Mass Renormalization and Radiation Damping for a Charged Particle in Uniform Circular Motion

TIMOTHY H. BOYER
Department of Physics
City College of the City University of New York
New York, New York 10031
(Received 7 February 1972; revised 30 May 1972)

The example of a charged point particle in uniform circular motion is treated from a pedagogical point of view within classical electrodynamics. The elementary analysis is of interest because it clearly separates out the finite radiation reaction force from the divergent mass renormalization term, and also because the particle is always treated as a point charge with the self-force found from evaluating the point charge fields over a surrounding spherical surface.

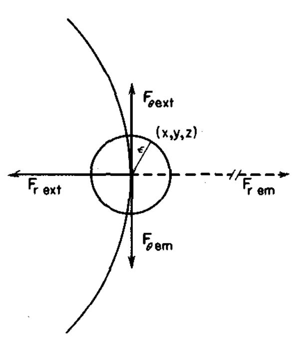

FIG. 1. Forces on a point charge in uniform circular motion.

A classical charged particle in uniform circular motion provides a convenient and elementary example of some aspects of mass renormalization and radiation damping in classical electromagnetism. The ideas required for the analysis are available to students of course in intermediate-level electromagnetism. In this note, we will sketch one possible approach to this problem which very clearly separates out a finite radiation damping force from the divergent mass renormalization force. For uniform circular motion, these two forces are at right angles to each other.

Furthermore, the approach appears surprising to some instructors in electromagnetism. The account works with a *point* charge and averages the fields due to that point charge over a small surrounding surface so as to define the self-force. This procedure using a point charge is quite different from the traditional textbook approach through a charge distribution of finite size whose various parts interact so as to give a self-force on the distribution.

NEWTON'S LAWS, ENERGY CONSERVATION, AND FORCES.

External centripetal force. We consider a particle of mass m and charge e moving with uniform angular velocity ω in a circle of radius R, and ask what are the external forces required to maintain this motion. The initial response from any student of mechanics is simply a centripetal force

$$F_{r\ \mathrm{ext}} = -m\omega^2 R, \tag{1}$$

required to provide the centripetal acceleration $-\omega^2 R = -v^2/R$.

External tangential force. However, we now remark that the charged particle, because it is accelerated, must radiate away energy into the electromagnetic field. What external source provides this energy? Clearly the centripetal force $F_{r\ \mathrm{ext}}$ can not supply energy because it is always radial and hence perpendicular to the motion of the particle. Thus energy conservation requires that there be an external force $F_{\theta\,\mathrm{ext}}$ in the direction of particle motion.

Radiation damping force. The argument is still not complete because a moment's reflection reminds us that an external force $F_{\theta\,\mathrm{ext}}$ in the direction of particle motion will try to accelerate the particle in the direction of motion. However, we assume that the particle was in uniform circular motion. Hence the net acceleration in the direction of motion must vanish, and also the net force on the particle in the direction of motion must vanish. We must have a further force $F_{\theta\,\mathrm{em}}$ which holds the particle back and balances $F_{\theta\,\mathrm{ext}}$,

$$F_{\theta\,\mathrm{em}}=-F_{\theta\,\mathrm{ext}}. \tag{2}$$

This additional force $F_{\theta\,\mathrm{em}}$ is the radiation damping force provided by the electromagnetic field to which the charged particle is tied.

RADIATION DAMPING AND MASS RENORMALIZATION RELATED TO THE LORENTZ SELF-FORCE

Electromagnetic and nonelectromagnetic forces. Having found the necessity of a radiation damping force $F_{\theta\,\mathrm{em}}$, we now attempt to understand the force in an elementary way. First of all, we remark that the external force given by the radial and centripetal components, $F_{r\,\mathrm{ext}}$ and $F_{\theta\,\mathrm{ext}}$, may be thought of as nonelectromagnetic in origin, and even imagined in terms of little men pushing the particle toward the center of the circle and along its circumference. However, all electromagnetic forces are specified in terms of the Lorentz force,

$$\mathbf{F}=e(\mathbf{E}+\mathbf{v}\times\mathbf{B}). \tag{3}$$

The radiation damping force is clearly an electromagnetic force; how can we reconcile it with the Lorentz force?

