The dynamics of a charged sphere and the electron

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A brief review of the history of the classical equations of motion of charged particles is given leading up to the present time. Emphasis is placed on the important role of the validity limits of the theory. The little known relativistic equations of motion of a surface-charged sphere are presented. Their proof and validity limits are discussed. An appendix gives the history and eventual solution of the stability problem (the ''4/3 problem''). The emphasis is on understanding rather than on technical details.

I. PURPOSE OF THIS PAPER

This year is the centennial of the discovery of the electron by J. J. Thomson. It is therefore most appropriate to review the attempts at a dynamics for this particle as it has been developed during the past 100 years. In the context of a very brief historical survey, two lessons learned from that tortuous history are the main topics. The first lesson is of a general nature. It elaborates on the fact that in a strict sense the notion of a ''classical point charge'' is an oxymoron because ''classical'' and ''point'' contradict one another: classical physics ceases to be valid at sizes at or below a Compton wavelength and thus cannot possibly be valid for a point object. The lesson is thus:

The dynamics of point charges is an excellent example of the importance of obeying the validity limits of a physical theory. When these limits are exceeded the predictions of the theory may be incorrect or even patently absurd. In the present case, the **classical** *equations of motion have their validity limits where quantum mechanics becomes important: they can no longer be trusted at distances of the order of (or below) the Compton wavelength.*

The second lesson refers to the equations of motion of an extended particle (rather than a point particle), a sphere with a uniformly distributed surface charge. In their nonrelativistic form, they have been written down almost 80 years ago; in their relativistic form, over 40 years ago.

The relativistic equations of motion of a surface-charged sphere are in excellent approximation a set of nonlinear first-order differential-difference equations. In the limit when the radius vanishes, they become the Lorentz–Abraham–Dirac equations.

The title of the present paper is purposely ambiguous. It can mean either (the dynamics of a charged sphere) and the electron, or the dynamics of (a charged sphere and an electron). If we heed the first of the above lessons, the first meaning is correct: An electron is a quantum mechanical particle and can therefore—strictly speaking—not be described by the dynamics of a classical object, a charged sphere. However, the dynamics has been used in the second sense. And this is justified if the distances involved are all large compared to quantum mechanical distances; the classical theory for *point* charges will then be an excellent approximation. Also, when the distances involved are large compared to the radius a of a charged sphere, the results will be independent of a; they will then be the same as those from a point charge theory. In fact, the equations of motion for a charged sphere reduce to the equations of motion for a point charge in the limit as a goes to zero. But one cannot use classical dynamics for a point charge when quantum mechanical distances are involved: quantum dynamics then becomes necessary.

It should be clear that in a short review like the present, it is impossible to do justice to the many papers written on the subject over the years. Many important papers must, unfortunately, remain unmentioned.

II. THE POINT CHARGE DYNAMICS

The electromagnetic force density on a volume element of charge, $\rho(\mathbf{x})$, was first given by Lorentz[1] in 1892,

$$\mathbf{f}_L = \rho(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \tag{2.1}$$

It was later[2] applied by him to the electron discovered by Thomson[3] in 1897, which he assumed to be a point of total charge e,

$$\mathbf{F}_L = e(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \tag{2.2}$$

This is the Lorentz force on a point charge.[4] But he argued that when the charge is accelerated, there are additional forces acting due to the charge's own electromagnetic field. The equation of motion for the electron, he argued, is therefore in first approximation,

$$m\dot{\mathbf{v}} = \mathbf{F}_L + \tfrac{2}{3}e^2\ddot{\mathbf{v}} \tag{2.3}$$

where a dot indicates a time derivative. This equation is known as the Lorentz equation.

In the same year in which the electron was discovered, Larmor[5] provided the formula for the radiation rate of an accelerated charge,

$$\frac{dE_R}{dt} = \frac{2}{3}\, e^2\dot{\mathbf{v}}^2. \tag{2.4}$$

This formula was soon generalized to the relativistic case by Heaviside,[6]

$$\frac{dE_R}{dt} = \frac{2}{3}\, e^2\dot{v}^{\alpha}\dot{v}_{\alpha}. \tag{2.5}$$

It is written here in relativistic notation: $v^0 = \gamma$, $v^k = (\gamma\mathbf{v})^k$ ($\alpha = 0,1,2,3$, $k = 1,2,3$, metric tensor tr $\eta = +2$), and τ denotes the proper time, $d\tau = dt/\gamma$ and $\gamma = (1 - \mathbf{v}^2)^{-1/2}$. In this as in all covariant equations, the dot indicates differentiation with respect to τ.

