SOLUTIONS

Assignment 5

Due date: Wednesday, March 6

1. H&F 4.14

2. H&F 4.17

3. Repeat the calculation in lecture, of the two orbital frequencies ω_θ (orbit completion) and ω_r (radial oscillation) when the force law is slightly different from a pure inverse square. Specifically, take as the potential $U(r)$ — which includes the "centrifugal barrier" — the function
$$U(r) = \frac{B}{r^2} - \frac{A}{r} - \frac{C}{r^3}.$$
As before, $A = GM_1M_2$ and $B = L_z^2/2\mu$. The last term, with strength C, arises when the mass distribution within one of the bodies is non-spherical. For example, a moon or spacecraft in orbit within the equatorial plane of Jupiter will experience a potential of exactly this form (a gravitational "quadrupole"). It is important that you treat C as a tiny correction. In fact, you should in all your calculations **keep only the lowest order correction caused by (nonzero)** C. Here is a guide to your calculations:

   (a) Calculate the new, corrected, equilibrium radius
$$r_1 = r_0 + \cdots,$$
   where $r_0 = 2B/A$ and $\cdots$ is a single term proportional to C.

   (b) Calculate the correction to the curvature of U at the new equilibrium point:
$$K = U''(r_1) = \frac{A}{r_0^3} + \alpha\frac{C}{r_0^5}.$$
   Here we have given you the answer up to a numerical factor α (which you need to find).

   (c) Calculate the corrections to ω_θ and ω_r (again, keeping only terms proportional to C).

   (d) Form the ratio
$$\omega_r/\omega_\theta = 1 + \beta\frac{C}{Br_0}$$

(yes, you need to find the number β) and from this determine the excess or deficit angle $\delta\theta$ by which the orbit fails to close. In other words, you will have found that the axis of the orbit (defined by radial motion) precesses by $\delta\theta$ over the course of one radial oscillation, *i.e.* from one periapsis to the next.

4. Two stars in a bound binary system have mass ratio $M/m = 2$ and their distance of furthest and nearest separation has ratio $r_{\max}/r_{\min} = 3$. Make a diagram that traces out the orbit of each star in the rest frame of the system, identifying a few pairs of corresponding positions including the positions of furthest and nearest separation. Base your plots on the orbit of the relative position vector.

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(4.95)

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Problem 12*: *(Explosion of projectile)* A projectile in outer space subjected to no external forces suddenly explodes into three pieces, which have Cartesian coordinates $\vec{r}_1, \vec{r}_2, \vec{r}_3$ with respect to an inertial reference frame. The three masses are m_1, m_2, and m_3. Assume that the forces during the explosion are not known, but it is believed that they can be derived from potentials depending only on the distances between pairs of the particles:

$$V = V_A(|\vec{r}_1 - \vec{r}_2|) + V_B(|\vec{r}_2 - \vec{r}_3|) + V_C(|\vec{r}_3 - \vec{r}_1|). \tag{4.96}$$

a) Show that the center of mass moves at the same constant velocity it had before the explosion.
b) Show that, in the center of mass reference frame, the three fragments lie in a plane after the explosion. Hint: Prove that the total momentum in this reference frame is zero.
c) Derive what happens to the center of mass after the explosion if instead of being "external force-free," the system also had a constant gravitational force on it. Does the result depend on the force being a constant? Is the total momentum of the fragments still zero in the noninertial center of mass reference frame?

Central Force Problems

Problem 13: *(Massive particle moving on a cone)* A massive particle moves under the acceleration of gravity, g, and without friction on the surface of a cone of revolution with half angle α. Find the Lagrangian in plane polar coordinates. Also find the equation of motion for r and the effective potential $V_{\text{eff}}(r)$. If the particle is launched horizontally with velocity v_0 at a height z_0, prove that the condition for circular motion is $v_0^2 = gz_0$.

