Assignment 6

Due date: Wednesday, March 13

1. H&F 5.4
2. H&F 5.5
3. H&F 5.6
4. H&F 5.7
5. Repeat the derivation of the conserved quantity I when the Lagrangian has a continuous symmetry (Noether's theorem) to allow for a slight generalization. The text and our lecture considered the case where the Lagrangian itself is unchanged when the continuous symmetry parameter s is varied:

$$\frac{d}{ds}L(Q_1(s),\ldots,\dot{Q}_1(s),\ldots)|_{s=0} = 0.$$

A more comprehensive definition of "symmetry" in the context of mechanics is the property that the symmetry-transformed motion (a continuous transformation applied to the trajectory) is also a valid solution to the equations of motion. But the equations of motion — the Euler-Lagrange equations — follow from Hamilton's principle, which actually applies to the *time integral* of the Lagrangian (the action). The symmetry transformation may therefore add a time derivative to the Lagrangian without having any effect on the action and consequently no effect on the equations of motion. Summarizing, we can generalize the condition on the Lagrangian to be the following:

$$\frac{d}{ds}L(Q_1(s),\ldots,\dot{Q}_1(s),\ldots)|_{s=0} = \frac{dF}{dt},$$

where F is an arbitrary function of the generalized coordinates and velocities. Now it's your turn: determine how the formula for the conserved quantity I is modified by the term involving F.

b) The parameter $t(\theta)$ is now to be regarded as a second dynamical variable. Prove that the momentum conjugate to t is

$$p_t = L + t'\frac{\partial L}{\partial t'} = -H, \tag{5.97}$$

where H is the ordinary Hamiltonian. The time has as its conjugate momentum the negative of the Hamiltonian. Phase space has been enlarged to four dimensions by adding time and energy.

c) Show that the momentum conjugate to q is unchanged by the transformation of the independent variable.

d) Find the Hamiltonian and Hamilton's equations of motion assuming that θ is the independent variable.

Problem 7: *(Particle in a 2-D central force)* Find the Lagrangian for a point particle in a 2-D central force. Work in only two dimensions, using plane polar coordinates. Are there any ignorable coordinates? Find the conjugate momenta. Then find the Hamiltonian and Hamilton's equations of motion. Prove that you obtain equations that are equivalent to (4.38, 4.41).

Problem 8: *(Particle on a cylinder)* Imagine a particle confined to an open cylinder of radius R and bound to the origin by a spring with spring constant k, as shown in Figure 5.11.

a) Prove that the Lagrangian is

$$L = \frac{1}{2}m((R\dot{\theta})^2 + \dot{z}^2) - \frac{1}{2}k(R^2 + z^2). \tag{5.98}$$

b) Next find the conjugate momenta, the Hamiltonian, and Hamilton's equations of motion. Based on these equations, what type of motion do you expect for the particle? Will there be oscillatory motion? How about linear motion?

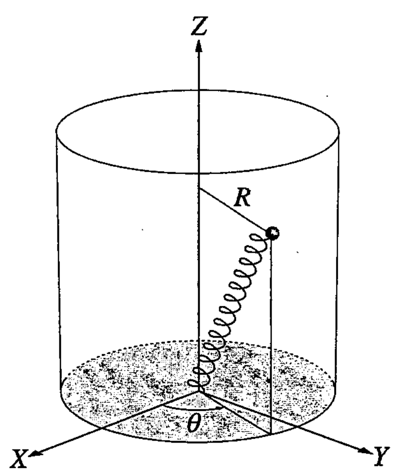

FIGURE 5.11

..., p_N space. It has $2N$
n in phase space of a large
nian. Phase space volume
pressible fluid.
Hamiltonian has a simple

(5.91)

transform away the first-
ing into a frame rotating

ree of freedom)

efined in Equation (5.13)
us transformation ($j = 1$)

$\Big|_{s=0}$. (5.92)

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explicit form for I in this

roblem)

problem (4.38, 4.41) that

(5.93)

nd μ the reduced mass), is

for an elliptical orbit.
essary to evaluate it at one
symmetry transformation?