Heaviside derived (2.5) before the theory of relativity was available because he used the Maxwell equations which are relativistic. Note that (2.5) is a Lorentz scalar. Further progress was made by Abraham[7] when he derived the *rate* of momentum carried away from the charge by radiation. Nonrelativistically, it is just the radiation rate (2.4) times the velocity $\mathbf{v}$. Its relativistic generalization, when combined with (2.5) gives the four-vector

$$\frac{dP^{\mu}}{d\tau}=\frac{2}{3}\, e^{2}\dot{v}^{\alpha}\dot{v}_{\alpha}v^{\mu}. \tag{2.6}$$

This four-vector represents the rate at which energy and momentum is carried away from the charge by radiation. Its negative is the *radiation reaction,* and it will necessarily play an important role in the equations of motion. Note that the 0-component of (2.6) is just γ times (2.5).

Abraham[8] then succeeded in deducing the equations of motion of a rigid sphere of charge e and radius a in the approximation of small a. Assuming a bare mass, m, which he assumed to vanish, his equations can be written

$$m\,\frac{d}{dt}\,(\gamma\mathbf{v})=\mathbf{F}_{L}-m_{\mathrm{ed}}\,\frac{d}{dt}\,(\gamma\mathbf{v})+\Gamma, \tag{2.7a}$$

$$\Gamma=\tfrac{2}{3}e^{2}\gamma^{2}\{[\ddot{\mathbf{v}}+3\gamma^{2}\mathbf{v}\cdot\dot{\mathbf{v}}\dot{\mathbf{v}}+\gamma^{2}[\mathbf{v}\cdot\ddot{\mathbf{v}}+3\gamma^{2}(\mathbf{v}\cdot\dot{\mathbf{v}})^{2}]\mathbf{v}]\}. \tag{2.7b}$$

Here m_{ed} is the electrodynamic energy

$$m_{\mathrm{ed}}=\frac{2e^{2}}{3a} \tag{2.8}$$

The term containing m_{ed} is an inertial term and, when so interpreted, can be combined with the left-hand side. The sum $m+m_{\mathrm{ed}}=m_{0}$ is then identified with the observed rest mass of the sphere.

The mass m involves a bare mass of unknown size as well as, in the case of a macroscopic charged sphere, the mass of the uncharged insulator. The mass of the latter, which must, of course, always be included on the left side of the equations of motion will play no further role in the later discussion and will not be explicitly mentioned below.

Before discussing the complicated result (2.7b), let us jump ahead historically and give the result found by von Laue.[9] He showed that Γ is just the relativistic generalization of the last term in the Lorentz equation (2.3) and can be identified with the space part (times $1/\gamma$) of the four-vector

$$\Gamma^{\mu}=\tfrac{2}{3}e^{2}(\ddot{v}^{\mu}-\dot{v}^{\alpha}\dot{v}_{\alpha}v^{\mu}). \tag{2.9}$$

One sees that the von Laue vector Γ^{μ} consists of two terms of which the second one is just the radiation reaction (2.6). The first one is sometimes called the Schott term; its space part has a nonrelativistic limit which is (apart from a factor γ) just the last term of the Lorentz equation (2.3). Unfortunately, in the earlier literature the *whole* expression Γ^{μ} has at times been called ''radiation reaction,'' this has caused much confusion in understanding and resulted in incorrect papers in the literature.

In addition to the rate of momentum change which yields (2.7), Abraham also derived the rate of energy change.[10] Assuming a bare mass, his equation becomes

$$m\,\frac{d\gamma}{dt}=\frac{dE}{dt}=\mathbf{F}_{L}\cdot\mathbf{v}-m_{\mathrm{ed}}\,\frac{d}{dt}\left(\gamma-\frac{1}{4\gamma}\right)+\frac{2}{3}\,e^{2}\gamma^{4}[\mathbf{v}\cdot\ddot{\mathbf{v}}+3\gamma^{2}(\mathbf{v}\cdot\dot{\mathbf{v}})^{2}]. \tag{2.10}$$