Problem 14*: *(Two connected masses)* Two masses m_1 and m_2 are connected by a weightless string of fixed total length l_0. Mass m_1 rests on a frictionless table, which has a small hole cut into it. Mass m_2 hangs down vertically from this hole. Assume that m_2 can only move in the vertical direction, so the problem has two degrees of freedom.

a) Assuming that the acceleration of gravity is g, find the Lagrangian and the equations of motion for this system. (Use plane polar coordinates.)
b) The total energy $E = T + V$ is a constant of the motion. How can you see this by inspection of the Lagrangian? There is a second constant of the motion. Explain how to find it, and prove that it is indeed constant. (Call this constant l). What is the physical interpretation of l?
c) Is there a case where the motion of the mass m_1 is a circle of constant radius r_0 from the hole in the table? Find the radius of this circle in terms of l, m_1, m_2, and g. Let $E(r_0)$ be the total energy in this case. Prove that $E(r_0) = \frac{3}{2}m_2 g r_0$. Why is $E(r_0)$ the minimum possible total energy E?

d) For $E > E(r_0)$, solve the radial equation of motion. Put the solution in terms of E and l, the constants of the motion. Express the solution as an integral ("solution by quadratures") that gives the time as a function of r. Is this sufficient to specify the solution completely? How would you find the turning points of the motion? The period?

e) Suppose you treated the EOM for the radial equation as generated by a fictitious 1-D potential? What would be this potential? Find the effective one-dimensional potential $V_{\text{eff}}(r)$ and draw a graph of V_{eff} versus r.

Problem 15: *(Arbitrary central force)* Suppose you have an arbitrary central force potential $V(r)$. Make the $r = \frac{1}{u}$ transformation and find the differential equation for $u(\phi)$. Work out the explicit form of the differential equation in u if $V(r) = -\frac{k}{r^\beta}$ with k, β constants.

Problem 16: *(Using Maupertuis' Principle instead of $u = \frac{1}{r}$ transformation)* Maupertuis' Principle states that $\Delta \int \sqrt{T}\, ds = 0$, where Δ is a variation between fixed end points that leaves the total energy E constant, T is the kinetic energy, and ds is the element of arc length. Recall that, for 2-D motion in the plane, $ds^2 = dr^2 + r^2 d\phi^2$. In the Euler–Lagrange equation it doesn't matter what the independent variable is, so use ϕ. Prove that Maupertuis' Principle gives the same equation for the orbit we obtained by the $u = \frac{1}{r}$ transformation. The potential is an arbitrary central potential $V(r)$.

Problem 17*: *(Tether ball)* A mass m is attached to a weightless string which initially ~~has the length~~ s_0. The other end of the string is attached to a post of radius a. Neglecting the effect of gravity, suppose the string is set into motion, with an initial velocity tangential to the string of v_0. Find the Lagrangian and the equation of motion for the length $s(t)$ of the string. Prove that the time it takes for the string to wind up on the post so that $s = 0$ is $t_{\text{wrap}} = \frac{s_0}{|v_0|}$. Notice that t_{wrap} does not depend on the post radius.

Gravity and Planetary Orbits

Problem 18: *(Elliptic orbits)* For elliptic orbits, prove that the distance from the ellipse center to the focus of the ellipse (position of the Earth–Sun center of mass) is $a\epsilon$, where a is the semimajor axis and ϵ is the eccentricity.

Problem 19*: *(Weighing the Sun, Earth, and Moon)* Kepler's Third Law in its exact form (4.61) allows you to "weigh" the Sun but not the Earth.

a) Determine the solar mass M_S from the length of the year and the mean radius of the Earth's orbit, neglecting the small eccentricity. Use $\bar{R} = 1.49 \times 10^8$ km, and the gravitational constant $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}.

b) You can find th
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determined?