Legendre Transformations

Problem 3: *(Routhians are "reduced" Lagrangians)* The coordinate q_N is ignorable if the Lagrangian contains only the time derivative of the Nth coordinate.

$$L = L(q_1, q_2, \ldots, q_{N-1}, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_{N-1}, \dot{q}_N, t). \tag{5.94}$$

By using a Legendre transformation, we create a new function, the *Routhian* R (1.71).

$$R(q_1, \ldots, q_{N-1}, \dot{q}_1, \ldots, \dot{q}_{N-1}) \equiv L - p_N\dot{q}_N. \tag{5.95}$$

Since q_N is ignorable in the original Lagrangian, $p_N \equiv \frac{\partial L}{\partial \dot{q}_N}$ is a constant. Prove that the problem is reduced to $N-1$ degrees of freedom by using the Routhian as a new Lagrangian and showing that the Routhian obeys the Euler–Lagrange equations in the $N-1$ dynamical variables $q_1, \ldots, q_{N-1}$.

Examples of Hamiltonian Dynamics

Problem 4*: *(Motion along a spiral)* A particle of mass m moves in a gravitational field along the spiral $z = k\theta$, r = constant, where k is a constant, and z is the vertical direction. Find the Hamiltonian $H(z, p)$ for the particle motion. Find and solve Hamilton's equations of motion. Show in the limit $r \to 0$, $\ddot{z} = -g$.

Problem 5*: *(Two particles connected by a spring)* Two particles of different masses m_1 and m_2 are connected by a massless spring of spring constant k and equilibrium length d. The system rests on a frictionless table and may both oscillate and rotate. Find Lagrange's equations of motion. Are there any ignorable coordinates? What are the conjugate momenta? Find the Hamiltonian and Hamilton's equations of motion.

Problem 6: *(Changing the independent variable; time as a dependent variable)* In the theory of special relativity, time is treated on the same basis as the space coordinates x, y, z. We no longer regard time as the independent variable, but instead we choose for that role another parameter, which we will call θ here. Then, in a particular reference frame, the trajectory of a particle would be given parametrically as $x(\theta), y(\theta), z(\theta), t(\theta)$. This can also be done in prerelativity mechanics, although there is no compelling reason to do it. Nevertheless it provides some interesting insights.

a) Let the time be an arbitrary function $t(\theta)$. If $L(q, \dot{q}, t)$ is the Lagrangian of a system with one degree of freedom, show that the Lagrangian corresponding to using θ as the independent variable is

$$L_\theta = t'L\left(q, \frac{q'}{t'}, t\right) \tag{5.96}$$

($t' \equiv \frac{dt}{d\theta}$, $q' \equiv \frac{dq}{d\theta}$). Show using Hamilton's Principle that this Lagrangian leads to the (two) Euler–Lagrange equations with θ as the independent variable.

P318 HW6 SOLUTIONS

Due 9 March

corrections? pt267@cornell.edu

1. (H&F 5.4) Spiral motion

$$L = \frac{1}{2}m(\cancel{\dot{r}^2} + \underbrace{r^2\dot{\theta}^2}_{\frac{R^2}{K^2}\dot{z}^2} + \dot{z}^2) - mgz \qquad \begin{cases} r = R, \text{ const} \\ z = K\theta \end{cases}$$

($\dot{r}^2 \to 0$)

$$= \frac{1}{2}m\left(1+\frac{R^2}{K^2}\right)\dot{z}^2 - mgz$$

$$P_z \equiv \frac{\partial L}{\partial \dot{z}} = m\left(1+\frac{R^2}{K^2}\right)\dot{z}$$

$$H(z,P_z) = \frac{1}{2}m\left(1+\frac{R^2}{K^2}\right)\dot{z}(z,P_z)^2 + mgz$$

$$= \boxed{\frac{P_z^2}{2m(1+R^2/K^2)} + mgz}$$

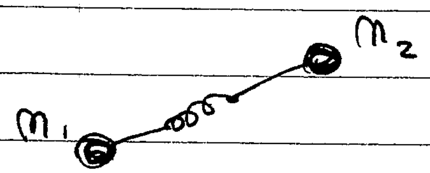
$$\dot{z} = \frac{\partial H}{\partial P_z} = \frac{1}{m(1+R^2/K^2)}P_z$$

$$\dot{P}_z = -\frac{\partial H}{\partial z} = -mg$$

$$\boxed{\ddot{z} = \frac{-g}{1+R^2/K^2}}$$

indeed $\ddot{z} \longrightarrow -g$ as $R \to 0$.