The last term together with the last term of (2.7a) is (except for a factor γ) just the four-vector Γ^{μ} in (2.9). But the m_{ed} term does not form a four-vector with the m_{ed} term of (2.7a). In the nonrelativistic limit, these two m_{ed} terms are the time derivatives of m_{es} and $4/3m_{\mathrm{es}}\mathbf{v}$, respectively, where

$$m_{\mathrm{es}}=\frac{1}{2}\,\frac{e^{2}}{a} \tag{2.11}$$

is the electrostatic energy of a surface-charged sphere. Because these are the correct expressions for the rest energy and the kinetic energy except for the factor 4/3, this is known as the ''4/3 problem.'' It has a long history that extends throughout most of the last 90 years. The main point for correcting the problem is the observation first made by Poincaré that one must take into account the forces that hold the (otherwise exploding) charged sphere together. These are called the ''Poincaré stresses,'' or, for a classical sphere, the ''binding forces'' that hold the surface charge to the insulator. I shall sketch the history and solution to the 4/3 problem in the Appendix. It yields the Lorentz invariant combination of (2.7) and (2.10),

$$m\dot{v}^{\mu}=F_{L}^{\mu}-m_{\mathrm{es}}\dot{v}^{\mu}+\Gamma^{\mu},\quad F_{L}^{\mu}=eF^{\mu\alpha}v_{\alpha}, \tag{2.12}$$

with Γ^{μ} given by (2.9). This is the relativistic equation of motion of a surface-charged sphere as it results from the work of Abraham, Lorentz, and Poincaré.

Much later, in 1939, after the development of quantum mechanics, Dirac[11] derived this same classical equation (2.12) for point charges using the Maxwell equations, the conservation laws, and simplicity (read: neglecting higher order terms). His derivation is manifestly covariant throughout. But he did it for a point charge so that the m_{es} term[12] is necessarily infinite $(a\to 0)$. Equation (2.12) should therefore be appropriately called the *Lorentz–Abraham–Dirac equation.*

It is instructive to derive this equation by a very simple argument. Starting with the relativistic form of Newton's equations for a particle of bare mass m subject to the Lorentz force, $m\dot{v}^{\mu}=F_{L}^{\mu}$, one observes that both sides of this equation are orthogonal to v^{μ}. Thus, if one wants to guess what extra term must be added in order to account for the effects of the self-field, one can write that term as

$$X^{\mu}=P^{\mu\nu}Y_{\nu},\quad P^{\mu\nu}=\eta^{\mu\nu}+v^{\mu}v^{\nu}. \tag{2.13}$$

$P^{\mu\nu}$ is the projection into the hyper-plane orthogonal to v^{μ}. Now the simplest choice for Y^{μ}, restricting oneself to linear terms, is

$$Y^{\mu}=av^{\mu}+b\dot{v}^{\mu}+c\ddot{v}^{\mu}.$$

The first term vanishes because $P^{\mu\nu}v_{\nu}=0$, the rest gives $X^{\mu}=b\dot{v}^{\mu}+c(v^{\mu}-v^{\mu}\dot{v}^{\alpha}\dot{v}_{\alpha})$ because $v^{\alpha}\ddot{v}_{\alpha}=-\dot{v}^{\alpha}\dot{v}_{\alpha}$. The first term is an inertial term so that b is identified with the rest energy of the charge, which for our sphere is just m_{es}. It can be combined with the left-hand side (this process can be called ''renormalization'') to yield the empirical rest mass $m_{0}=m+m_{\mathrm{es}}$. The coefficient c is determined by the requirement that the radiation rate four-vector (2.6) must appear as a radiation reaction (hence the minus sign) in the equation of motion; thus $c=2e^{2}/3$. The result is (2.12). This must surely be the simplest possible argument leading to the Lorentz–Abraham–Dirac equation. A similar argument will play a role again later on.

The Lorentz–Abraham–Dirac equation (2.12) as well as its nonrelativistic limit, the Lorentz equation (2.3), have solutions that are pathological. These solutions include the force-free case ($F_L=0$) which, in addition to the physical solution $v=$constant, has a solution according to which the particle accelerates indefinitely (the so-called runaway solutions); and there are also solutions which *anticipate* a change in the external force and according to which the particle accelerates in advance of the application of a force (preacceleration solutions). These solutions are discussed in various texts.

