1. GRIFFITHS 7.8

(a) (7.19) $\gamma^\mu p_\mu - m = 0$

$\gamma^0 p^0 - \vec{\gamma}\cdot\vec{p} = m$

set $c \equiv 1$, like any reasonable particle physicist

$\gamma^0 p^0 = m + \vec{\gamma}\cdot\vec{p}$

$p^0 = \gamma^0 (m + \vec{\gamma}\cdot\vec{p})$ since $(\gamma^0)^2 = \mathbb{1}$

$\Rightarrow \boxed{H = \gamma^0 (m + \vec{\gamma}\cdot\vec{p})}$ since $p^0 = E$ $(c \equiv 1)$

(b) $\vec{L} = \vec{r} \times \vec{p}$

$$[H, \vec{L}] = [\gamma^0 m + \gamma^0 \vec{\gamma}\cdot\vec{p}, \vec{r}\times\vec{p}]$$

$$= \underbrace{[\gamma^0 m, \vec{r}\times\vec{p}]}_{=0} + [\gamma^0 \vec{\gamma}\cdot\vec{p}, \vec{r}\times\vec{p}]$$

much easier if we write using Einstein summation convention

$$= [\gamma^0 \gamma^i p^i, \varepsilon^{abc} r^b p^c] \qquad i, a, b, c \in \{1,2,3\}$$

by the way, there is an implied "$\mathbb{1}$" (spinor space) here, in case you are concerned whether we can commute these objects.

$$= \gamma^0 \gamma^i \varepsilon^{abc} [p^i, r^b p^c]$$

$$= \gamma^0 \gamma^i \varepsilon^{abc} [p^i, r^b] p^c \qquad \text{since } \vec{p} \text{ commutes with itself}$$

$$= \gamma^0 \gamma^i \varepsilon^{abc} (-i\hbar \delta^{ib}) p^c$$

$$= -i\hbar \gamma^0 (\varepsilon^{abc} \gamma^b p^c)$$

$$= \boxed{-i\hbar \gamma^0 \vec{\gamma}\times\vec{p}}$$

(c) (7.48) $\vec{S} = \hbar/2 \, \vec{\Sigma} = \hbar/2 \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$

recall that $\vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$ (7.17)

$$[H, \vec{S}] = [\underbrace{\gamma^0 m}_{} + \gamma^0 \gamma^i p^i, \hbar/2 \begin{pmatrix} \sigma^j & \\ 0 & \sigma^j \end{pmatrix}]$$

γ^0 is DIAGONAL, COMMUTATOR OF THIS TERM $= 0$

now (mathematically, at least) the nontrivial commutator comes from the matrix structure.

$$= \left[\gamma^0 \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} p^i, \frac{\hbar}{2} \begin{pmatrix} \sigma^j & 0 \\ 0 & \sigma^j \end{pmatrix}\right]$$

$$= \frac{\hbar p^i}{2} \gamma^0 \left[\begin{pmatrix} 0 & \sigma^i \sigma^j \\ -\sigma^i \sigma^j & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma^j \sigma^i \\ -\sigma^j \sigma^i & 0 \end{pmatrix}\right]$$

$$[H, \vec{S}] = \frac{\hbar p^i}{2}\gamma^0 \begin{pmatrix} 0 & [\sigma^i, \sigma^j] \\ -[\sigma^i, \sigma^j] & 0 \end{pmatrix}$$

$$= \frac{\hbar p^i \gamma^0}{2} \begin{pmatrix} 0 & 2i\epsilon^{ijk}\sigma^k \\ -2i\epsilon^{ijk}\sigma^k & 0 \end{pmatrix}$$

$$= i\hbar\, \gamma^0 \epsilon^{ijk} p^i \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}$$

$$= i\hbar\, \gamma^0 \underbrace{\epsilon^{ijk} p^i \gamma^k}_{= \epsilon^{jki}\gamma^k p^i}$$

$$= \boxed{i\hbar\gamma^0\, \vec{\gamma} \times \vec{p}}$$

ARE WE SURPRIZED THAT $[H, \vec{L}]$ LOOKS LIKE $[H, \vec{S}]$ (UP TO A SIGN)? NO, THEY'RE BOTH ANGULAR MOMENTA, BUT IT'S CUTE TO SEE THE 'SAME' RESULT COME OUT OF OPERATORS & OUT OF MATRIX MULTIPLICATION.

