here only holds for non-gravitational theories on flat space.

Together (B.3) and (B.3) imply that for $D > 2$ $\epsilon(x)$ is at most quadratic in x,

$$\epsilon_\mu(x) = \alpha_\mu + b_{\mu\nu} x^\nu + c_{\mu\nu\rho} x^\nu x^\rho. \tag{B.4}$$

The constant term α_μ is easily understood as a translation in spacetime. Substituting the linear $b_{\mu\nu}$ term into (B.2) we obtain

$$b_{(\mu\nu)} = \frac{2}{D} b^\rho{}_\rho \eta_{\mu\nu} \tag{B.5}$$

so that $b_{\mu\nu}$ is the sum of an antisymmetric part and a pure trace,

$$b_{\mu\nu} = \lambda \eta_{\mu\nu} + \omega_{\mu\nu}. \tag{B.6}$$

These are also rather familiar spacetime transformations: dilation by λ and rotations (Lorentz transformations) ω. Sometimes authors will refer to dilations as 'dilatations,' but this sounds like a Don King-ism ("Dilatations are the most *splendiferous* transformations in the world!"). Finally, the $c_{\mu\nu\rho}$ term can be shown to satisfy

$$c_{\mu\nu\rho} = \eta_{\mu\rho} b_\nu + \eta_{\mu\nu} b_\rho - \eta_{\nu\rho} b_\mu \tag{B.7}$$

for some parameter $b_\mu = (1/d)c^\rho{}_{\rho\mu}$. This is a new kind of transformation that one might not be familiar with from a first field theory course. It is called a **special conformal transformation** (SCT) and can be interpreted as the sequence of transformations:

$$\text{inversion} \circ \text{translation by } b^\mu \circ \text{inversion}, \tag{B.8}$$

where inversions take $x^\mu \to x^\mu/x^2$. Together these transformations are the generators of the D-dimensional ($D > 2$) conformal group. Let us summarize their generators and finite forms:

$$\begin{array}{rll}
\text{translation} & P_\mu = -i\partial_\mu & x'^\mu = x^\mu + a^\mu \\
\text{dilation} & D = -ix^\mu\partial_\mu & x'^\mu = \lambda x^\mu \\
\text{Lorentz} & L_{\mu\nu} = i\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right) & x'^\mu = \Lambda^\mu{}_\nu x^\nu \\
\text{SCT} & K_\mu = -i\left(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu\right) & x'^\mu = \dfrac{x^\mu - b^\mu x^2}{1 - 2b\cdot x + b^2x^2}.
\end{array}$$

We remark that we are, of course, implicitly assuming that the fields themselves do not transform under a conformal transformation so that we are only studying the transformation on spacetime. Indeed, we know that for general field theories the fields must transform: this is the content of the renormalization group. For good reviews connecting the renormalization group and scale transformations see chapter 3 of [18], Stevenson [19], or Hollowood [20]. We will construct appropriate representations of the fields under the conformal transformations in the following section.

First, however, let us write the conformal algebra satisfied by the generators above. They can be calculated straightforwardly.

$$[D, P_\mu] = iP_\mu \tag{B.9}$$
$$[D, K_\mu] = -iK_\mu \tag{B.10}$$
$$[K_\mu, P_\nu] = 2i(\eta_{\mu\nu}D - L_{\mu\nu}) \tag{B.11}$$
$$[K_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu) \tag{B.12}$$
$$[P_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu) \tag{B.13}$$
$$[L_{\mu\nu}, L_{\rho\sigma}] = i(\eta_{\nu\rho}L_{\mu\sigma} + \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\rho}). \tag{B.14}$$

If one stares at this algebra long enough, one will notice that these can be arranged into a more compact form:

$$J_{\mu\nu} = L_{\mu\nu} \qquad J_{(-1)\mu} = \frac{1}{2}\left(P_\mu - K_\mu\right) \qquad J_{(-1)\mu} = DJ_{0\mu} = \frac{1}{2}\left(P_\mu + K_\mu\right), \tag{B.15}$$

with all of the Js antisymmetric in their indices. Here we have used the P, D, and K generators to extend the Lorentz algebra to J_{ab} with $a, b \in -1, 0, \cdots D$. Thus if we work in a spacetime with p spacelike directions and q timelike directions ($\mathbb{R}^{p,q}$), then the conformal algebra is precisely SO($p+1$,$q+1$). We will almost always work in Wick-rotated coordinates so that the $L_{\mu\nu}$ generate D-dimensional rotations. In this case the conformal algebra is SO(D+1,1).

We remark that the usual Poincaré group with dilations form a subgroup of the conformal group. It turns out, however, that any consistent scale-invariant quantum field theory also is conformally invariant.