Electromagnetic field at the particle as an average. In order to use the Lorentz force, we must know the value of the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ at the position of the charge e. Now a point charge involves fields falling off as r^{-2} where r is the distance from the charge. These fields diverge as $r\to 0$, and so do not provide meaningful answers at $r=0$. However, intuition suggests that one should average the fields surrounding the particle, and define this average value as the field at the particle. The Lorentz force then follows just as in Eq. (3). This intuitive procedure can indeed be carried out in a rigorous way[1] and is equivalent to the more familiar treatments of self-forces and mass renormalization.

Averaging the electric field over a sphere. Returning to the specific problem of a charge in uniform circular motion, we may, in the nonrelativistic approximation, average the electromagnetic fields over a small sphere surrounding the charge. (The fields are taken as those given by the retarded Green's function.) The actual calculations are provided in a short appendix at the end of this note.

Calculated electromagnetic retarding force. The θ component of the electric field at the charged particle, obtained by averaging over a sphere of radius ϵ and allowing $\epsilon\to 0$, is just

$$E_{\theta\,\mathrm{self}}=-\tfrac{2}{3}e\omega^3R/c^3. \tag{4}$$

(See the last equation in the appendix). Thus the component in the direction of particle motion for the Lorentz force on the charge due to its own electric fields is

$$\begin{aligned}F_{\theta\,\mathrm{em}}&=eE_{\theta\,\mathrm{self}}\\&=-\tfrac{2}{3}(e^2\omega^3R/c^3)\\&=\tfrac{2}{3}e^2(\dddot{\mathbf{x}})_\theta/c^3,\end{aligned} \tag{5}$$

which is the familiar radiation damping force appearing in electromagnetism textbooks.[2] Since Newton's second law required the relationship between the external force and the radiation damping given in Eq. (2), we have that the power expended by the external force is just

$$P=F_{\theta\,\mathrm{ext}}\omega R=\tfrac{2}{3}e^2\omega^4R^2/c^3. \tag{6}$$

The power radiated by a charged particle into the electromagnetic field appears in textbooks as[3]

$$P=\tfrac{2}{3}(e^2/c^3)(\ddot{x})^2$$
$$=\tfrac{2}{3}(e^2/c^3)(\omega^2R)^2. \qquad (7)$$

Thus indeed the external force in the direction of motion provides the power radiated away by the charge.

In the case that the external force $F_{\theta\ \mathrm{ext}}$ is not present, then the self-force $F_{\theta\ \mathrm{em}}$ will slow the particle motion. This is the case for realistic situations involving a charged particle moving in a uniform magnetic field, or a charged particle in an attractive central potential. The energy loss due to radiation is compensated by the change in particle potential and kinetic energy.

Calculated electromagnetic centrifugal force. The radial component of the electric field when averaged over a sphere of radius ϵ is of the form

$$E_{r\ \mathrm{self}}=(e\omega^2/\epsilon)\ \mathrm{const}+O(1)$$

where the constant is positive. Thus the field $E_{r\ \mathrm{self}}$ at the position of the particle is divergent, and the Lorentz self-force is divergent. It is precisely this divergent self-force which is involved in the concept of mass renormalization. The self-force behaves as ω^2, and hence as a term involving the particle acceleration $\ddot{\mathbf{x}}$. The (divergent) centripetal self-force is regarded as combined with a (divergent) negative bare mass for the particle to give a finite renormalized mass m which responds to the external force $F_{r\ \mathrm{ext}}$ as in Eq. (1).

Energy and momentum singularities and the self-forces. On an intuitive level, it seems clear why the self-force $F_{\theta\ \mathrm{em}}$ is finite for this example, while $F_{r\ \mathrm{em}}$ diverges. The energy in the electromagnetic field has a singularity at the point charge. However, since the speed of the particle is constant, there are no changes in this singular energy, but only in the radiated energy which involves no singularities. Thus $F_{\theta\ \mathrm{em}}$ is finite and is tied to finite energy changes. However, the acceleration of the point charge involves changes in the direction of the electromagnetic momentum which is singular at the position of the point charge. The divergent self-force $F_{r\ \mathrm{em}}$ is required to account for the change in the direction of the singular electromagnetic momentum.

CONCLUSION

The example of a point charge in uniform circular motion is a convenient source of illustrations for some of the concepts and complications of radiation damping and mass renormalization in classical electromagnetism.