Problem 20*: *(M*
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c) Note that in
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Problem 22: *(H*

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b) Why does it

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Problem 23: *((*
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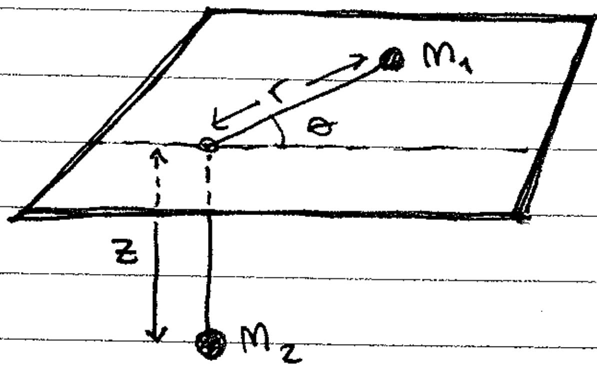
assuming the c
Sun, calculate w
and compare it

P318 SOLUTIONS: HW #5 DUE 6 March

CORRECTIONS? pt267@cornell.edu

1. [H&M #14] connected masses.



constraint:
$l_0 = r + z$

arbitrary choice of origin

a) $L = \frac{1}{2}m_1(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}m_2\dot{z}^2 - m_2 g(l_0 - z)$
$= \frac{1}{2}m_1\dot{r}^2 + \frac{1}{2}m_1 r^2\dot{\theta}^2 + \frac{1}{2}m_2\dot{r}^2 - m_2 g r$

$$\boxed{L = \frac{1}{2}(m_1+m_2)\dot{r}^2 + \frac{1}{2}m_1 r^2\dot{\theta}^2 - m_2 g r}$$

EOM's:
$$\boxed{\begin{aligned}(m_1+m_2)\ddot{r} &= m_1 r\dot{\theta}^2 - m_2 g \\ m_1 r^2\ddot{\theta} &= -m_1 r\dot{r}\dot{\theta}\end{aligned}}$$

b) In this case, $E = H$. H is conserved because L has no explicit time dependence. L is also independent of θ, so

$\boxed{l = \frac{\partial L}{\partial \dot{\theta}}} = m_1 r^2\dot{\theta}$ is also conserved.

It is the angular momentum of m_1.

c) Suppose such a configuration were possible. Then: $\dot{r} = \ddot{r} = 0$, plug into r EOM:

$$m_1 r_0 \dot{\theta}^2 = m_2 g$$

$$m_1 r_0 \frac{\ell^2}{m_1^2 r_0^4} = m_2 g \Rightarrow \boxed{r_0^3 = \frac{\ell^2}{m_1 m_2 g}}$$

Solution satisfies EOM. ✓

$$E = \tfrac{1}{2}(m_1+m_2)\cancelto{0}{\dot{r}_0^2} + \tfrac{1}{2}m_1 r_0^2 \dot{\theta}^2 + m_2 g r_0$$

$$= \frac{1}{2} m_1 r_0^2 \frac{\ell^2}{m_1^2 r_0^4} + m_2 g r_0$$

$$= \frac{1}{2m_1}\frac{\ell^2}{r_0^2} + m_2 g r_0$$

$$= \frac{1}{2\cancel{m_1}}\left(r_0^{\cancel{3}} \cancel{m_1} m_2 g\right)\frac{1}{\cancel{r_0^2}} + m_2 g r_0$$

$$= \boxed{\frac{3}{2} m_2 g r_0}$$ ✓

THIS IS THE MINIMUM POSSIBLE ENERGY: FOR ANY INITIAL $\dot{\theta} = \dot{\theta}_0 \Rightarrow \ell$, IT MINIMIZES V_{eff} WITH $T = 0$.

