Lec: Grav 2-body III

4 March

- constructing orbits from $r(\theta)$
- "scattering kinematics" (UNBOUND ORBITS)
- Gravity assist

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start w/ Page ★

?

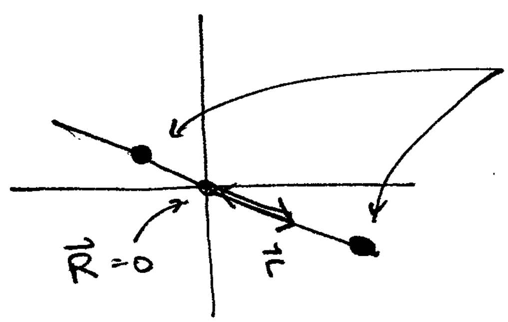
<u>Center of mass frame</u> : $\dot{\vec{R}} = 0$

<u>choose</u> coordinates s.t. $\vec{R} = 0$

$$\vec{r}_1 = \frac{m_2}{m_1+m_2}\vec{r}$$

$$\vec{r}_2 = \frac{-m_1}{m_1+m_2}\vec{r}$$

Positions of each body are parallel to $\vec{r}$



which one is closer? more massive one closer to origin

maybe start w/ "[illegible]"

<u>Simple example</u> : binary stars w/ $m_1 = m_2$

$$\vec{r}_1 = -\vec{r}_2 = \tfrac{1}{2}\vec{r}$$

ECCENTRICITY :