(d) $$(\vec{S})^2 = \frac{\hbar^2}{4} \vec{\Sigma} \cdot \vec{\Sigma}$$

$$= \frac{\hbar^2}{4} \begin{pmatrix} \sigma^i\sigma^i & 0 \\ 0 & \sigma^i\sigma^i \end{pmatrix} \quad \leftarrow (\sigma^i)^2 = \mathbb{1}$$

$$= \frac{3\hbar^2}{4} \mathbb{1}$$

$\vec{S}^2 \propto \mathbb{1}$, SO EVERY DIRAC SPINOR IS AN EIGENSTATE OF $\vec{S}^2$, WITH EIGENVALUE $3\hbar^2/4$.

IF EIGENVALUE $= \hbar^2 s(s+1) \Rightarrow \boxed{s = \frac{1}{2}}$, AS ONE WOULD EXPECT!

2. GRIFFITHS 7.9

$$C\psi = i\gamma^2\psi^* \qquad \text{where} \qquad i\gamma^2 = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix}$$

from (7.46) our plane wave solutions are (up to overall normalization)

$$u^{(1)} = \begin{pmatrix} 1 \\ 0 \\ p_z/E+m \\ p_x+ip_y/E+m \end{pmatrix} \qquad u^{(2)} = \begin{pmatrix} 0 \\ 1 \\ p_x-ip_y/E+m \\ -p_z/E+m \end{pmatrix}$$

$$i\gamma^2 u^{*(1)} = \begin{pmatrix} p_x-ip_y/E+m \\ -p_z/E+m \\ 0 \\ 1 \end{pmatrix} = v^{(1)}$$

$$i\gamma^2 u^{*(2)} = \begin{pmatrix} -p_z/E+m \\ -(p_x+ip_y)/E+m \\ -1 \\ 0 \end{pmatrix} = v^{(2)}$$

3. GRIFFITHS 7.22 (USING DEFINITIONS ABOVE)

$$\sum_{s=1,2} u^{(s)}\bar{u}^{(s)} = \sum_{s=1,2} u^{(s)}(u^{\dagger(s)}\gamma^0) \qquad \leftarrow \gamma^0 = \begin{pmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{pmatrix}$$

$$= N^2\{(1,0,p_z/E+m,\ p_x+ip_y/E+m)^T \cdot (1,0,-p_z/E+m,\ -(p_x-ip_y)/E+m)$$
$$+ (0,1,p_x-ip_y/E+m,\ -p_z/E+m)^T \cdot (0,1,-(p_x+ip_y)/E+m,\ p_z/E+m)\}$$

(7.47) $N^2 = E+m$

$$= \begin{pmatrix} (E+m) & 0 & -p_z & -(p_x-ip_y) \\ 0 & 0 & 0 & 0 \\ p_z & 0 & -p_z^2/E+m & -p_z(p_x-ip_y)/E+m \\ (p_x+ip_y) & 0 & -p_z(p_x+ip_y)/E+m & -(p_x^2+p_y^2)/E+m \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & (E+m) & -(p_x+ip_y) & p_z \\ 0 & (p_x-ip_y) & -(p_x^2+p_y^2)/E+m & p_z(p_x-ip_y)/E+m \\ 0 & -p_z & p_z(p_x+ip_y)/E+m & -p_z^2/E+m \end{pmatrix}$$

$$= \begin{pmatrix} E+m & 0 & -p_z & -(p_x - ip_y) \\ 0 & E+m & -(p_x+ip_y) & p_z \\ p_z & p_x - ip_y & -\vec{p}^2/E+m & 0 \\ p_x+ip_y & -p_z & 0 & -\vec{p}^2/E+m \end{pmatrix}$$