As emphasized earlier, the appearance of pathological solutions is not surprising since (except for the self-energy) all powers of a have been neglected in the derivation so that this *classical* equation describes effectively a *point* particle. We shall see in Sec. III that when *all* higher order terms are included, pathological solutions indeed do not occur. However, for the Lorentz–Abraham–Dirac equation (2.12) the problem of the pathological solutions can be solved without going to the higher, size-dependent terms in the expansion. For the runaway solutions, one can simply impose asymptotic conditions to the effect that in the distant future the acceleration ceases.[13] This preserves only the physical solution $v=$constant in the force-free case. For preacceleration, the problem can be solved by removing its origin. This was done by Yaghjian.[14]

He observed that the expansion made in the derivation requires the function that is expanded to be analytic. Since one integrates over the retarded time, this analyticity is violated when the force changes too quickly (within a time short compared to the time it takes a light ray to cross the particle). The step function onset is an extreme example of this. The problem can be solved by introducing a smooth function $\eta(\tau)$, which vanishes for $\tau<0$ and becomes l at about $\tau=2a$. Equation (2.12) should then be written as

$$m\dot{v}^{\mu}=F_L^{\mu}-m_{\text{es}}\dot{v}^{\mu}+\eta\Gamma^{\mu}. \tag*{$(2.12)_{\text{Y}}$}$$

This equation has no preacceleration solutions. I must refer to the excellent book by Yaghjian for details.

Of considerable interest is the fact that so many efforts have been made to propose substantial modifications of the classical Lorentz–Abraham–Dirac equation in order to remove the pathological solutions. These attempts were not replacements of the classical equations by quantum equations, as should have been expected after quantum mechanics became well established. Nor did these authors argue that the solutions of a classical equation cannot be valid for small distances and certainly not for *point* particles in compliance with the first lesson given above. Rather, they wrote many an ingenious paper suggesting modifications of (2.12) by replacing the $\ddot{v}^{\mu}$ term by terms involving fewer derivatives, or even suggesting modifications of the Maxwell equations.

III. THE CHARGED SPHERE

The fact that the electromagnetic self-energy of a charged sphere diverges in the limit to a point charge, $a\to 0$, has persuaded Abraham and others since the beginning of the century to attempt a derivation of the equations of motion in which the finite size is fully taken into account. An excellent review of charged particle dynamics including these attempts was given by Erber.[15] Other valuable reviews emphasize nonradiating solutions,[16] point particles with spin[17] and with multipole moments.[18]

Particularly extensive work on extended charges was carried out by Sommerfeld.[19] I shall limit the discussion here to the sphere with uniform surface charge; the case of a volume charge is considerably more complicated and adds nothing to the understanding of the problem. Sommerfeld showed that in the nonrelativistic case, such a sphere obeys in good approximation the equation[20]

$$m\dot{\mathbf{v}}=\mathbf{F}_L+m_{\text{ed}}\frac{1}{2a}[\mathbf{v}(t-2a)-\mathbf{v}(t)]. \tag{3.1}$$

It is not difficult to derive this equation. The equation of motion can be written

$$m\dot{\mathbf{v}}=\mathbf{F}_L+\mathbf{F}_S, \tag{3.2}$$

where $\mathbf{F}_S$ is the self-force. As shown in Jackson,[21] Sec. 17.3, if one neglects nonlinear terms, $\mathbf{F}_S$ is given by an infinite series, Eq. (17.28). Each term of that series contains an integral that is easily evaluated for a surface-charged sphere,

$$\int\int d^3x d^3x'\rho(\mathbf{x})|\mathbf{x}-\mathbf{x}'|^{n-1}\rho(\mathbf{x}')=2e^2\frac{(2a)^{n-1}}{n+1}, \tag{3.3}$$

so that the infinite series becomes

$$\mathbf{F}_S=\frac{2}{3}e^2\frac{1}{a}\sum\frac{(-1)^n}{n!}\frac{1}{n+1}\left(\frac{2a\partial}{\partial t}\right)^n\mathbf{v}(t). \tag{3.4}$$

This series can be summed and the result is (3.1).

Equation (3.1) is a first-order differential-difference equation of the retarded type.[22] It implies that the electromagnetic field attached to the charge causes a self-force which has a delayed effect on its motion. The delay time is exactly the time it takes a light ray to cross the diameter of the sphere. If one expands (3.1) for small a (which is actually an expansion in powers of the derivative ad/dt), one finds

$$m\dot{\mathbf{v}}=\mathbf{F}_L-m_{\text{ed}}\dot{\mathbf{v}}+\tfrac{2}{3}e^2\ddot{\mathbf{v}}+O(a), \tag{3.5}$$

which is, of course, just the Lorentz equation but with the m_{ed} term instead of the m_{es} term that occurs in the Lorentz–Abraham–Dirac equation (2.12). But what matters dynamically is only the renormalized equation,

$$(m_0-m_{\text{ed}})\dot{\mathbf{v}}=\mathbf{F}_L+m_{\text{ed}}\frac{1}{2a}[\mathbf{v}(t-2a)-\mathbf{v}(t)], \tag*{$(3.1)'$}$$

and that agrees, when expanded, with the renormalized Lorentz equation. Both the inertial term as well as the Schott term arise from the expansion of the square bracket on the right side. These two terms are therefore both the result of the surrounding field, and are of dynamic origin; they hint at the delay effect explicit only in (3.1).