$$\varepsilon^2 = \left(E \ast \frac{2\ell^2}{\mu k^2} - 1\right)$$

CONSTANTS

$V = -k/r$

$0 \le \varepsilon < 1$

CIRCLE

Hyperbola

suppose $\varepsilon = \sqrt{3}/2$

recall: $\dfrac{a}{b} = \dfrac{1}{\sqrt{1-\varepsilon^2}} = \dfrac{2}{1}$

major & minor axes of ellipse

Aspect ratio

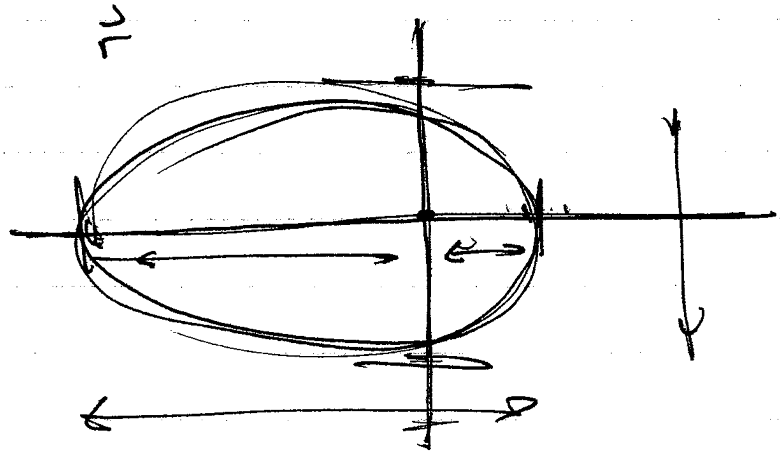
So let's draw the orbit

$$r_{max} = \frac{p}{1+\varepsilon} \qquad p = \frac{l^2}{\mu k}$$

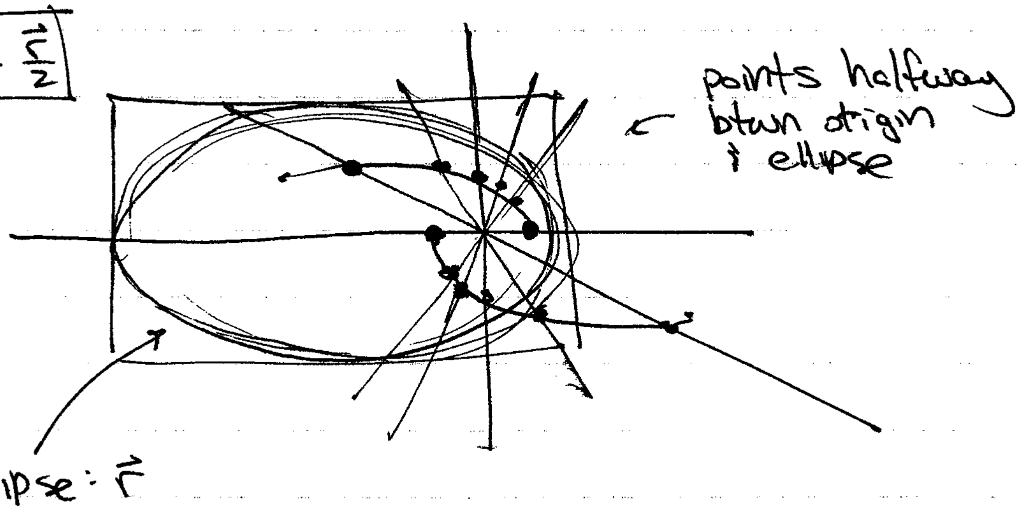
$$r_{min} = \frac{p}{1-\varepsilon}$$

the origin $\vec{R} = 0$ divides the major axis according to the ratio

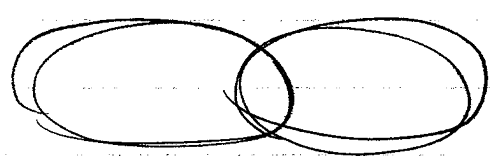
$$\frac{r_{max}}{r_{min}} = \frac{1+\varepsilon}{1-\varepsilon} \approx 14$$



$$\boxed{\vec{r}_1 = -\vec{r}_2 = \frac{\vec{r}}{2}}$$



If you took a time lapse photo, the binary star system would look like:



remark: you'll do this in problem #4!

$k = GM_1 M_2$

<u>Next</u>: "orbits" with $\varepsilon > 1$

$$\Leftrightarrow \text{Energy} = (\varepsilon^2 - 1)\frac{A}{2r_0} = (\varepsilon^2-1)\frac{\mu k^2}{2l^2}$$

> 0 → "ESCAPE TO INFINITY"

by the way: what does this mean?
TOTAL ENERGY is only defined up to an overall constant, no?

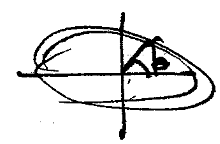
→ w/rt <u>FREE PARTICLE</u>

in GR, constants matter

→ cosmological constant

(supersymmetry : ~~susy~~ vacuum)

<u>Recall</u>: ellipse $\boxed{r_0 = r + \varepsilon \underbrace{r\cos\theta}_{x}}$



$$\to (r_0 - \varepsilon x)^2 = r^2 = x^2 + y^2$$

$$\to r_0^2 = \boxed{(1-\varepsilon^2)x^2 + 2\varepsilon r_0 x + (1-\varepsilon^2)x_0^2} + y^2$$

$+ (1-\varepsilon^2)x_0^2$

$$x_0 = \frac{-\varepsilon}{1-\varepsilon^2} r_0$$

$$r_0\left(1 + \frac{\varepsilon^2}{1-\varepsilon^2}\right) = \frac{r_0^2}{1-\varepsilon^2}$$

$$= (1-\varepsilon^2)(x-x_0)^2 + y^2$$

$$\frac{r_0^2}{1-\varepsilon^2} = (1-\varepsilon^2)(x-x_0)^2 + y^2$$

$$\rightarrow 1 = \underbrace{\frac{(1-\varepsilon^2)^2}{r_0^2}}_{1/a^2}(x-x_0)^2 + \underbrace{\frac{1-\varepsilon^2}{r_0^2}}_{1/b^2} y^2$$

$$\rightarrow \boxed{a = \frac{r_0}{1-\varepsilon^2} \qquad b = \frac{r_0}{\sqrt{1-\varepsilon^2}}}$$

For ~~ELLIPSE~~ HYPERBOLA: $\varepsilon \geq 1$

$$-\frac{r_0^2}{\varepsilon^2-1} = -(\varepsilon^2-1)(x-x_0)^2 + y^2$$

$$\rightarrow 1 = \underbrace{\frac{(\varepsilon^2-1)^2}{r_0^2}}(x-x_0)^2 - \underbrace{\frac{\varepsilon^2-1}{r_0^2}} y^2$$

$$\boxed{a = \frac{r_0}{\varepsilon^2-1} \qquad b = \frac{r_0}{\sqrt{\varepsilon^2-1}}}$$

$$\boxed{1 = \frac{(x-x_0)^2}{a^2} - \frac{y^2}{b^2}}$$

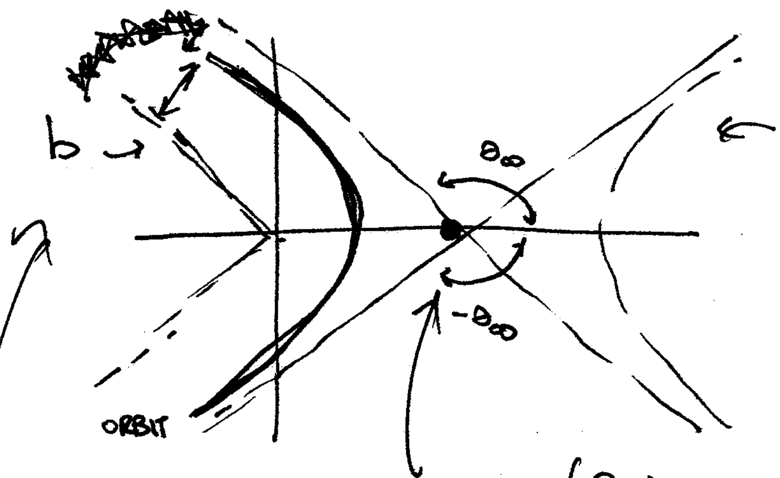
hyperbola

Hyperbola: 2 disconnected branches
which one ?