$$\vec{p}^2 = E^2 - m^2 = (E+m)(E-m)$$

$$= \begin{pmatrix} E+m & 0 & -p_z & -(p_x - ip_y) \\ 0 & E+m & -(p_x+ip_y) & p_z \\ p_z & p_x - ip_y & -E+m & 0 \\ p_x+ip_y & -p_z & 0 & -E+m \end{pmatrix}$$

$$= \begin{pmatrix} E\cdot\mathbb{1} & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -E\cdot\mathbb{1} \end{pmatrix} + \begin{pmatrix} m\mathbb{1} & \\ & m\mathbb{1} \end{pmatrix}$$

$$= \boxed{\gamma^\mu p_\mu + m} \checkmark$$

NOW WE CAN USE THIS & THE RESULT OF #2 TO PROVE THE COMPLETENESS OF THE $v^{(s)}$ SOLUTIONS.

from PROBLEM 2: $v^{(s)} = i\gamma^2 u^{*(s)}$ $\quad s = 1, 2$

$$\sum_{s=1,2} v^{(s)}\overline{v}^{(s)} = \sum_{s=1,2} (i\gamma^2 u^{*(s)})\overline{(i\gamma^2 u^{*(s)})}$$

$$= -\sum_s \gamma^2 u^{*(s)}\,\overline{u^{*(s)}}\,(\gamma^2)^\dagger \leftarrow \gamma^{2\dagger} = \gamma^2$$

$$= -\sum_s \gamma^2 (u^{(s)}\overline{u}^{(s)})^* \gamma^2$$

$$= -\gamma^2(\gamma^\mu p_\mu + m)^*\gamma^2$$

$$= -\underbrace{\gamma^2\gamma^{\mu *}\gamma^2}\, p_\mu - m \quad \leftarrow (\gamma^2)^2 = \mathbb{1} \text{ \& } m \in \mathbb{R}$$

$$\gamma^2\gamma^{\mu *}\gamma^2 = -\gamma^\mu$$ ← YOU CAN CHECK THIS (note: the c.c. ONLY AFFECTS $\mu = 2$)

$$= \boxed{\gamma^\mu p_\mu - m}$$

(much easier than writing out matrices again!)

in fact, if one were really clever, no explicit matrix manipulation is necessary! (CONSIDER WORKING IN THE REST FRAME & THEN BOOSTING)

4. $P\psi = \gamma^0 \psi$

(a) $\gamma^0 u^{(1)}_{p=0} = \begin{pmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = u^{(1)}_{p=0}$ $\boxed{\lambda^u_P = 1}$

(b) $\gamma^0 v^{(1)}_{p=0} = \begin{pmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = -v^{(1)}_{p=0}$ $\boxed{\lambda^v_P = -1}$

(c) IF WE TOOK $P = -\gamma^0$

$\boxed{\begin{array}{l} \lambda^u_P = -1 \\ \lambda^v_P = +1 \end{array}}$

(d) SCALAR: $P(\bar{\psi}\psi) = (P\psi)^\dagger \gamma^0 (P\psi)$

$= \psi^\dagger \gamma^{0\dagger} \gamma^0 \gamma^0 \psi$ $\gamma^{0\dagger} = \gamma^0$

$= \psi^\dagger \gamma^0 \psi$

$= \bar{\psi}\psi$

$\Rightarrow \boxed{\lambda^{\bar{\psi}\psi}_P = 1}$

AS DONE IN GRIFFITHS (7.62) TRANSFORMS AS WE EXPECT FOR A SCALAR UNDER PARITY

PSEUDO-SCALAR: $P(\bar{\psi}\gamma^5\psi) = (P\psi)^\dagger \gamma^0 \gamma^5 P\psi$

