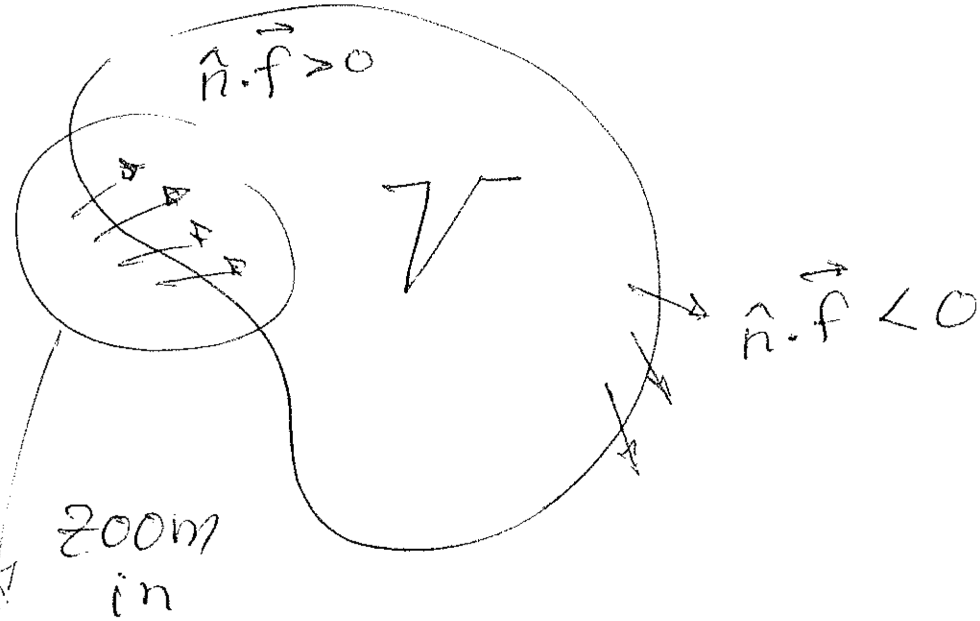
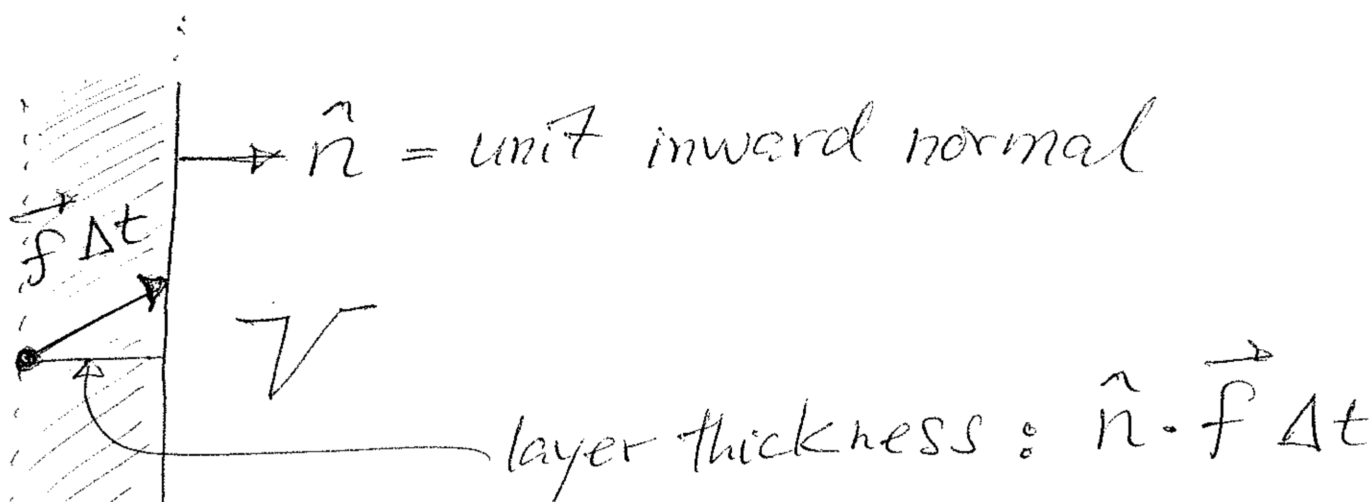
Dynamics in phase space

- consequences of Liouville's thm.
- Poincaré recurrence thm.
- action principle in phase space