Since the Lorentz equation has pathological solutions, one wonders whether Eq. (3.1) also has these undesirable solutions. The answer was given by Moniz and Sharp.[23] They proved that *both* the self-acceleration solutions *and* the preacceleration solutions are absent provided the radius of the sphere $a>\tau_0$, where τ_0 is 2/3 the classical electron radius e^2/m_0. Equivalently, this condition can also be stated as

$$m_0>m_{\text{ed}}. \tag{3.6}$$

Note that the preacceleration solutions are absent because *all* orders of a have been included here within the linear approximation. No expansion that requires analyticity is reflected in (3.1). An example of how the solutions of Eq. (3.1) compare with those of the Lorentz–Abraham–Dirac equation was given by Levine, Moniz, and Sharp.[24]

Thus one sees that within the validity limits (3.6), the equations of motion (3.1) of a finite size sphere provide a fully acceptable classical nonrelativistic dynamics. Is there a relativistic generalization of these equations that corresponds to the relativistic generalization of the Lorentz equation to the Lorentz–Abraham–Dirac equation? The answer to this question was first given more than 40 years ago by Caldirola.[25] He conjectured the equation (here written in slightly different form)

$$m\dot{v}^{\mu}=F_L^{\mu}+m_{\mathrm{ed}}\frac{1}{2a}\left[v^{\mu}(\tau-2a)\right.$$
$$\left.+v^{\mu}(\tau)v^{\alpha}(\tau)v_{\alpha}(\tau-2a)\right]. \tag{3.7}$$

But he could not prove it. The proof was given much later by Yaghjian (see Ref. 14, Appendix D).

His proof starts from first principles: the Maxwell equations and the conservation laws. The approximations made are the same as those in the derivation of Jackson's (17.28): the nonlinear terms are neglected, except that here the fully relativistic nature of the theory is kept. The same trick of working in the instantaneous rest frame of the sphere is also used. This yields for the self-force

$$\mathbf{F}_S(t)=\int\int d\mathbf{E}(\mathbf{x},t)\rho(\mathbf{x})d^3x$$
$$=-\frac{2}{3}\int\int\frac{\mathbf{v}(t-R)}{R}\rho(\mathbf{x}')d^3x'\rho(\mathbf{x})d^3x,$$

where $R=|\mathbf{x}(t)-\mathbf{x}'(t)|$. It is now easy to expand $\mathbf{v}(t-R)$ in powers of R. After integration over a surface charge one obtains an infinite series for $\mathbf{F}_s(t)$ which—fortunately—can be summed. The result is the simple expression

$$\mathbf{F}_S(t)=m_{\mathrm{ed}}\frac{1}{2a}\mathbf{v}(t-2a). \tag{3.8}$$

This is the relativistic self-force in the instantaneous rest frame in the linear approximation. One can Lorentz transform it to a general frame. Alternatively, one can apply the argument used near the end of Sec. II to derive the Lorentz–Abraham–Dirac equation. Note that the projection tensor $P^{\mu\nu}$ in the instantaneous rest frame is just the identity tensor for the space part and zero for the time part: $P^{kl}=\delta^{kl}$, $P^{\mu 0}=0$. Therefore (3.8) generalizes to

$$F_S{}^{\mu\nu}(\tau)=m_{\mathrm{ed}}\frac{1}{2a}P^{\mu\nu}(\tau)v_{\nu}(\tau-2a)$$
$$=m_{\mathrm{ed}}\frac{1}{2a}\left[v^{\mu}(\tau-2a)+v^{\mu}(\tau)v^{\nu}(\tau)v_{\nu}(\tau-2a)\right], \tag{3.9}$$

which is exactly what is needed to yield the Caldirola equation (3.7).