$= \psi^\dagger \gamma^0 \gamma^0 \gamma^5 \gamma^0 \psi$

$= \psi^\dagger \gamma^5 \gamma^0 \psi$

$= -\psi^\dagger \gamma^0 \gamma^5 \psi$ $\{\gamma^\mu, \gamma^5\} = 0$

$= -\bar{\psi}\gamma^5\psi$

$\boxed{\lambda^{\bar{\psi}\gamma^5\psi}_P = -1}$

VECTOR: $P(\bar{\psi}\gamma^\mu\psi) = \psi^\dagger \gamma^\mu \gamma^0 \psi$ $\{\gamma^0, \gamma^i\} = 0$ for $i = 1,2,3$

$= \pm \bar{\psi}\gamma^\mu\psi$

$\begin{cases} + & \text{if } \mu = 0 \\ - & \text{if } \mu \in \{1,2,3\} \end{cases}$

SO THE SPATIAL PART OF A 4-VECTOR HAS $\lambda_P = -1$, WHILE THE TEMPORAL PART HAS $\lambda_P = +1$, AS ONE WOULD EXPECT.

$\Rightarrow$ VECTORS ARE NOT PARITY EIGENSTATES.

PSEUDO-VECTOR: $\mathbb{P}(\bar{\psi}\gamma^\mu\gamma^5\psi) = \psi^\dagger\gamma^\mu\gamma^5\gamma^0\psi$

$= -\psi^\dagger\gamma^\mu\gamma^0\gamma^5\psi \qquad \{\gamma^5,\gamma^0\} = 0$

now anticommute if $\mu > 0$

$$= \begin{cases} -\bar{\psi}\gamma^\mu\gamma^5\psi & \text{IF } \mu = 0 \\ +\bar{\psi}\gamma^\mu\gamma^5\psi & \text{OTHERWISE} \end{cases}$$

NOT A PARITY EIGENSTATE.

ANTISYMMETRIC TENSOR: $\mathbb{P}(\bar{\psi}\sigma^{\mu\nu}\psi) = \psi^\dagger\left(\frac{i}{2}[\gamma^\mu,\gamma^\nu]\right)\gamma^0\psi$

by now we see how this works.
we want to send γ^0 to the left of $(\cdots)$,
this involves going past a product of γ^μ & γ^ν.

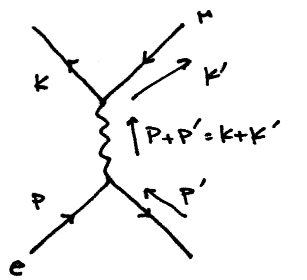
IF EITHER $\mu = 0$ OR $\nu = 0$, BUT NOT BOTH,
then PICK UP A (-1) ANTICOMMUTING PAST $\gamma^i \quad i \neq 0$

OTHERWISE, EITHER PICK UP TWO (-1)'s OR NONE.

$$\Rightarrow \quad \mathbb{P}(\bar{\psi}\sigma^{\mu\nu}\psi) = \begin{cases} +\bar{\psi}\sigma^{\mu\nu}\psi & \text{IF } \mu = \nu = 0 \text{ OR } (\mu \neq 0 \text{ AND } \nu \neq 0) \\ -\bar{\psi}\sigma^{\mu\nu}\psi & \text{OTHERWISE.} \end{cases}$$

$\Rightarrow$ NOT A PARITY EIGENSTATE

5. (FROM PESKIN & SCHROEDER *An Introduction to Quantum Field Theory* CHAPTER 5.1, P. 131)



$$= \bar{v}(p')(-ie\gamma^\mu)u(p)\left(\frac{-ig_{\mu\nu}}{(p+p')^2}\right)\bar{u}(k)(-ie\gamma^\nu)v(k')$$

$$|M|^2 = \frac{e^4}{(p+p')^4}\left(\bar{v}(p')\gamma^\mu u(p)\bar{u}(p)\gamma^\nu v(p')\right) \times \left(\bar{u}(k)\gamma_\mu v(k')\bar{v}(k')\gamma_\nu u(k)\right)$$