d) Follow the same steps leading to H&F (4.4)

$$E = \tfrac{1}{2}(m_1+m_2)\dot{r}^2 + \underbrace{\tfrac{1}{2}m_1 r^2\dot{\theta}^2} + m_2 g r$$

$$\tfrac{1}{2}m_1 r^2 \frac{\ell^2}{m_1^2 r^4} = \frac{1}{2}\frac{\ell^2}{m_1 r^2}$$

$$= \tfrac{1}{2}(m_1+m_2)\dot{r}^2 + \frac{1}{2m_1}\frac{\ell^2}{r^2} + m_2 g r$$

solve for $\dot{r} = \frac{dr}{dt}$

$$\left(\frac{dr}{dt}\right)^2 = \frac{2}{m_1+m_2}\left(E - \frac{1}{2m_1}\frac{\ell^2}{r^2} - m_2 g r\right)$$

$$\boxed{t = \sqrt{\frac{m_1+m_2}{2}}\int_{r_0}^{r} dr' \left(E - \frac{1}{2m_1}\frac{\ell^2}{(r')^2} - m_2 g r'\right)^{-1/2}}$$

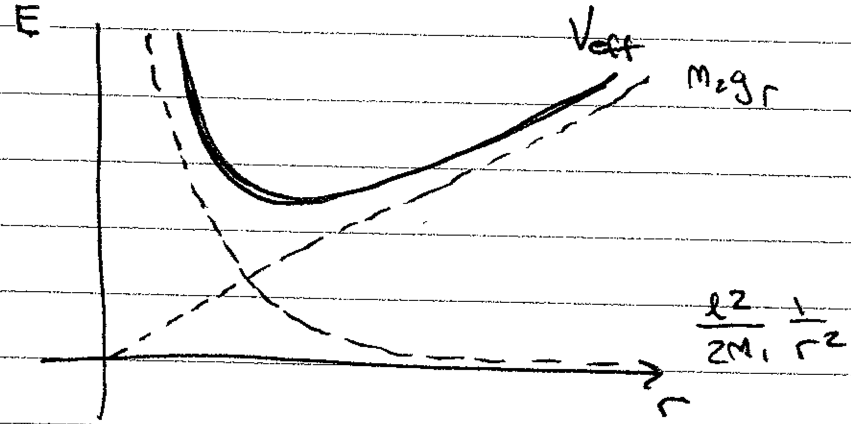
THIS GIVES $t(r)$ UPON SPECIFYING INITIAL CONDITIONS (r_0, $\dot{\theta}_0 \Leftrightarrow \ell$, E, θ_0). INVERT TO GET $r(t)$, $\theta(t)$ IS OBTAINED USING ℓ = CONST AND $r(t)$.

Turning points: $\underbrace{\text{kinetic energy}}_{\frac{1}{2}(m_1+m_2)\dot{r}^2 \text{ only!}} = 0 \Leftrightarrow r$ s.t. $E = V_{eff}(r)$

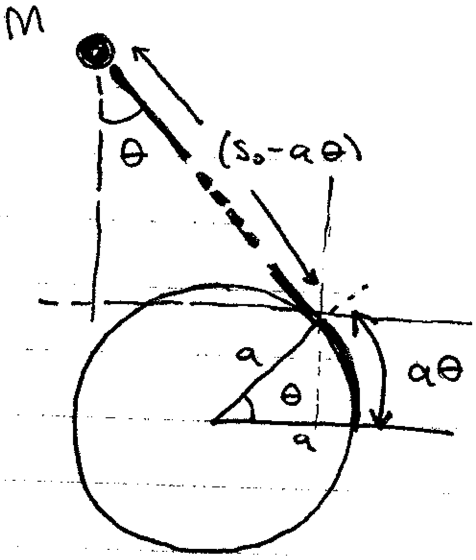
ℓ^2 term lives in V_{eff}

Period: $\tau = 4\times$(time between turning points)

e) $\boxed{V_{eff} = m_2 g r + \frac{l^2}{2M_1}\frac{1}{r^2}}$



2. (H&F 4.17) Tetherball



$x = a\cos\theta - (s_0 - a\theta)\sin\theta$

$y = a\sin\theta + (s_0 - a\theta)\cos\theta$

$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2)$

FREE PARTICLE LAGRANGIAN BUT SUBJECT TO CONSTRAINT

$\dot{x} = -a\cancel{(\sin\theta)\dot{\theta}} + a\cancel{(\sin\theta)\dot{\theta}} - (s_0 - a\theta)(\cos\theta)\,\dot{\theta}$

