Lecture 3

- Kinetic energy
- conservative forces
- Lagrangian

The notion of generalized forces came up in connection with work, a scalar quantity. The kinetic energy T is another scalar quantity, to which we now turn.

Our system is comprised of point masses m_i at positions $\vec{r}_i$. Previously we saw how the velocities $\dot{\vec{r}}_i$ are expressed in terms of the generalized coordinates and their velocities. From that we know that the kinetic energy always takes the following form:

$$T = \sum_i \frac{1}{2} m_i \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i$$

$$= T(q_1, \ldots, q_N, \dot{q}_1, \ldots, \dot{q}_N, t)$$

Suppose our "ladder" is nearly massless compared to the burly firefighter of mass M clinging to the middle rung (at point $\vec{r}_1$)

The kinetic energy is then,

$$T = \tfrac{1}{2} M \left(\dot{x}^2 + \dot{y}^2 \right)$$

$$= \tfrac{1}{2} M \left(\left(\tfrac{L}{2}\right)^2 \underbrace{\dot{\theta}^2}_{\dot{q}} + L \underbrace{\cos\theta}_{q} \, \underbrace{\dot{\theta}}_{\dot{q}} \, \underbrace{\dot{w}}_{t} + \underbrace{\dot{w}^2}_{t} \right)$$

We now derive an important identity involving partial derivatives of the kinetic energy.

$$\frac{\partial T}{\partial q_k} = \sum_i m_i \dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial q_k}$$

$$= \sum_i \vec{P}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial q_k}$$

Here $\vec{P_i}$ is (standard) momentum of point mass i. Here's the other kind of partial derivative:

$$\frac{\partial T}{\partial \dot{q}_k} = \sum_i \vec{P_i} \cdot \frac{\partial \dot{\vec{r_i}}}{\partial \dot{q}_k}$$

$$= \sum_i \vec{P_i} \cdot \frac{\partial \vec{r_i}}{\partial q_k}$$

We used the identity for generalized velocity derived previously. The identity we are after involves the total time derivative of the last expression:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = \sum_i \left(\dot{\vec{P_i}} \cdot \frac{\partial \vec{r_i}}{\partial q_k} + \vec{P_i} \cdot \frac{\partial \dot{\vec{r_i}}}{\partial q_k}\right)$$

For the second term we can substitute what we found at the bottom of page 3; for the ~~second~~ first term we make use of an actual principle of physics: Newton's Second Law

$$\dot{\vec{P}}_i = \vec{F}_i \, . \quad *$$

over

Making both of these substitutions we get:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = \underbrace{\sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k}}_{\tilde{F}_k} + \frac{\partial T}{\partial q_k}$$

$\tilde{F}_k$ (our old friad, the generalized force)

Kinetic energy identity:

$$\tilde{F}_k = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_{r_k}}\right) - \frac{\partial T}{\partial q_{r_k}}$$

We now turn to the important case where the (non-constraint) forces in our system are <u>conservative</u>. This allows for the introduction of another scalar, the potential energy function

$$V = V(\vec{r}_1, \vec{r}_2, \ldots)$$

which determines the forces by

$$\vec{F}_i = -\vec{\nabla}_i V$$

gradient w.r.t. position of point mass i

Recall, we are aiming for a formulation that involves only the generalized coordinates (the minimum necessary), and so we would like to use only gradients with respect to these. Since ~~the $\vec{r}_i$'s only~~ V only depends on the q's through the $\vec{r}_i$'s, we use the chain rule like this:

$$-\frac{\partial V}{\partial q_k} = -\sum_i \cancel{\frac{\partial V}{\partial \vec{r}_i}} \vec{\nabla}_i V \cdot \frac{\partial \vec{r}_i}{\partial q_k}$$

$$= \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k} = \tilde{F}_k$$

And since the $\vec{r}_i$'s do <u>not</u> depend on $\dot{q}$'s, we know

$$\frac{\partial V}{\partial \dot{q}_k} = 0$$

To finish up, go back to our kinetic energy identity and substitute our potential energy gradient for the generalized force:

$$-\frac{\partial V}{\partial q_k} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k}$$

We've achieved our goal, of using only scalars (V & T) and the minimal set of variables (generalized coord's and velocities).

To tidy-up we introduce a new scalar quantity called "the Lagrangian":

$$L = T - V$$

$$0 = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial}{\partial q_k}(T - V)$$

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k}$$

where we used the fact that

$$\frac{\partial V}{\partial \dot{q}_k} = 0.$$