In order to interpret the Caldirola equation, it is desirable to write it in the form

$$m\dot{v}^{\mu}=F_L^{\mu}+F_C^{\mu}+F_R^{\mu}, \tag{3.10a}$$

$$F_C^{\mu}=m_{\mathrm{ed}}\frac{1}{2a}\left[v^{\mu}(\tau-2a)-v^{\mu}(\tau)\right], \tag{3.10b}$$

$$F_R^{\mu}=m_{\mathrm{ed}}\frac{1}{2a}\left[v^{\mu}(\tau)+v^{\mu}(\tau)v^{\nu}(\tau)v_{\nu}(\tau-2a)\right]. \tag{3.10c}$$

The force F_C^{μ} is due to the comoving (nonradiative) field. In the point limit, it reduces to the electromagnetic inertial term and the Schott term; these terms now receive an explanation: they are part of the self-force due to the comoving field. As Lorentz predicted a long time ago, his v^{μ} is only the first approximation of something much more complicated. The force F_R^{μ} is due to radiation leaving the sphere and is properly called radiation reaction; it is a generalization of (2.6).

Equation (3.7) can be renormalized,

$$(m_0-m_{\mathrm{ed}})\dot{v}^{\mu}=F_L^{\mu}+m_{\mathrm{ed}}\frac{1}{2a}\left[v^{\mu}(\tau-2a)\right.$$
$$\left.+v^{\mu}(\tau)v^{\alpha}(\tau)v_{\alpha}(\tau-2a)\right]. \tag{3.7$'$}$$

Caldirola showed that this equation has the correct limits: it reduces to the Lorentz–Abraham–Dirac equation in the limit $a\to 0$ and to (3.1) in the nonrelativistic limit. He also showed that his equation has no pathological solutions, no runaway, and no preacceleration solutions. This is, of course, not surprising because these same pathologies would already appear in the nonrelativistic case (3.1), and for that case it has already been shown that they are absent.[23]

The validity limits of the Caldirola equation are determined by the fact that (1) it is a classical equation and therefore is not valid for quantum mechanical sizes of the radius a (from the analysis of the pathological solutions[23] this means $m_0>m_{\mathrm{ed}}$ or $a>2r_0/3$, where r_0 is the classical electron radius, $r_0=e^2/m_0$), and (2) because of the approximations made in its derivation, it is valid only whenever nonlinear terms in the instantaneous rest frame are negligible.

IV. CONCLUDING REMARKS

If the dynamics of a *classical* surface-charged sphere is not valid in the quantum domain, what is the *correct* dynamics at a very small radius or even at zero radius? To answer this question, a nonrelativistic calculation was carried out by Moniz and Sharp.[23] They derived equations of motion for a finite size *quantum mechanical* charge. They found an infinite order differential equation, i.e., an infinite series of derivatives that apparently cannot be summed. The first term of that series is a finite electromagnetic inertial term corresponding to a finite (nondivergent) mass m_{MS}. If, for a finite a one takes the limit of that term to classical physics ($\hbar\to 0$), one obtains $m_{\mathrm{MS}}=m_{\mathrm{ed}}$ of (2.8). But if one takes the point particle limit of these *quantum mechanical* equations, the expression for m_{MS} does *not* diverge. In fact, in the point particle limit the result is $m_{\mathrm{MS}}=0$! A nonrelativistic *quantum* point particle has *no* electromagnetic mass.[26]

Parenthetically, it is good to know that the nonrelativistic quantum dynamics for a charged sphere is also free of pathological solutions[23] as long as the fine structure constant satisfies $\alpha<1$.

Returning to the overview of classical charged particle dynamics, one can summarize the present situation as very satisfactory: for a charged sphere there now exist equations of motion both relativistically and nonrelativistically that make sense and that are free of the problems that have plagued the theory for most of this century; these equations have no unphysical solution, no runaways, and no preaccelerations. For the electron which is physically a point charge, quantum mechanics must be used (and gives satisfactory results). Only when all distances involved are in the classical domain is classical dynamics acceptable for electrons.