APPENDIX

Averaging the Self-Field Over a Small Sphere Around the Charged Particle

In this appendix, we will sketch an unsophisticated calculation of the self-fields of a point charge in uniform circular motion. The electric field of a point charge is given by[4]

$$\mathbf{E}(\mathbf{x},t)=e[(\hat{n}-\boldsymbol{\beta})(1-\beta^2)/K^3r^2]_{\mathrm{ret}} + (e/c)[(\hat{n}/K^3r)\times\{(\hat{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}]_{\mathrm{ret}}$$

where $K=1-\hat{n}\cdot\boldsymbol{\beta}$, and the expressions are evaluated at the retarded time. We wish to evaluate this expression for the case of a charge in uniform circular motion when the field point (x,y,z) is very close to the point charge. Referring to Fig. 1 indicating the point charge at time $t=0$, we will take the y axis as upwards and the x axis to the right so that the particle position $\mathbf{r}_e$ at time t is

$$x_e=R\cos\omega t-R,$$
$$y_e=R\sin\omega t,$$
$$z_e=0.$$

Since the field point (x,y,z) is a small distance ϵ from the charge, we will expand the expression for the field at time $t=0$ in terms of $\epsilon=(x^2+y^2+z^2)^{1/2}$. Although it is not difficult, (only tedious), to carry all terms in (v/c) for the fields, it is not instructive for this note. Hence we will make the second approximation that we will keep only the lowest contributing terms in the particle frequency ω.

Expanding the sine and cosine functions about $t=0$, we require for the velocity fields

$$\mathbf{r}_e = (-\omega^2 R t^2/2)\hat{\imath} + \omega R t \hat{\jmath},$$

$$\boldsymbol{\beta} = -(\omega^2 R t/c)\hat{\imath} + (\omega R/c)\hat{\jmath}.$$

Then

$$\hat{n} = (\mathbf{x} - \mathbf{r}_e)/|\mathbf{x} - \mathbf{r}_e|,$$

$$\hat{n} - \boldsymbol{\beta} = [(x - \tfrac{1}{2}\omega^2 R t^2)\hat{\imath} + y\hat{\jmath} + z\hat{k}]/(-ct),$$

and the velocity field is

$$e[(\hat{n} - \boldsymbol{\beta})/r^2]_{\text{ret}} = e\epsilon^{-3}\{[x - (\omega^2 R \epsilon^2/2c^2)]\hat{\imath} + y\hat{\jmath} + z\hat{k}\}$$

where we have approximated the retarded time t by

$$-ct = r \cong \epsilon.$$

For the radiation fields it is sufficient to take

$$\hat{n} = (x\hat{\imath} + y\hat{\jmath} + z\hat{k})/(-ct),$$

$$\boldsymbol{\beta} = (-\omega^2 R\hat{\imath} - \omega^3 R t\hat{\jmath})/c$$

giving

$$(e/c)[\hat{n} \times (\hat{n} \times \boldsymbol{\beta})/r]_{\text{ret}}$$

$$= \epsilon^{-3}\frac{e}{c}\left\{\frac{(\epsilon^2 - x^2)\omega^2 R}{c}\hat{\imath} + \left[\frac{-xy\omega^2 R}{c} + \frac{(y^2 - \epsilon^2)\omega^3 R\epsilon}{c^2}\right]\hat{\jmath} + \left(\frac{-xz\omega^2 R}{c} + \frac{yz\omega^3 R\epsilon}{c^2}\right)\hat{k}\right\}.$$

Both contributions to the electric field diverge as $\epsilon \to 0$. However, when averaged over the sphere of radius ϵ centered on the particle at time $t=0$, then all terms which are odd in x, y, or z vanish. Recalling that x^2 or y^2 or z^2 gives the value $\frac{1}{3}\epsilon^2$ when averaged over the sphere of radius ϵ, we have

$$\mathbf{E}_{\text{self}} = \frac{e\omega^2 R}{6c^2\epsilon}\hat{\imath} - \frac{2}{3}\frac{e\omega^3 R}{c^3}\hat{\jmath},$$

just as required in the note.

[1] The idea of defining the fields at a charged particle by averaging the self-fields has occurred to a number of persons before the present author. Recently an analysis has been published by C. Teitelboim [Phys. Rev. D **4**, 345 (1971)], which gives references to several people who published similar ideas still earlier.

[2] See, for example, J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962), p. 582, Eq. (17.8).

[3] See Ref. 2, p. 469, Eq. (14.22), or J. R. Reitz and F. J. Milford, *Foundations of Electromagnetic Theory* (Addison-Wesley, Reading, Mass., 1967), 2nd ed., p. 354, Eq. (16–161).

[4] See Ref. 2, p. 467, Eq. (14.14).