recall : $r(\theta) = \dfrac{r_0}{1 + \varepsilon \cos\theta}$ ← before squaring

r, r_0 POSITIVE $\Rightarrow$ $\cos\theta > -1/\varepsilon$

so the asymptotes are $\theta_\infty = \pm \cos^{-1}(-1/\varepsilon)$



what about this? relic of squaring orbit equation.

$x_0 = \left(\dfrac{\varepsilon}{\varepsilon^2 - 1}\right) r_0$ ← not the origin

shift of asymptotes: IMPACT PARAMETER
(not semi-minor axis), from $l \neq 0$.

What is this <u>impact parameter</u>?
consider asymptotic expressions for l & E.

$$l = b\mu v_\infty = (\vec{r} \times \vec{p})_z$$
$$E = \frac{1}{2}\mu v_\infty^2$$

$v_\infty = |\dot{\vec{r}}|$ in lim $|\vec{r}| \to \infty$

$$\frac{l^2}{E} = 2\mu b^2 \Rightarrow \boxed{b^2 = \frac{l^2}{2\mu E}} = \frac{A\mu r_0}{\frac{A\mu}{r_0}(\varepsilon^2-1)} = \boxed{\frac{r_0^2}{\varepsilon^2-1}}$$

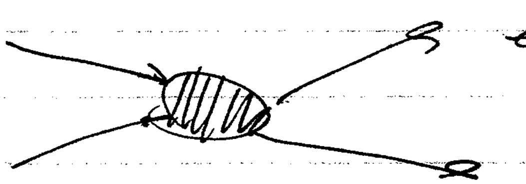
recall: $E = (\varepsilon^2-1)\frac{A}{2r_0} = (\varepsilon^2-1)\frac{\mu k^2}{2l^2}$

compare to hyperbola equation — in fact,

IMPACT PARAM. → b <u>is</u> the "b" in hyperbola!

$ <u>Scattering</u>: one state of uniform (unacc.) motion
↓
another state ——— " ———

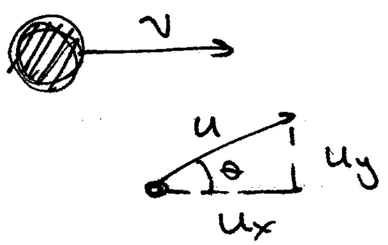
"asymptotic states"



asymp free, non-interacting

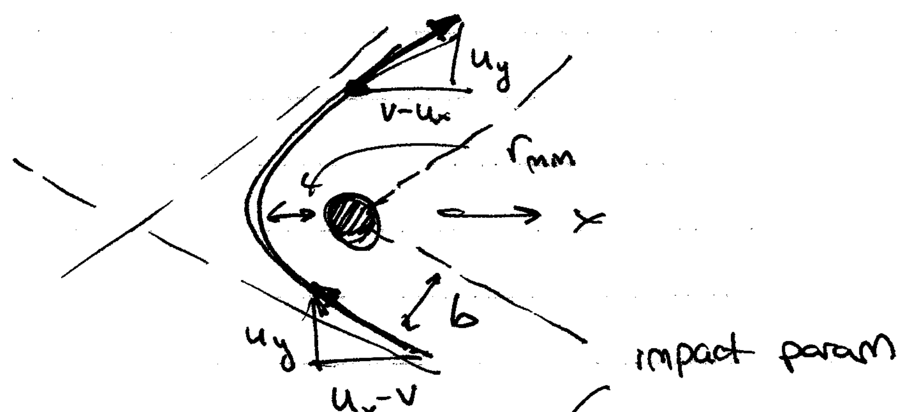
eg: Gravity assist: spacecraft boosted by a well-orchestrated sequence of "scatterings" from planets

consider: spacecraft in near collision course w/ planet
[eg: STAR CONTROL

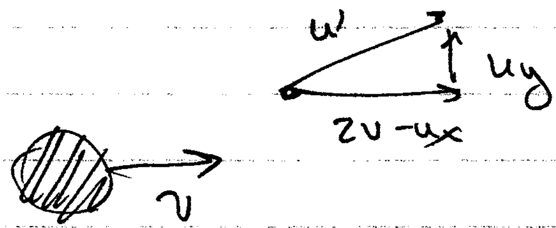


$V > u_x$
so planet is "catching up"