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APPENDIX

The force equation (2.7) and the power equation (2.10) each contain contributions from the self-field (the last two terms in these equations). In order to make these equations frame independent, special relativity requires force and power to form a four-vector (when derivatives are taken with respect to τ rather than t). In addition, this four-vector must be orthogonal to v^μ as was pointed out preceding Eq. (2.13). Since the last terms of (2.7) and (2.10) form Γ^μ, (2.9), which *is* a four-vector and which *is* orthogonal to v^μ, the remaining concern is with the inertial terms $d\mathbf{P}_A/d\tau$ and $dE_A/d\tau$, where

$$\mathbf{P}_A = m_{\mathrm{ed}}\gamma\mathbf{v} \quad E_A = m_{\mathrm{ed}}\gamma[1-1/(4\gamma^2)] \tag{A.1}$$

are the momentum and energy due to the mass m_{ed}. $\mathbf{P}_A$ and E_A do not form a four-vector. In the nonrelativistic limit they are

$$\mathbf{P}_A = m_{\mathrm{ed}}\mathbf{v} = 4/3\ m_{\mathrm{es}}\mathbf{v}, \quad E_A = m_{\mathrm{es}} \tag*{(A1)$_{\mathrm{NR}}$}$$

The offending factor of 4/3 gave this problem its name, ‘‘the 4/3 problem.’’ But the four-vector character was not a problem for Abraham when he derived his equations before relativity.

The key to its solution *as well as a solution* had been provided some 90 years ago. It was given by Poincaré in 1906.[27] He pointed out that a spherical shell of charge is not stable and would explode unless it is stabilized by centripetal forces (Poincaré stresses). For a macroscopic charged insulator these are the forces that hold the surface charge to the insulator. There is also an associated binding energy. In the rest frame, the Poincaré binding force per unit surface charge is

$$\mathbf{f}_P = -(\mathbf{r}/r)e/(8\pi a^2). \tag{A.2}$$

For a moving frame, the sphere becomes a spheroid and one must integrate $\mathbf{f}_P$ and the work done by it over that surface. That calculation was carried out by Yaghjian (Ref. 14, Sec. 4.1) in the spirit of Abraham and Lorentz. But there can be, in addition, a binding energy that is already present in the rest frame, $E_P(0)$. Thus, one finds $\mathbf{P}_P=0$, $E_P=E_P(0)+(1/4)m_{\mathrm{ed}}(1/\gamma-1)$. When this is added to (A.1), one obtains a four-vector

$$P^\mu_{\mathrm{total}} = Mv^\mu \tag{A.3}$$

with $M=m_{\mathrm{ed}}$, provided $E_P(0)$ is chosen to be $m_{\mathrm{ed}}/4$.

The physical relationship between the stability of the charged sphere (or electron) and the frame independence of the expression for energy and momentum is reflected mathematically in the relationship between the vanishing of the divergence of the energy–momentum–stress tensor of a closed system and the four-vector character of its integral, the energy–momentum: $P^\mu = \int T^{\mu\nu} d\sigma_\nu$ is independent of the surface σ and is therefore a four-vector, if and only if $\partial_\nu T^{\mu\nu}=0$.

More than 30 years after Abraham, Poincaré, and Lorentz, a calculation was made by Dirac that was manifestly covariant throughout. Dirac *completely ignored* the stabilizing force and still found a covariant answer! This answer was the result of the manifest covariance which he preserved throughout his calculation. Proceeding in this way, a covariant result is thus guaranteed even when the physical considerations (stability of the charged object) are ignored. Note, however, that he obtained (A.3) with $M=m_{\mathrm{es}}$ rather than m_{ed}; his inertial term is thus just the Lorentz boosted electrostatic energy of a spherical shell in the rest frame, $P^0_D(0)=m_{\mathrm{es}}$.

Dirac's work teaches two things: (1) stabilizing forces are not necessary for obtaining a covariant result (though they are, of course, physically necessary for stability); and (2) the separation into electromagnetic and stabilizing forces can be done either noncovariantly (as was done by Abraham and Poincaré above) or covariantly. A noncovariant separation makes good sense because the stabilizing forces are physically complicated atomic and molecular forces that hold extra electrons to neutral atoms and molecules, or that hold positive ions to the neutral solid substrate. They are not necessarily all of classical electromagnetic origin—they could be of quantum mechanical nature such as covalent bonds, for example—and they are therefore physically not cleanly separable from classical electromagnetic forces. But if one is primarily interested in covariant relativistic equations of motion and one is willing to ignore the problem of how the charge is attached to the spherical insulator, a *covariant* formulation of the charged shell is preferable. And that is exactly what Dirac has done in 1938 (in the limit of a point charge) and what was adopted in (2.12) above.[28]

If one *does* want to include the stabilizing forces, one can proceed *either* noncovariantly *or* manifestly covariant. The former has the advantage of permitting detailed physical considerations more easily, the latter is mathematically simpler. Both have been used in the literature and they lead, of course, to identical results. Also, one must realize that the Poincaré choice (A.2) is not unique; physically, one might expect the cohesive forces to be associated with an energy as was observed by Abraham. Depending on whether the coherence energy is chosen to be zero or nonzero, the results for P^μ_{total} will differ. But both ways of computing give a four-vector if the calculation is done covariantly (see below).