B.2 Representations of the conformal group in D dimensions

Let's now proceed to play a familiar game, the method of induced representations. This is how we determine the field representations of an extended spacetime symmetry. We study the subgroup of the conformal symmetry which preserves the origin $x = 0$. We can then form representations of this group (or, more properly, the universal cover of this group) and translate the generators to other positions on spacetime.

In the usual case of the Poincaré group we would define the matrix representation of the Lorentz group (the subgroup that leaves the origin invariant), this is just a choice of spin representation: $L_{\mu\nu}|_{x=0} = S_{\mu\nu}$. This only holds at the identity, but can be translated to any point in spacetime via

$$e^{ix\cdot P} L_{\mu\nu} e^{-ix\cdot P} = S_{\mu\nu} - x_\mu P_\nu + x_\nu P_\mu. \tag{B.16}$$

Here we've used the Poincaré algebra and the Hausdorff formula for operators,

$$e^{-A} B e^{A} = B + [B, a] + \frac{1}{2!}\left[[B, A], A\right] + \frac{1}{3!}\left[\left[[B, A], A\right], A\right] + \cdots . \tag{B.17}$$

The punchline is that we now have a representation of the Poincaré group generators on all of spacetime,

$$P_\mu = -i\partial_\mu \tag{B.18}$$

$$L_{\mu\nu} = i\left(x_\mu \partial_\nu - x_\nu \partial_\mu\right) + S_{\mu\nu}. \tag{B.19}$$

Now let's summarize the results of this procedure applied to the conformal group, whose identity-preserving subgroup is composed of Lorentz transformations, dilation, and special conformal transformations. For details see [21]. We define the representations of these generators at the origin: $S_{\mu\nu}$ under Lorentz transformations, $\widetilde{\Delta}$ under scaling, and κ_μ under special conformal transformations. These representations must obey the relevant commutators of the conformal algebra. The remaining commutators give us a way to translate the generators via the Hausdorff formula. In addition to (B.18) and (B.18), we end up with

$$D = -ix^\nu \partial_\nu + \widetilde{\Delta} \tag{B.20}$$

$$K_\mu = \kappa_\mu + 2x_\mu \widetilde{\Delta} - x^\nu S_{\mu\nu} - 2ix_\mu x^\nu \partial_\nu + ix^2 \partial_\mu. \tag{B.21}$$

Now there's a nice little simplification. Requiring that our fields transform as irreducible representations of the Lorentz group, then Schur's lemma tells that any operator which commutes with $S_{\mu\nu}$ must be a multiple of the identity with respect to the spin indices. This means that κ_μ must vanish. Further, $\widetilde{\Lambda}$ is just a number which is equal to $-i\Delta$, where Δ is the usual **scaling dimension** of the field. That is to say, under $x \to \lambda x$, $\Phi(x) \to \Phi'(x') = \lambda^{-\Delta}\Phi(x)$. Note that $\widetilde{\Lambda}$ is not Hermitian, a reflection that representations of dilations on classical fields are not unitary.

For conformal transformations that take $g_{\mu\nu}(x) \to \Lambda(x) g_{\mu\nu}(x)$, we may write the scale factor $\Lambda(x)$ in terms of the Jacobian of the transformation,

$$\left|\frac{\partial x'}{\partial x}\right| = \Lambda(x)^{-d/2}. \tag{B.22}$$

Thus a spinless field transforms under a conformal transformation as

$$\phi(x) \to \phi'(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\Delta/D} \phi(x). \tag{B.23}$$

This is straightforward since the field is irreducibly spinless so has no Lorentz transformation and κ_μ vanishes. The scaling factor in the above equation is just the transformation that one obtains for dilation. This exhausts the subgroup that preserves the origin. A field which transforms according to (B.23) is called **quasi-primary**.

B.3 The energy-momentum tensor in D dimensions

If conformal invariance is a symmetry of a field theory, then Noether's theorem tells us that there should be a conserved conformal current. It should be no surprise that this current is the stress-energy (or energy-momentum) tensor since we already know that translations are part of the conformal group. Let's review the basic features that might already be familiar from field theory.

