PHYSICS 121 | FLIP'S SECTION #2 | 4/18/2006

FLIP TANEDO
flipt@stanford.edu
EUCALYPTO 240

AGENDA: ANNOUNCEMENTS, TENSORS, WAVES

ANNOUNCEMENTS

1) OFFICE HOURS THIS WEEK: THU. AFTERNOON OR BY APPOINTMENT
→ NO OH / email CONTACT THIS WEEKEND!

TENSORS - A FIRST PASS

"PHYSICISTS ARE MATHEMATICIANS IN A HURRY." (B. MANDELBROT)

Meaning: PHYSICISTS & MATHEMATICIANS SPEAK THE SAME LANGUAGE, BUT HAVE VERY DIFFERENT DIALECTS.

I WILL DESCRIBE A PHYSICIST'S* UNDERSTANDING OF TENSORS, AT LEAST A FIRST PASS.
(* - NOT COUNTING GENERAL RELATIVISTS, WHO ARE REALLY DIFFERENTIAL GEOMETERS IN DISGUISE!)

→ MATHEMATICIANS HAVE THEIR OWN FANCY DEFINITIONS & MACHINERY TO UNDERSTAND TENSORS. FOR NOW THIS IS ALL UNNECESSARY.
→ THOSE OF YOU WHO HAVE TAKEN A COURSE IN GENERAL RELATIVITY KNOW ALL OF THIS ALREADY.

QUICK & DIRTY DEF.: TENSORS ARE GENERALIZATIONS OF VECTORS & MATRICES.

1) SCALARS - JUST NUMBERS
UNDER A ROTATION OF COORDINATES, SCALARS DO NOT CHANGE (THEY ARE INVARIANT)

2) VECTORS - COLLECTIONS OF SCALARS

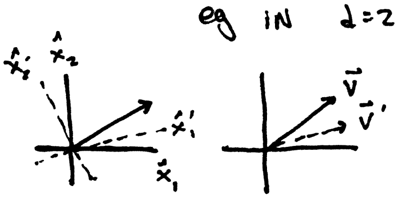
$\vec{V} = (V_x, V_y, V_z)$ or $= (V_1, V_2, V_3, \ldots, V_d)$ for d-dim

WE CAN USE TENSOR (INDEX) NOTATION: V_i

EINSTEIN SUM. CONVENTION

UNDER A ROTATION W/ MATRIX R, $\vec{V} \to \vec{V}' = R\vec{V}$
$V_i \to V_i' = \sum_j R_{ij} V_j = \boxed{R_{ij} V_j}$

eg IN $d=2$: $\vec{V} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$ ROTATE BY θ ⇓



$\vec{V} \to R\vec{V}$ $\quad R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$V_1' = \cos\theta\, V_1 + \sin\theta\, V_2$
$V_2' = -\sin\theta\, V_1 + \cos\theta\, V_1$

⇒ "COVARIANT" (mathy types will say "contravariant")

3) <u>MATRICES</u> - 2d GRID OF #'s (COLLECTION OF VECTORS)
→ think of as a LINEAR TRANSFORMATION

$M = m_{ij}$ ← TWO INDICES: i^{th} ROW, j^{th} COLUMN

Q. HOW DOES A LIN. TRANSF. CHANGE UNDER A $\underbrace{\text{CHANGE OF BASIS}}_{\text{IN OUR CASE, A ROTATION}}$, S?
A. RECALL THAT $M \longmapsto M' = SMS^T$
WHERE THE CHANGE OF BASIS TAKES
VECTORS, $\vec{v}$, TO $\vec{v}' = S\vec{v}$

DO YOU SEE WHY? $(M\vec{v})$ IS A VECTOR, SO UNDER TRANSF.
$(M\vec{v}) \longleftrightarrow (M\vec{v})' = SM\vec{v})$
SINCE $S^T = S^{-1}$ FOR ROT, THEN $\vec{v} = S^T\vec{v}'$
$\Rightarrow (M\vec{v})' = SMS^T\vec{v}'$
$M'\vec{v}' = (SMS^T)\vec{v}'$ ✓

OR, ALTERNATIVELY: $\vec{w}^T M\vec{v}$ IS A SCALAR, SO:

$$\vec{w}^T M\vec{v} \longleftrightarrow (\vec{w}')^T M' \vec{v}' = \vec{w}^T M\vec{v}$$
$$= (S^T\vec{w}')^T M (S^T\vec{v}')$$
$$= (\vec{w}')^T \underbrace{SMS^T}_{M'} \vec{v}'$$