$\dot{y} = a\cancel{(\cos\theta)\dot{\theta}} - a\cancel{(\cos\theta)\dot{\theta}} - (s_0 - a\theta)(\sin\theta)\,\dot{\theta}$

$$\boxed{L = \frac{1}{2}m(s_0 - a\theta)^2\dot{\theta}^2}$$

Solve using energy conservation

$$\cancel{\tfrac{1}{2}m}V_0^2 = \cancel{\tfrac{1}{2}m}V^2 = \cancel{\tfrac{1}{2}m}(s_0 - a\theta)^2\dot{\theta}^2$$

$$\Rightarrow \dot{\theta} = \frac{\pm v_0}{s_0 - a\theta}$$

pick +

$$\int_0^{s_0/a}(s_0 - a\theta)\,d\theta = \pm v_0 t_{wrap}$$

$$= s_0\theta - \tfrac{1}{2}a\theta^2 \Big|_0^{s_0/a} = v_0 t_{wrap}$$

$$\Rightarrow V_0 t_{wrap} = \frac{s_0^2}{a} - \frac{1}{2}\frac{s_0^2}{a}$$

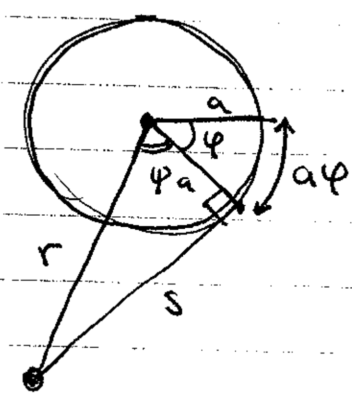
$$\boxed{t_{wrap} = \frac{s_0^2}{2aV_0}}$$

Alternately:

$$v_0 = \pm\underbrace{\sqrt{\dot r^2 + r^2\dot\theta^2}}_{v}$$

algebra & trig

$$= \pm \frac{s\dot s}{a}$$



$$\begin{cases} r^2 = a^2 + s^2 \\ s_0 = s + a\varphi \\ \tan\psi = s/a \\ \theta = \varphi + \psi \end{cases}$$

$$\Rightarrow \frac{ds}{dt} = \frac{\pm a v_0}{s}$$

$$\Rightarrow s\,ds = \pm a v_0\,dt$$

$$\tfrac{1}{2}(s^2 - s_0^2) = -a v_0 t$$ ← PICK MINUS SIGN (S DECREASING)

$$s^2 = s_0^2 - 2a v_0 t$$

$$0 \text{ @ } t_{WRAP} = \boxed{\frac{s_0^2}{2a v_0}}$$ ✓

3. Gravitational quadrupole

$$U(r) = \frac{B}{r^2} - \frac{A}{r} - \frac{C}{r^3}$$

$A = G m_1 m_2$
$B = \ell^2/2\mu$
$C =$ "tiny correction"