2 (H&F 5.5) "<u>spring theory</u>"



observe: this is a 2D <u>CENTRAL FORCE</u> problem, we know how to reduce this to a 1D system (cf CH. 4)

$$\vec{R}_{cm} = \frac{1}{m_1+m_2}\left(m_1\vec{r}_1 + m_2\vec{r}_2\right) \qquad \text{eq. (4.24)}$$
$$\vec{r} = \vec{r}_1 - \vec{r}_2$$
$$\mu^{-1} = m_1^{-1} + m_2^{-1} \qquad \text{eq. (4.27)}$$
$$M = m_1 + m_2$$

$$L = \tfrac{1}{2}M\dot{\vec{R}}^2 + \tfrac{1}{2}\mu\dot{\vec{r}}^2 - K\vec{r}^2 \qquad \text{eq. (4.28)}$$

Lagrange EoM: $\boxed{\begin{aligned}\ddot{\vec{R}} &= 0\\ \ddot{\vec{r}} &= -\frac{k}{\mu}\vec{r}\end{aligned}}$ ← cm motion is trivial

Because $\ddot{\vec{R}} = 0$, we can forget about $\vec{R}$ in the cm frame. further, $L(\vec{r}) = L(r,\dot{r},\dot{\theta})$ so that we can ignore θ:

$$\boxed{P_\theta = \frac{\partial L}{\partial \dot\theta}} \xrightarrow{\text{is conserved}} P_\theta = \frac{\partial}{\partial\dot\theta}\left(\frac{1}{2}\mu(\dot{r}^2 + r^2\dot\theta^2) - kr^2\right)$$
$$= \boxed{\mu r^2\dot\theta} = \ell$$

So the dynamics boils down to:

$$\boxed{\begin{aligned}P_\theta &= \mu r^2\ddot\theta = \text{const}\\ \ddot r &= -2\frac{k}{\mu}r + \cancel{k}\, r\dot\theta^2\end{aligned}}$$
$$\vec{R}_{cm} = 0 \quad \text{in cm frame}$$

$$\cancel{\mu}\, r\left(\frac{\ell^2}{\mu^2 r^4}\right) = \frac{\ell^2}{\mu^2 r^3}$$

Now let's do this using the Hamiltonian formalism. In addition to p_θ above, we need:

$$\boxed{\begin{aligned} p_r &= \frac{\partial L}{\partial \dot{r}} = \mu \dot{r} \\ \vec{P}_{cm} &= \frac{\partial L}{\partial \dot{\vec{R}}_{cm}} = M \dot{\vec{R}}_{cm} \end{aligned}}$$

$$H = \frac{1}{2M}\vec{P}_{cm}^{\,2} + \frac{1}{2\mu}p_r^2 + Kr^2 + \frac{p_\theta^2}{2\mu r^2}$$

$$\left.\begin{aligned} \dot{\vec{R}}_{cm} &= \frac{\partial H}{\partial \vec{P}_{cm}} = \frac{1}{M}\vec{P}_{cm} \\ \dot{\vec{P}}_{cm} &= -\frac{\partial H}{\partial \vec{R}_{cm}} = 0 \end{aligned}\right\} \ddot{\vec{R}}_{cm} = 0$$

$$\left\{\begin{aligned} \dot{r} &= \frac{\partial H}{\partial p_r} = \frac{1}{\mu}p_r \\ \dot{p}_r &= -\frac{\partial H}{\partial r} = -2Kr + \frac{p_\theta^2}{\mu r^3} \end{aligned}\right.$$

$$\left.\begin{aligned} \dot{\theta} &= \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{\mu r^2} \\ \dot{p}_\theta &= -\frac{\partial H}{\partial \theta} = 0 \end{aligned}\right\} \begin{aligned} &\ddot{\theta} = 0 \\ &p_\theta = \text{const.} \end{aligned}$$

$$\ddot{r} = -2\frac{K}{\mu}r + \frac{p_\theta^2}{\mu^2 r^3} \checkmark$$

3. (H&F 5.6) Time as a dependent var

a) Suppose $L(q, \dot{q}, t)$ is the Lagrangian for a system w/ 1 DOF, q. How do we write the Lagrangian when we consider time a dependent parameter?