Imagine many initial conditions for our system, as points uniformly distributed with density $\rho(0)$ in phase space. Using Liouville's theorem we will show the same points, after a short time Δt, will have moved such that their distribution is still uniform. Our strategy is to calculate the flux of points through the boundary of an arbitrary subvolume $\mathcal{V}$:





layer volume $= dA\ \hat{n}\cdot\vec{f}\,\Delta t$

dA = "area" element of $\mathcal{V}$ - boundary

("area" = volume in one lower dimension)

points in this layer flow into $\mathcal{V}$

density of points $= \rho(0)$

$$\begin{pmatrix}\text{change in} \\ \text{number of} \\ \text{points within} \\ V\end{pmatrix} = \Delta N$$

$$= \oint_{V\text{-boundary}} \rho(0)\, \hat{n}\cdot\vec{f}\, \Delta t\, dA$$

$$\Rightarrow \frac{\Delta N}{\Delta t} = \rho(0) \oint_{V\text{-bound.}} \vec{f}\cdot\hat{n}\, dA$$

divergence theorem

$$= \rho(0) \int_{V} \vec{\nabla}\cdot\vec{f}\, dV$$

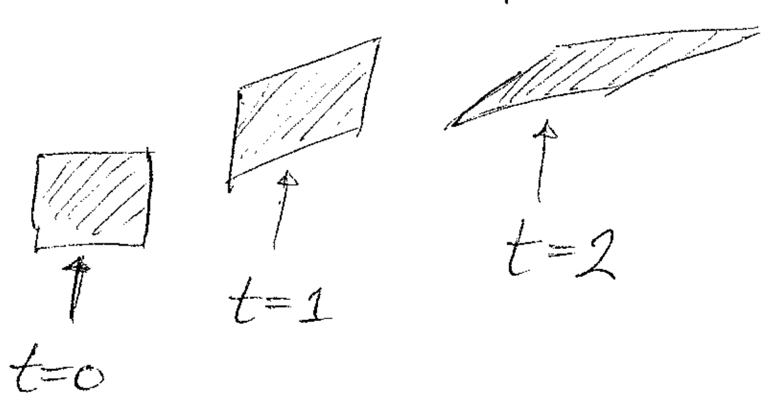
↑ "volume" element (2N dimensions)

$$= 0 \quad \text{(Liouville)}$$

So the number of points within V stays constant. And since V was arbitrary and can be arbitrarily small (as long as we make Δt sufficiently small, $\rho(0)$ sufficiently large) this can only be true if the density of points stays constant, i.e.

$$\rho(\Delta t) = \rho(0).$$

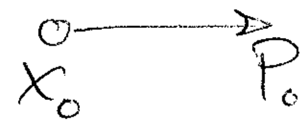
Now suppose we track all the points that lie initially within some region of phase space:



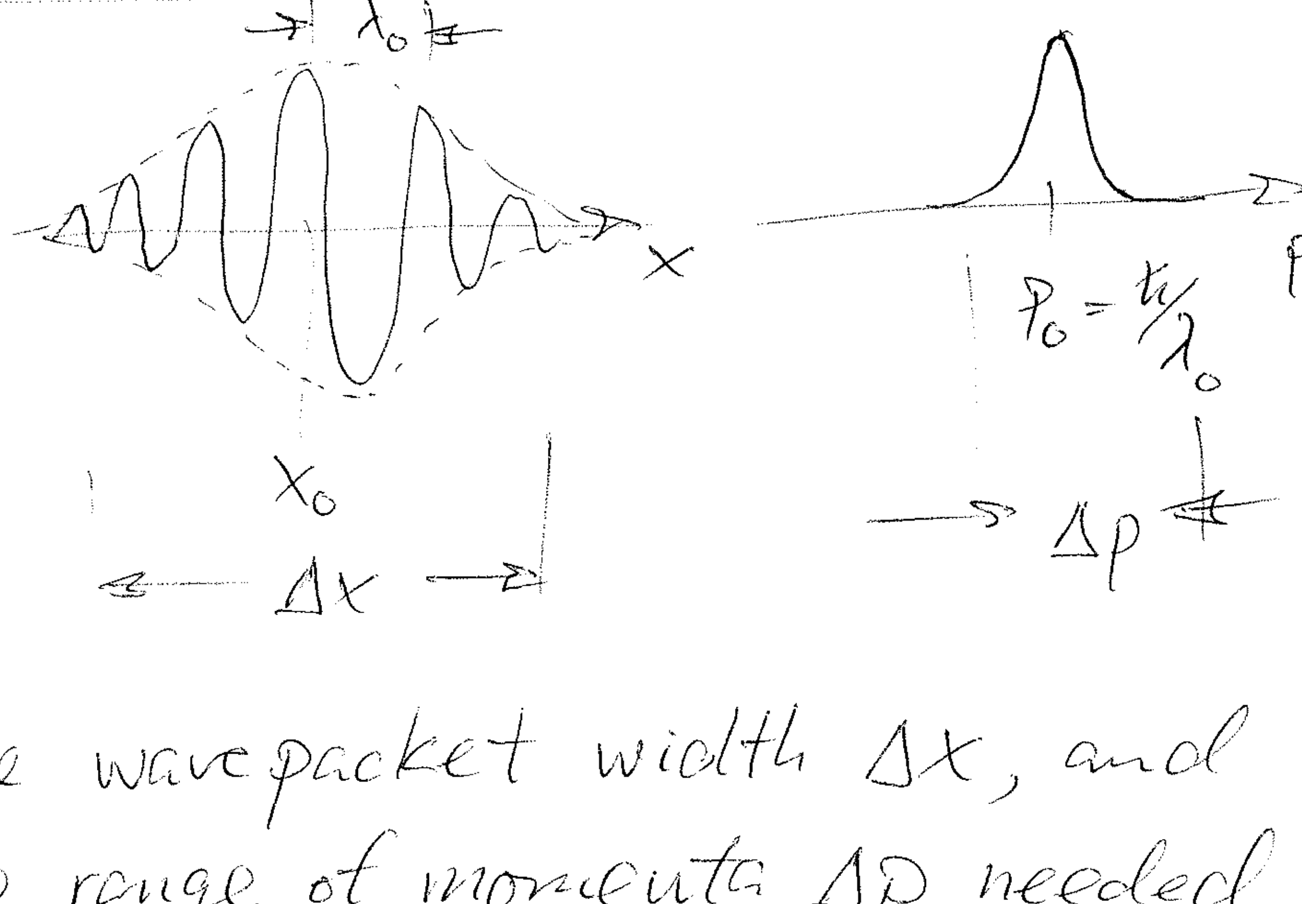
From what we just proved, we know that the volumes (in phase space) are exactly equal (while their shapes are free to change). If we think of phase space as a substance ("Hamiltonium"), then it flows as the perfect incompressible fluid!

There is a nice consistency between the incompressibility of phase space and what quantum mechanics teaches us about the nature of states. The very notion of "point" in phase space is incompatible with one of the most basic notions of Q.M., that position and momentum cannot both be precisely specified. Recall the correspondence between classical and quantum states of a particle:

<u>classical</u>



<u>quantum</u>

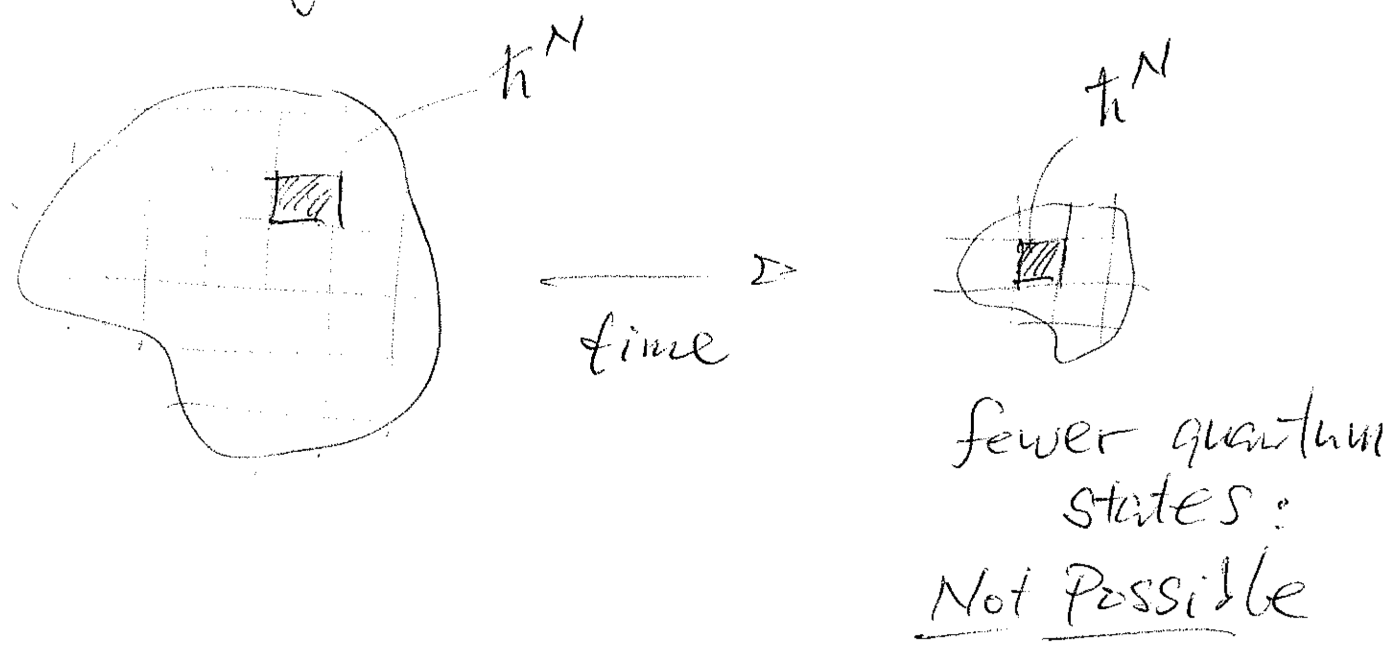


The wavepacket width Δx, and the range of momenta Δp needed in forming the superposition of plane waves, always satisfy

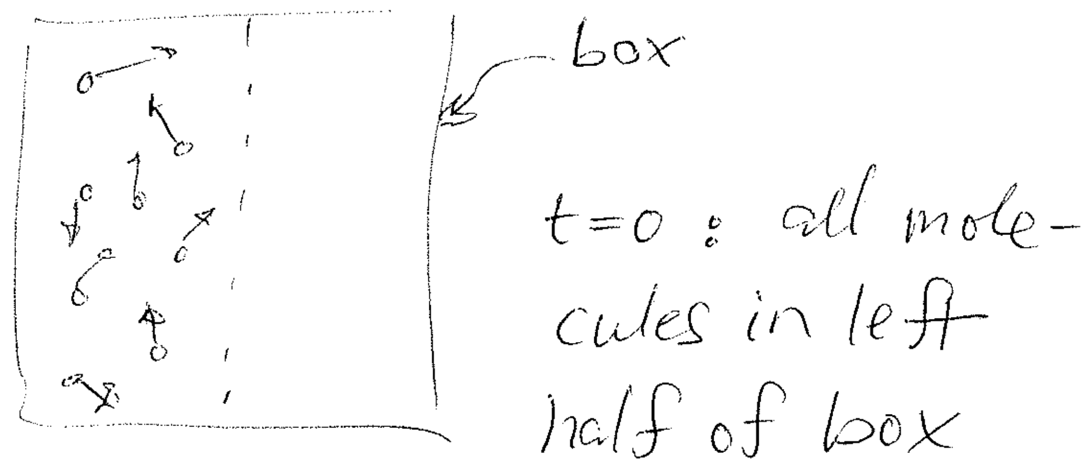
$$\Delta p \, \Delta x \geq \hbar .$$

In other words, every quantum states requires a minimum area in phase space for each position/momentum pair. A particle moving through space, with $N=3$ degrees of freedom, requires phase space volume $\hbar^3$ for each of its quantum states. In general, if the system has N degrees of freedom then each quantum state occupies volume $\hbar^N$ in phase space.

We can see now why the classical incompressibility of phase space is a good thing. If phase space volumes could shrink with time, there wouldn't be enough volume to accomodate all the original quantum states:



We can use the volume preserving property of Hamiltonian flow to show that, given enough time, dynamical systems will return arbitrarily close to their initial conditions. The proof requires one mild condition, that the volume of phase space accessible to the system is <u>finite</u>. A box filled with molecules satisfies this condition since (1) the total kinetic energy is finite so each molecule's momentum (p) is bounded, and (2) the box bounds the available positions (q) of the molecules. The theorem we are about to prove then asserts that even highly unusual initial conditions,



return close to that initial condition after a sufficiently long time (i.e. back in left half of box).