a) Solve for equilibrium radius r_1 as a 1^{st} order perturbation in C

$$0 = U'(r) = -2\frac{B}{r^3} + \frac{A}{r^2} + 3\frac{C}{r^4}$$

$$\Rightarrow \quad 0 = +Ar^2 - 2Br + 3C$$

$$r_1 = \frac{2B \pm \sqrt{4B^2 - 4A(3C)}}{2A}$$

PICK UP + SIGN SOLUTION SINCE THAT GIVES

$$= \frac{2B}{2A} + \frac{2B}{2A}\sqrt{1 - \frac{3AC}{B^2}}$$

$\sqrt{1+\varepsilon} = 1 + \frac{1}{2}\varepsilon$

$$= \underbrace{\boxed{\frac{2B}{A} - \frac{3C}{2B}}}_{r_0} + \mathcal{O}(C^2)$$

b) $U''(r) = 6\frac{B}{r^4} - 2\frac{A}{r^3} - 12\frac{c}{r^5}$

$$r_1 = r_0 - \frac{3}{2B}c$$

Use: $(r+\varepsilon)^{-n} = r^{-n} - n\varepsilon r^{-n-1}$

$$\begin{aligned} U''\left(r_0 + \frac{3c}{2B}\right) = \; & 6\frac{B}{r_0^4} - 6B\cdot 4\left(\frac{-3c}{2B}\right)\frac{1}{r_0^5} \\ & -2\frac{A}{r_0^3} + 2A\cdot 3\left(\frac{-3c}{2B}\right)\frac{1}{r_0^4} \\ & -12\frac{c}{r_0^5} + \mathcal{O}(c^2) \end{aligned}$$

Plug in $r_0 = 2B/A$ to massage this into the form in the problem:

$$\begin{aligned} U''(r_1) = & \boxed{\begin{aligned} & 6\frac{B}{r_0^3}\cdot\left(\frac{A}{2B}\right) \\ & -2\frac{A}{r_0^3} \end{aligned}} + 36\frac{c}{r_0^5} \\ & \qquad\qquad\qquad\quad \underline{- 9\frac{Ac}{B}\cdot\frac{1}{r_0^5}\cdot\left(\frac{2B}{A}\right)} \\ & -12\frac{c}{r_0^5} + \mathcal{O}(c^2) \end{aligned}$$

$$\frac{3A}{r_0^3} - \frac{2A}{r_0^3} = \frac{A}{r_0^3} \quad \checkmark \qquad\qquad -18\frac{c}{r_0^5}$$

$$= \boxed{\frac{A}{r_0^3} + 6\frac{c}{r_0^5}}$$

$$\boxed{\alpha = 6}$$

c) corrections to ω_θ :

as before: $\dot{\theta} = \omega_\theta = \dfrac{\ell}{\mu r_1^2} = \dfrac{\ell}{\mu r_0^2} - 2\left(\dfrac{-3c}{2B}\right)\dfrac{\ell}{\mu r_0^3}$

$$\Delta\omega_\theta = (\omega_\theta - \omega_\theta(c=0)) = \boxed{+\frac{3c}{B}\frac{\ell}{\mu r_0^3}}$$

corrections to ω_r :

PERTURB AROUND $r = r_1$: $r(t) = r_1 + \delta r(t)$

$$U(r) = \underbrace{U(r_1)}_{\#} + \cancel{U'(r_1)}^{\,0}(r-r_1) + \frac{1}{2}\underbrace{U''(r_1)}_{\frac{A}{r_0^3} + 6\frac{c}{r_0^5}}(r-r_1)^2$$

EOM : $\mu\,\delta\ddot{r} = -U'(r) = -U''(r_1)\,\delta r$

$$\omega_r = \sqrt{\frac{U''(r_1)}{\mu}} = \sqrt{\frac{1}{\mu}\left(\frac{A}{r_0^3} + \frac{6c}{r_0^5}\right)}$$

$$= \omega_r(c=0) + \frac{1}{2}\cdot\frac{6c}{\mu r_0^5}\sqrt{\frac{\mu r_0^3}{A}}$$

$$\Delta\omega_r = \omega_r - \omega_r(c=0) = \boxed{+\frac{1}{2}\,\frac{6\,c}{\mu r_0^5}\sqrt{\frac{\mu r_0^3}{A}}}$$

$$\cancel{\sqrt{X+\varepsilon} = \sqrt{X}\sqrt{1+\varepsilon/X} = \sqrt{X}\left(1+\frac{1}{2}\frac{\varepsilon}{X}\right) = \sqrt{X} + \frac{1}{2}\frac{1}{\sqrt{X}}\varepsilon}$$