IN INDICES:

$$M \to SMS^T$$
$$m_{ij} \to S_{ik} M_{kl} (S^T)_{lj}$$
$$= S_{ik} M_{kl} S_{jl}$$
$$= \boxed{S_{ik} S_{jl} M_{kl}}$$

the usual book notation
write $S = R$

4) CAN WE GENERALIZE THIS?
<u>n-tensor</u> : n dim grid of #'s (collection of (n-1) tensors)

$T = T_{i_1 i_2 \dots i_n}$ ← n INDICES

VECTOR: $V_i \longrightarrow R_{ij} V_j$
MATRIX: $M_{ij} \longrightarrow R_{ik} R_{jl} M_{kl}$
⋮
n-TENSOR $T_{i_1 \dots i_n} \longrightarrow R_{i_1 k_1} R_{i_2 k_2} \cdots R_{i_n k_n} T_{k_1 \dots k_n}$

think of this as: WE HAVE TO ROTATE EA. INDEX
why? ANALOG TO MATRIX
- CAN "CONTRACT" (will define later) $T_{i_1 \dots i_n}$ w/ n VECTORS TO FORM A SCALAR. NEED n ROTATIONS TO "COUNTER" THE ROTATIONS OF EACH VECTOR

TENSORS - A SECOND PASS

NOW WE'VE MOTIVATED WHAT A TENSOR IS - LET'S BE SLIGHTLY MORE FORMAL

INDEX NOTATION (EINSTEIN SUMMATION CONVENTION)

- WRITE TENSORS IN TERMS OF AN ARBITRARY ELEMENT (eg v_i FOR $\vec{v}$)

→ THIS WILL MAKE MANIPULATIONS MORE TRANSPARENT

- SUM OVER REPEATED INDICES:

$$M_{ij} V_j \equiv \sum_j M_{ij} V_j \qquad (= M\vec{v})$$

$$V_j V_j \equiv \sum_j V_j V_j \qquad (= \vec{v}\cdot\vec{v})$$

sanity check: make sure you understand this is equiv. to "old" matrix mult. rule!

$\{ \vec{v}^T \vec{v}$ IN MATRIX MULT ? $\}$

SUMMING THIS WAY IS CALLED <u>CONTRACTION</u>.

HOWEVER, THERE ARE ACTUALLY 2 KINDS OF VECTORS:

- COLUMN VECTORS : $\vec{v}$
- ROW VECTORS : $\vec{v}^T$

DISTINCTION IS IMPORTANT IN "OLD WAY" OF THINKING SINCE WE USE THE MATRIX MULTIPLICATION RULE:

$$(v_1, v_2, \ldots, v_n) \begin{pmatrix} M_{11} & \cdots & \\ \vdots & & \\ & & M_{nn} \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

$\vec{v}^T M \vec{v}$ ⇗

SO LET'S NOW DISTINGUISH BTWN. ROW & COLUMN VECTORS BY USING UPPER & LOWER INDICES:

- COLUMN VECTOR : v^i → "CONTRAVARIANT"
- ROW VECTOR : v_i → "COVARIANT"

<u>NOTICE</u>

- JUST AS $\vec{v}\vec{v}$ DOESN'T PRODUCE A SCALAR IN MAT. MULT
  $v^i v^i$ DOESN'T EITHER

  → BUT $\vec{v}^T\vec{v}$ IS A SCALAR $(= \vec{v}\cdot\vec{v} = v^2)$
  AND SO IS $v^i v_i$

⇒ SO OUR NEW EINSTEIN RULE IS THAT WE SUM OVER UPPER INDICES "CONTRACTED" W/ LOWER INDICES

⟶ NOTE $v^i w_i = v_i w^i = w^i v_i = w_i v^i$

- MATRIX MULT: $\vec{v}^T M \vec{v} \rightarrow v_i M^i{}_j v^j$

MATRICES HAVE UPPER & LOWER INDEX

NOTE FURTHER THAT THIS WHOLE CONTRAVARIANT/COVARIANT BUSINESS IS A RESULT OF THE FACT THAT ROW/COL VECTORS TRANSFORM DIFFERENTLY UNDER ROTATIONS!

$$\vec{V} \longrightarrow R\vec{V} \qquad\qquad v^i \to R^i{}_j v^j$$

$$\vec{V}^T \longrightarrow (R\vec{V})^T = \vec{V}^T R^T \qquad\qquad v_i \to R^j{}_i v_j$$

TRANSFORMS W/ THE INVERSE MATRIX!