Noncovariant calculations of the electromagnetic momentum and energy for *stabilized* charged spheres (and other charged objects) have been made for several different models.[29] All models yield covariant results, i.e., four-vectors for the total energy–momentum, $P^\mu_{\mathrm{total}}=Mv^\mu$. But the value of M depends on the model.

Similarly, *manifestly covariant* calculations have been made for different models. Again, the value of M in P^μ_{total} depends on the model; but for a given model they are the *same* as obtained for a noncovariant separation.[30]

An instructive example is the manifestly covariant calculation made by Schwinger.[31] He constructed energy–momentum–stress tensors that assure stability: he observed that there are two possible choices,

$$ {}^{1}T_{s}^{\mu\nu}=P^{\mu\nu}t, \quad {}^{2}T_{s}^{\mu\nu}=\eta^{\mu\nu}t, \quad t=\frac{e^{2}}{8\pi a^{4}}\,\theta(a^{2}-x_{\perp}^{2}), \tag{A.4} $$

where t corresponds to (A.2), θ is the step function, and $x_{\perp}^{\mu}=P^{\mu\nu}x_{\nu}$, and $P^{\mu\nu}$ is defined in (2.13). That tensor can be added to the electromagnetic tensor and one finds with $d\sigma_{\nu}=v_{\nu}d\sigma$,

$$ P_{\text{total}}^{\mu}=\int (T_{em}^{\mu\nu}+T_{s}^{\mu\nu})v_{\nu}d\sigma=Mv^{\mu}. \tag{A.5} $$

The result is either ${}^{1}M=m_{\text{es}}$ or ${}^{2}M=m_{\text{ed}}$. In the first case, the stabilizing tensor contributes nothing because $P^{\mu\nu}$ is orthogonal to v^{μ}. That is the case where $E_{P}=0$. In the second case, $E_{P}=m_{\text{es}}/3$ in the rest frame. One can, of course, choose the coherence energy density to be only a fraction, f, of t. In that case, $T_{s}^{\mu\nu}=(1-f)\,{}^{1}T_{s}^{\mu\nu}+f\,{}^{2}T_{s}^{\mu\nu}$, and one finds $M=(1+f/4)m_{\text{es}}$. The argument over whether m_{es} or $m_{\text{ed}}=4m_{\text{es}}/3$ is the ''right'' answer is thus resolved: again, it depends on the model; either value as well as any value in between is possible. But in all cases, one obtains a four-vector for the stabilized charged sphere.

ACKNOWLEDGMENT

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[28] This is also one reason why I wanted to formulate the electromagnetic four-momentum in a manifestly covariant way. The actual value of M is here unimportant. See my book, Ref. 13, p 90, and my ''Electromagnetic energy, momentum, and mass,'' Am. J. Phys. **38**, 1310–1316 (1970).

[29] See, for example, T. H. Boyer, ''Classical model of the electron and the definition of the electromagnetic field momentum,'' Phys. Rev. D **25**, 3246–3250 (1982); E. Comay, ''Lorentz transformation of electromagnetic systems and the 4/3 problem,'' Z. Natuforschlung **46a** 377–383 (1991); I. Bialynicki-Birula, ''Classical model of the electron; exactly solvable example,'' Phys. Rev. D **28**, 2114–2117 (1983). The very recent paper by V. Hnizdo, ''Hidden momentum and the electromagnetic mass of a charge and current carrying body,'' Am. J. Phys. **65**, 55–65 (1997), is especially instructive.

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NEWTON, FORGIVE ME

Enough of this. Newton, forgive me; you found the only way which, in your age, was just about possible for a man of highest thought- and creative power. The concepts, which you created, are even today still guiding our thinking in physics, although we now know that they will have to be replaced by others farther removed from the sphere of immediate experience, if we aim at a profounder understanding of relationships.

Albert Einstein, ''Autobiographical Notes,'' in *Albert Einstein: Philosopher—Scientist*, edited by Paul Arthur Schilpp (Harper, New York, 1951), Vol. I, pp. 31, 33.