cm frame: ≈ planet rest frame



mission control / cptn. picard: choose b s.t. x-axis of hyperbolic traj coincides w/ planet direction of motion back to solar-system frame



compare final & initial spacecraft speeds:

$$\left(\frac{u'}{u}\right)^2 = \frac{(2v - u_x)^2 + u_y^2}{u^2}$$

$$= \left(\frac{2v}{u}\right)^2 - 4\frac{v}{u}\underbrace{\frac{u_x}{u}}_{\cos\theta} + 1$$

$$= \left(\frac{2v}{u} - 1\right)^2 + 4\frac{v}{u}(1 - \cos\theta)$$

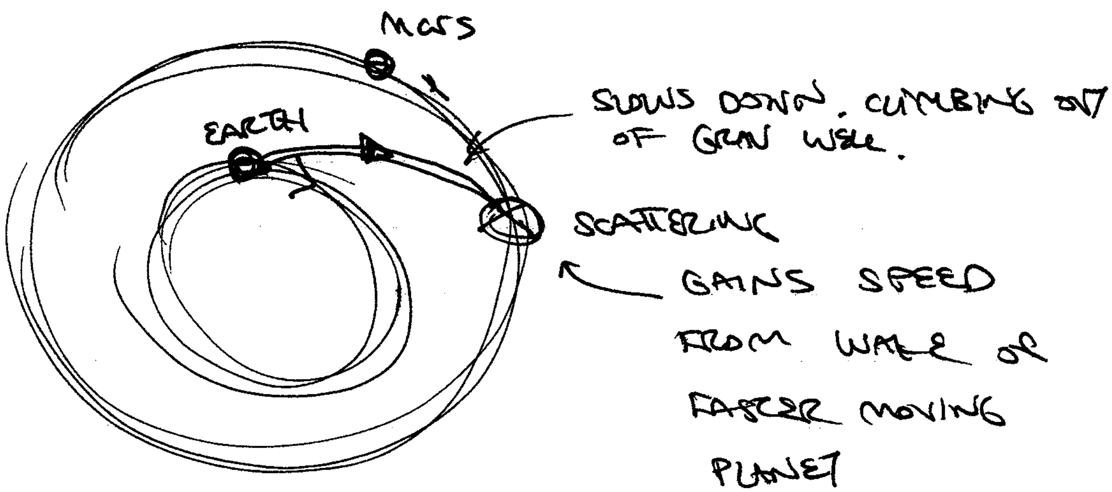
$u' > u$ for $v' > u$

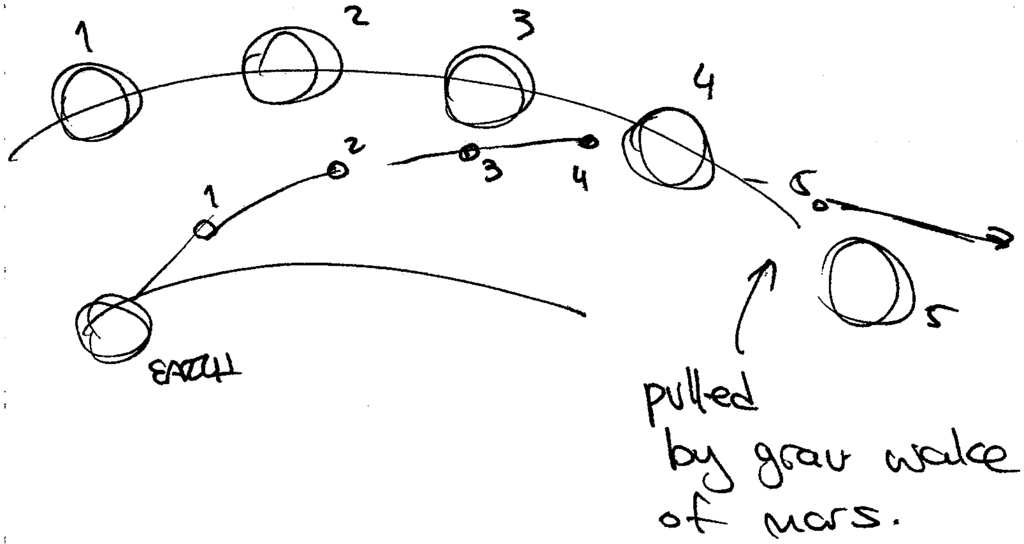
small angle → greater boost

Q: how is it that energy is conserved?

Voyager spacecraft

series of grav. assists to climb out of sun's gravitational well





ORBITS ANALYZED @ 2 LENGTH SCALES

large: ELLIPSES AROUND SUN

small: hyperbolae, scattering off mars