<u>SLIGHTLY MATHEMATICAL EXAMPLE</u> OF COVARIANT/CONTRAVARIANT VECTORS:

- THE PARTIAL DERIVATIVE OPERATOR IN THE x^i DIRECTION:

$$\left(\frac{\partial}{\partial x^i}\right) \longmapsto \underbrace{\frac{\partial x^j}{\partial y^i}}_{\text{CHANGE OF BASIS MATRIX, } M} \frac{\partial}{\partial x^j} \leftarrow \text{COVARIANT VECTOR}$$

- THE DIFFERENTIAL OF THE x^i COORDINATE

$$dx^i \longmapsto \underbrace{\frac{\partial y^i}{\partial x^j}}_{\text{TRANSF. AS THE INVERSE OF } M} dx^j \leftarrow \text{CONTRAVARIANT VECTOR}$$

<u>Aside</u>: in Quantum Mechanics it's the same thing w/ different names:

$|v\rangle$ = CONTRAVARIANT VECTOR ("KET")
$\langle v|$ = COVARIANT VECTOR ("BRA")

SCALAR $\langle v|v\rangle$ = "BRACKET" (= PROBABILITY^2)

as you know from Phys 70, $|v\rangle$ ACTUALLY REPRESENTS A (WAVE) <u>FUNCTION</u>!
THE LINEAR ALGEBRAIC POINT OF VIEW CALLS THIS SPACE A <u>HILBERT SPACE</u>.

WE NOTICED THAT COVARIANT & CONTRAVARIANT INDICES REQUIRE DIFFERENT (INVERSE) TRANSFORMATION MATRICES.

WE CAN NOW BUILD MORE COMPLICATED TENSORS. INSTEAD OF JUST "n-TENSORS", WE CALL THEM

$$\underset{\text{CONTRAVARIANT}}{n},\ \underset{\text{COVARIANT}}{m}\text{)- TENSORS} \quad : \quad T^{i_1 \cdots i_n}{}_{j_1 \cdots j_m}$$

HOW DO THEY TRANSFORM?
CONTRACT W/ APPROPRIATE TRANSF. MATRIX FOR EA INDEX

$$\underbrace{\left(\frac{\partial y^{i_1}}{\partial x^{k_1}}\right)\left(\frac{\partial y^{i_2}}{\partial x^{k_2}}\right)\cdots\left(\frac{\partial y^{i_n}}{\partial x^{k_n}}\right)}_{\text{TRANS OF CONTRAV. PT}} \underbrace{\left(\frac{\partial x^{l_1}}{\partial y^{j_1}}\right)\left(\frac{\partial x^{l_2}}{\partial y^{j_2}}\right)\cdots\left(\frac{\partial x^{l_m}}{\partial y^{j_m}}\right)}_{\text{TRANSF OF COVARIANT PT.}} T^{i_1 \cdots i_n}{}_{j_1 \cdots j_m}$$

WHAT ABOUT THE LENGTH OF A VECTOR? (aside)
USUALLY $|\vec{v}|^2 = \vec{v} \cdot \vec{v} = \vec{v}^T \vec{v}$

how do we make a covariant vector out of a contravariant vector?

<u>METRIC</u>: tensor $\{(0,2)$ or $(2,0)\}$ TO RAISE OR LOWER INDICES
→ ALLOWS THE DEF. OF A DOT PRODUCT / CONTRACTION / NORM
→ DEFINES LENGTH

$$\eta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{EUCLIDEAN 3-SPACE}$$

$$\eta_{ij} v^j = v_i \qquad \eta^{ij} v_j = v^i$$

$$\eta_{ij} T^{i_1 \cdots i_n j}{}_{k_1 \cdots k_m} = T^{i_1 \cdots i_n}{}_{i k_1 \cdots k_m}$$

$$\eta_{ij} \eta^{jk} = \delta_i{}^k$$

<u>ANYWAY</u>: THIS IS A LITTLE BIT EXTRA, PROBABLY MORE THAN YOU'RE USED TO. THE REASON I BRING IT UP IS THAT RELATIVITY (SPECIAL / GEN) IS ALL ABOUT NON-TRIVIAL METRICS!