SINCE $m_e \ll m_\mu$, SET $m_e = 0$

recall problem 8:

$$\left.\begin{aligned} \sum_s u^{(s)}(p)\,\bar{u}^{(s)}(p) &= \not{p} + m \\ \sum_s v^{(s)}(p)\,\bar{v}^{(s)}(p) &= \not{p} - m \end{aligned}\right\} \quad \text{where } \not{q} \doteq \gamma^\mu q_\mu$$

SUMMING & AVERAGING OVER SPINS (average over initial, Σ over final!)

$$\frac{1}{2}\sum_{s_{e^-}}\frac{1}{2}\sum_{s_{e^+}}\sum_{s_{\mu^-}}\sum_{s_{\mu^+}}|M|^2 = \frac{1}{4}\frac{e^4}{s^2}\mathrm{Tr}\left[\not{p}'\gamma^\mu\not{p}\gamma^\nu\right]\mathrm{Tr}\left[(\not{k}+m_\mu)\gamma_\mu(\not{k}'-m_\mu)\gamma_\nu\right]$$

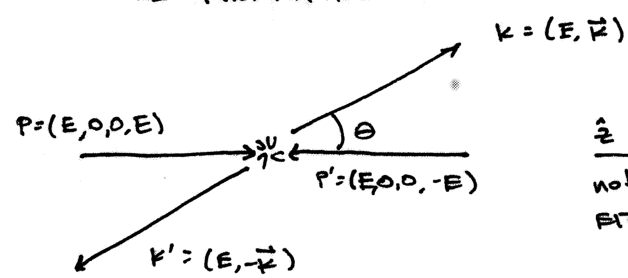
$p\cdot p' = p^\mu p'_\mu$ MASS OF μ^-, NOT A 4-VECTOR!

$$\frac{1}{4}\sum_{\text{spins}}|M|^2 = \frac{1}{4}\frac{e^4}{s^2}4\left[p'^\mu p^\nu + p'^\nu p^\mu - g^{\mu\nu}(p\cdot p')\right]4\left[k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu}(k\cdot k' + m_\mu^2)\right]$$

$$= \frac{8e^4}{s^2}\left[(p\cdot k)(p'\cdot k') + (p\cdot k')(p'\cdot k) + m_\mu^2(p\cdot p')\right]$$

NOW WE WORK OUT THE KINEMATICS (work in CM frame)

$|\vec{k}|^2 = E^2 - m_\mu^2$



$\hat{z} \rightarrow$

note: E is ENERGY OF EITHER INITIAL e; $s=(2E)^2$

$$p\cdot k = p'\cdot k' = E^2 - E|\vec{k}|\cos\theta$$
$$p'\cdot k = p\cdot k' = E^2 + E|\vec{k}|\cos\theta$$

$$\frac{1}{4}\sum_{\text{spins}}|M|^2 = \frac{8e^4}{16E^4}\left[E^2(E-|\vec{k}|\cos\theta)^2 + E^2(E+|\vec{k}|\cos\theta)^2 + 2m_\mu^2E^2\right]$$

$$= e^4\left[\left(1+\frac{m_\mu}{E^2}\right) + \left(1-\frac{m_\mu^2}{E^2}\right)\cos^2\theta\right] \leftarrow 1-\left(\frac{m_\mu}{E}\right)^2 = \frac{|\vec{k}|^2}{E^2}$$

$$= e^4\left[2 + (|\vec{k}|^2/E^2)\cos^2\theta\right] \qquad 1+\left(\frac{m_\mu}{E}\right)^2 \approx 2 \text{ (nonrelativistic)}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{8E^2}\frac{|\vec{k}|}{16\pi^2(2E)}\cdot\frac{1}{4}\sum_{\text{spin}}|M|^2$$

$$= \frac{\alpha^2}{16E^2}\frac{|\vec{k}|}{E}\left[2 + \frac{|\vec{k}|^2}{E^2}\cos^2\theta\right]$$