The key insight is that the independent parameter is just a dummy variable for the action integral. Treating t as $t(\theta)$ is simply a change of variables:

$$S = \int_{t_0}^{t_1} dt\, L(q, \dot{q}, t) = \int_{t_0(\theta)}^{t_1(\theta)} dt(\theta)\, L\left(q(t(\theta)), \frac{dq}{dt(\theta)}, t(\theta)\right)$$

$$dt = \frac{dt}{d\theta} d\theta = t' \, d\theta$$

$$\Rightarrow \frac{dq}{dt} = \frac{1}{t'} \frac{dq}{d\theta} = \frac{q'}{t'}$$

$$\Rightarrow S = \int_{t_0(\theta)}^{t_1(\theta)} d\theta \underbrace{\boxed{t' L\left(q, \frac{q'}{t'}, t\right)}}_{L_\theta}$$

✓

This gives two Euler-Lagrange equations:

$$\boxed{\frac{d}{d\theta}\frac{\partial L_\theta}{\partial q'} = \frac{\partial L_\theta}{\partial q}} \qquad \boxed{\frac{d}{d\theta}\frac{\partial L_\theta}{\partial t'} = \frac{\partial L_\theta}{\partial t}}$$

What do these eqs tell us? [NOT ASKED BY THE PROBLEM, BUT IT'S USEFUL TO CHECK]. NOTE $L_\theta = t' L(q, \dot{q}, t)$ WHERE $\dot{q} = q'/t'$.

$$\frac{d}{d\theta}\frac{\partial L_\theta}{\partial q'} = \frac{d}{d\theta}\left[t' \frac{\partial L}{\partial \dot{q}} \frac{\partial \dot{q}}{\partial q'}\right] = \frac{d}{d\theta}\left[t' \frac{\partial L}{\partial \dot{q}} \frac{1}{t'}\right] = \frac{d}{d\theta}\frac{\partial L}{\partial \dot{q}}$$

$$\frac{\partial L_\theta}{\partial q} = t' \, \partial L/\partial q$$

$$1^{st} \text{ EOM} \Rightarrow \frac{d}{d\theta}\frac{\partial L}{\partial \dot{q}} = t' \frac{\partial L}{\partial q} \Longrightarrow \underbrace{\frac{1}{t'}\frac{d}{d\theta}}_{d/dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$$

original EOM !

$$\frac{d}{d\theta}\frac{\partial L_\theta}{\partial t'} = \frac{d}{d\theta}\left[L + \frac{\partial L}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial t'}\right] = \frac{d}{d\theta}\left[L + t'\frac{\partial L}{\partial \dot{q}}\left(-\frac{q'}{(t')^2}\right)\right] = \frac{d}{d\theta}\left[L - \frac{\partial L}{\partial \dot{q}}\dot{q}\right]$$

$$\frac{\partial L_\theta}{\partial t} = t' \frac{\partial L}{\partial t}$$

$$2^{ND} \text{ EOM} \Rightarrow \frac{d}{d\theta}\Big[\underbrace{L - \frac{\partial L}{\partial \dot{q}}\dot{q}}_{-(\text{original } H)}\Big] = t'\frac{\partial L}{\partial t}$$

$$\Rightarrow \frac{d}{dt} H = -\frac{\partial L}{\partial t}$$

relates total time dependence of original H to explicit time dependence of original L.

b) $$P_t = \frac{\partial L_\theta}{\partial t'} = \frac{\partial}{\partial t'}\left(t'L\right) = \left(L + t'\underbrace{\frac{\partial L}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial t'}}\right)$$

we calculated this on prvs. page

$$= L + t'\frac{\partial L}{\partial \dot{q}}\left(-\frac{q'}{(t')^2}\right)$$

$$\boxed{\begin{aligned} &= L - \frac{\partial L}{\partial \dot{q}}\dot{q} \\ &= -H \end{aligned}}$$ ✓

familiar from Noether's thm! conserved "momentum" from time translations → energy!