There's a useful trick for deriving the stress energy tensor that holds for any theory. We shall closely follow the presentation by Tong [7]. Let us start in flat space, $g_{\mu\nu} = \eta_{\mu\nu}$. We can derive the conserved current associated with a symmetry by temporarily promoting the transformation parameter to a spacetime field $\epsilon \to \epsilon(x)$. Under a symmetry transformation with this faux-spacetime-dependent parameter, the action must change as

$$\delta S = \int d^d x \, J^\mu \partial_\mu \epsilon(x). \tag{B.24}$$

This is clear since $\delta S = 0$ when ϵ is constant. Now the important part: *when the equations of motion are satisfied* $\delta S = 0$, eve for non-constant $\epsilon(x)$. This is tautologically true since the equations of motion are nothing more and nothing less than the minimum of the action. Thus when the equations of motion are obeyed—i.e. classically—we know that J^μ is a conserved current $\partial_\mu J^\mu = 0$.

Consider translations (which are certainly a subset of conformal symmetries), $x^\mu \to x^\mu + \epsilon^\mu$. We promote the transformation parameter to a function of the spacetime $\epsilon^\mu \to \epsilon^\mu(x)$. Now here's another trick: let us *additionally* promote the flat metric to a dynamical background metric $\eta_{\mu\nu} \to g_{\mu\nu}(x)$. Now we can interpret the spacetime-dependent translation as a *diffeomorphism*. The upshot is that we can keep the theory invariant by making a corresponding transformation on the metric,

$$\delta g_{\mu\nu} = \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_m u. \tag{B.25}$$

The key insight to gain from this is that the $\delta g_{\mu\nu}$ transformation must change the action in such a way that it counteracts the change from the spacetime-depenedent translation. Thus the δS coming from a spacetime-dependent translation must just be *minus* the change in the action coming from (B.25),

$$\delta S = -\int d^d x \, \frac{\partial S}{\partial g_{\mu\nu}} \delta g_{\mu\nu} = -2 \int d^d x \, \frac{\partial S}{\partial g_{\mu\nu}} \partial_\mu \epsilon_\nu. \tag{B.26}$$

This is precisely the form that we wanted in (B.24) to identify the Noether current, which of course is simply proportional to the energy-momentum tensor $T_{\mu\nu}$.

That's great, but we haven't really learned anything new yet. The key difference between ordinary field theories and conformal field theories is that in the latter case the energy-momentum tensor can typically be made traceless. This is easy to see for conformal transformations (B.2)

$$\delta g_{\mu\nu} = f\, g_{\mu\nu} \tag{B.27}$$

so that

$$\delta S = \int d^d x\, \frac{\partial S}{\partial g_{\mu\nu}} \delta g_{\mu\nu} \propto - \int d^d x\, f\, T^\mu{}_\mu. \tag{B.28}$$

Let us make two important remarks. First, scale transformations tend to be anomalous in quantum field theories. (This is again related to the renormalization group.) Next, we have seen that the a traceless the energy-momentum tensor implies (at least classically) conformal invariance. However, it is not true that conformal invariance necessarily implies a traceless energy-momentum tensor because the variation $f(x) \propto \partial \cdot \epsilon(x)$ is not an arbitrary function, c.f. the form of $\epsilon_\mu(x)$ in (B.4). This is closely related to the statement that a scale and Poincaré invariant field theory must also, under certain conditions, be conformally invariant. (See, for example, chapter 4.2.2 of [11].)

B.4 Correlation functions in D dimensions

In conformal theories the only physical objects are correlation functions of the fields. It is perhaps worth noting that in most of the CFT literature, the term **field** needn't necessarily refer to the fundamental fields of the theory (i.e. those with their own path integral), but more generally can refer to any local construction of those fields and their derivatives. These correlation functions are, in a sense, the only things that one can calculate in a CFT.

Let us review the transformation of correlation functions under a general spacetime transformation.

$$\langle \phi(x_1') \cdots \phi(x_n') \rangle = \frac{1}{Z} \int [d\phi]\ \phi(x_1') \cdots \phi(x_n') e^{-S[\phi]} \tag{B.29}$$

$$= \frac{1}{Z} \int [d\phi']\ \phi'(x_1') \cdots \phi'(x_n') e^{-S[\phi']}, \tag{B.30}$$

where we've just changed the path integral variable. Invoking (B.23),

$$= \left| \frac{\partial x'}{\partial x} \right|^{-n\Delta/D} \langle \phi(x_1) \cdots \phi(x_n) \rangle. \tag{B.31}$$

More generally for different fields, each with their own scaling dimension,

$$\langle \phi_1(x_1') \cdots \phi_n(x_n') \rangle = \left| \frac{\partial x'}{\partial x} \right|^{-\Delta_1/D} \cdots \left| \frac{\partial x'}{\partial x} \right|^{-\Delta_n/D} \langle \phi(x_1) \cdots \phi(x_n) \rangle. \tag{B.32}$$

We can use conformal symmetry to constrain the form of these correlation functions. In two dimensions we will see that this turns out to be a very powerful constraint.

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