WHEN $c=0$, we know $\omega_r = \omega_\theta \equiv \omega_0$

FROM part c):

$$\omega_\theta = \omega_0\left(1 + 2 \cdot \frac{3c}{2B} \cdot \frac{1}{r_0}\right)$$

$$= \omega_0\left(1 + \frac{3c}{B} \cdot \frac{1}{r_0}\right)$$

$$\omega_r = \cancel{\omega_0}\sqrt{\frac{A}{\mu r_0^3}\left(1 + \frac{\alpha c}{A r_0^2}\right)}$$

$$= \omega_0\left(1 + \frac{1}{2}\frac{\alpha c}{A r_0^2}\right)$$

$r_0 = \frac{2B}{A}$

$$= \omega_0\left(1 + \frac{1}{2}\frac{\alpha c}{A r_0} \cdot \frac{A}{2B}\right)$$

$$= \omega_0\left(1 + \frac{\alpha c}{4 B r_0}\right)$$

$\alpha = -30$

$$\frac{\omega_r}{\omega_\theta} = \frac{1 + \frac{\alpha c}{4Br_0}}{1 + \frac{3c}{Br_0}}$$

$$= 1 + \frac{\alpha}{4}\frac{c}{Br_0} - 3\frac{c}{Br_0}$$

$$= 1 + \underbrace{\left(-3 + \frac{\alpha}{4}\right)}\frac{c}{Br_0}$$

$\alpha = 6$

$$\boxed{-3/2 = \beta}$$

note that $2\pi/\omega_\theta$ is the time it takes to make an "angular" cycle

then $\left(\frac{2\pi}{\omega_\theta}\right)\omega_r$ is the phase of the radial cycle.

$$2\pi \frac{\omega_r}{\omega_\theta} = 2\pi - \underbrace{\frac{3\pi c}{B r_0}}$$

$$\boxed{\delta\theta = -3\pi c / B r_0}$$

deficit angle (precession)

4. Binary system

<u>parameters</u>: $M/m = 2$, $r_{max}/r_{min} = 3$

$$\vec{r}_1 = \frac{M}{m+M}\vec{r} = \frac{2m}{m+2m} = \frac{2}{3}\vec{r}$$

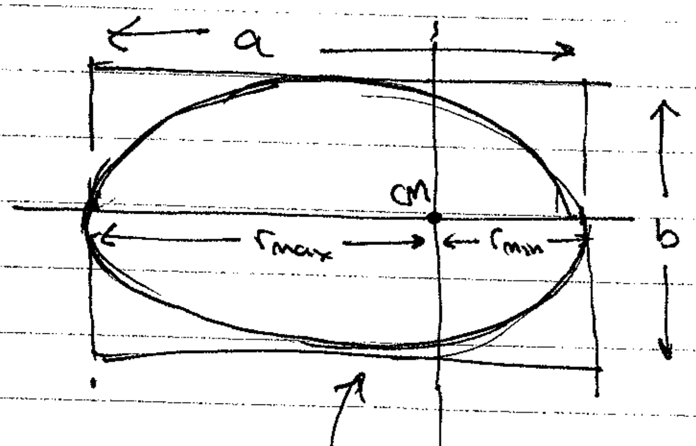
$$\vec{r}_2 = \frac{-m}{m+M}\vec{r} = \frac{-m}{3m} = \frac{-1}{3}\vec{r}$$

? or w/ overall minus sign if you assigned: $m_1 = M$, $M_2 = m$.

FIRST DRAW ELLIPSE. NEED ε:

$$\frac{r_{max}}{r_{min}} = 3 = \frac{1+\varepsilon}{1-\varepsilon} \Rightarrow 3-3\varepsilon = 1+\varepsilon \Rightarrow \underline{\varepsilon = 1/2}$$

$$\frac{a}{b} = \frac{1}{\sqrt{1-\varepsilon^2}} = \frac{1}{\sqrt{1-1/4}} \approx 1.15$$



path of relative separation, $\vec{r}$ in $\vec{R}_{cm} = 0$ frame

Now sketch using: $\vec{r}_1 = \frac{2}{3}\vec{r}$, $\vec{r}_2 = -\frac{1}{3}\vec{r}$