c) $$P_q = \frac{\partial L_\theta}{\partial q'} = t'\frac{\partial L}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial q'} = \boxed{\frac{\partial L}{\partial \dot{q}}}$$

observe that independent of how we parameterized $t(\theta)$, $P_q = \partial L/\partial \dot{q}$. In other words, $\dot{q}$ is some function of θ, eg q'/t', but no matter what this function is,

$$P_q = \frac{\partial}{\partial x} L(q, x, t)\Big|_{x=\dot{q}}.$$ ✓

d) $H_\theta = q' \frac{\partial L_\theta}{\partial q'} + t' \frac{\partial L_\theta}{\partial t'} - L_\theta$

$$= q' t' \frac{\partial L}{\partial \dot{q}} \frac{\partial \dot{q}}{\partial q'} + t' \left(L + t' \frac{\partial L}{\partial \dot{q}} \frac{\partial \dot{q}}{\partial t'} - L \right)$$

$$= q' t' P_q \frac{1}{t'} + (t')^2 P_q \left(\frac{-q'}{(t')^2} \right)$$

$$= \boxed{0}$$

Hamilton eqns of motion are simple:

$$q' = \frac{\partial H}{\partial P_q} = 0$$

$$P_q' = -\frac{\partial H}{\partial q} = 0$$

$$t' = \frac{\partial H}{\partial P_t} = 0$$

$$P_t' = -\partial H/\partial t = 0$$

4. (H&F 5.7) <u>2D central force</u>

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

θ is an ignorable coordinate

$$p_r = m\dot{r}$$
$$p_\theta = mr^2\dot{\theta}$$

$$H = \frac{1}{2m}p_r^2 + \frac{1}{2mr^2}p_\theta^2 + V(r)$$

$$\left.\begin{aligned} \dot{\theta} &= \frac{\partial H}{\partial p_\theta} = \frac{1}{mr^2}p_\theta \\ \dot{p}_\theta &= -\frac{\partial H}{\partial \theta} = 0 \end{aligned}\right\} \quad \ell = \text{const.}$$

$$\left.\begin{aligned} \dot{r} &= \frac{\partial H}{\partial p_r} = \frac{1}{m}p_r \\ \dot{p}_r &= -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} - V'(r) \end{aligned}\right\} \quad m\ddot{r} = \frac{p_\theta^2}{mr^3} - V'(r)$$

5. Noether generalized

Instead of the invariance of L, we can be more general & require only the invariance of $S = \int dt\, L$. In other words, our action is unchanged when L is shifted by a total time derivative, since $\int dt \cdot df/dt = 0$.

suppose: $\frac{d}{ds} L(Q_1(s), \cdots, \dot{Q}_1(s), \cdots) = \frac{d}{dt} F$

follow the usual Noether derivation on the LHS:

$$\sum_i \left(\frac{\partial L}{\partial Q_i} \frac{dQ_i}{ds} + \frac{\partial L}{\partial \dot{Q}_i} \frac{d\dot{Q}_i}{ds} \right) = \frac{d}{dt} F$$

$$= \sum_i \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}_i}\right) \frac{dQ_i}{ds} + \frac{\partial L}{\partial \dot{Q}_i} \frac{d}{dt} \frac{dQ_i}{ds} \right) = \frac{d}{dt} F$$

$$= \frac{d}{dt} \left(\sum \underbrace{\frac{\partial L}{\partial \dot{Q}_i} \frac{dQ_i}{ds}}_{\text{old } I \text{, conserved quantity}} \right) = \frac{d}{dt} F$$

combining RHS into LHS:

$$\boxed{\frac{d}{dt} \left[\underbrace{\left(\sum_i \frac{\partial L}{\partial \dot{Q}_i} \frac{dQ_i}{ds} \right) - F}_{} \right] = 0}$$

You may also plug in $P_i = \partial L / \partial \dot{Q}_i$ & expand about $Q_i(0) = q$

new conserved quantity, I, generalizing previous expression.