P3318 Section

29 March 2013

<u>TODAY</u>: CANONICAL TRANSFORMATIONS | RETURN HW

<u>NEXT WK</u>: MIDTERM REVIEW

↳ bring your questions

<u>RECALL (HW #2, i think)</u>:

LAGRANGIAN

LEOM ARE UNCHANGED UNDER THE TRANSFORMATION

$$\boxed{L' = \lambda L - \frac{dF}{dt}} \qquad (*)$$

RESCALING (CANCELS ON BOTH SIDES of EOM) ignore this.

total time derivative

BECOMES BOUNDARY TERM IN ACTION, BUT qs DON'T VARY ON THE BOUNDARY.

recall further: $\boxed{H = p\dot{q} - L}$ (implied Σ)

func of p, q

<u>GOAL</u>: WANT TO FIND NEW COORDINATES (P, Q) IN THE HAMILTONIAN PICTURE SUCH THAT HEOM ARE UNCHANGED

↳ "canonical transformation"

or H'

EXPLICITLY: WANT $Q(q,p)$ & $P(q,p)$ s.t. $\exists\ K$,
A NEW HAMILTONIAN, SUCH THAT:

$$\dot{Q}_i = \frac{\partial K}{\partial P_i} \qquad \dot{P}_i = -\frac{\partial K}{\partial Q_i}$$

IF THESE ARE GOOD COORDINATES, THEN THE LAGRANGIAN ASSOCIATED TO K MUST BE AT MOST DIFFERENT FROM L (ORIGINAL LAGRANGIAN) BY A TRANSFORMATION of THE FORM (*)

$$L' = \boxed{P\dot{Q} - K = \lambda(\underbrace{p\dot{q} - H}_{L}) - \frac{dF}{dt}}$$

for our purposes, set $\lambda = 1$

USUALLY VERY INTERESTING { (scale transformations are uninteresting in this context.)

RHS A PRIORI UNKNOWN

Philosophy: DON'T KNOW P, Q, K ∴ just specify F

$$\text{so:} \quad \boxed{p\dot{q} - H = P\dot{Q} - K + \frac{dF}{dt}}$$

F: GENERATING FUNCTION, CHARACTERIZES THE TRANSF. $\to P, Q, K$

simplest example: $F = F_1(\underline{q, Q}, t)$ ...

OTHER CASES: q, P / p, Q / p, P } require additional terms then: $p\dot{q} - H = P\dot{Q} - K + \frac{dF_1}{dt}$

$$= P\dot{Q} - K + \underbrace{\frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial q}\dot{q} + \frac{\partial F_1}{\partial Q}\dot{Q}}_{\text{came from } F_1(q, Q, t)}$$

$H(p,q)$ $\quad$ $K(P,Q)$

<u>note</u>: q & Q are separately independent
(@ this point we haven't specified $Q(p,q)$)

SO: COEFFICIENT OF $\dot{q}$ ON LHS & RHS MATCH:

$$\boxed{p = \frac{\partial F_1}{\partial q}}$$ → CAN hopefully! INVERT TO SOLVE FOR $Q(q,p,t)$

ALSO: COEFFICIENT OF $\dot{Q}$ ON LHS & RHS MATCH:

$$0 = P + \frac{\partial F_1}{\partial Q} \Rightarrow \boxed{P = -\frac{\partial F_1}{\partial Q}}$$

FURTHER: WHAT'S LEFT OVER MUST BE EQUAL:

$$-H = -K + \frac{\partial F_1}{\partial t} \Rightarrow \boxed{K = H + \frac{\partial F_1}{\partial t}}$$

<u>SO</u>: GIVEN $F_1(q,Q)$, WE HAVE FOUND THE
P, Q (implicitly), and K!

SIMPLE EXAMPLE

$$F_1 = qQ \Rightarrow \begin{cases} p = Q \\ P = -q \end{cases}$$

trivially gives Q

$$K = H$$

$$\Rightarrow \left.\begin{aligned} Q &= p \\ P &= -q \end{aligned}\right\} \text{ swapped } p \text{ \& } q!$$

PHYSICALLY THIS IS WEIRD!

~ ROTATING PHASE SPACE

$$H(p,q) \longrightarrow H(-q,p)$$

OTHER TYPES OF CANONICAL TRANSFORMATIONS
→ by LEGENDRE TRANSFORMS

eg. $F_3(p,Q,t) = F_1(q,Q,t) - qp$

LEGENDRE TRANSFORM $q \to p$
note: not the usual $\dot{q} \to p$!

$$F = F_1(q,Q,t) = \boxed{F_3(p,Q,t) + qp}$$

eg. $F_2(q,P,t) = F_1(q,Q,t) + QP$

$$F = F_1(q,Q,t) = \boxed{F_2(q,P,t) - QP}$$

note sign!

Summary:

$$F = F_1(q, Q, t)$$
$$F = F_2(q, P, t) - QP$$
$$F = F_3(p, Q, t) + qp$$
$$F = F_4(p, P, t) + qp - QP$$

why the signs? try wrong sign:

$$F = F_2(q, P) \pm QP$$

$$\Rightarrow p\dot{q} - H = P\dot{Q} - K + \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial q}\dot{q} + \left(\frac{\partial F_2}{\partial P} \pm Q\right)\dot{P} \pm P\dot{Q}$$

left w/: $-H = \underbrace{-K + \frac{\partial F_2}{\partial t}}_{\text{GOOD, LIKE BEFORE}} + \underbrace{P\dot{Q} \pm P\dot{Q}}_{\text{WANT THIS TO VANISH}}$

DONT WANT K TO BE A FUNCTION OF $\dot{Q}$!

$$\boxed{P = \frac{\partial F_2}{\partial q} \qquad Q = \frac{\partial F_2}{\partial P} \qquad K = H + \partial F_2/\partial t}$$

Simple example

$$F_2 = qP \longrightarrow p = \frac{\partial F_2}{\partial q} = P$$
$$Q = \frac{\partial F_2}{\partial P} = q$$
$$K = H$$
} trivial transf.

Non-trivial example $\quad \omega^2 = k/M$

$$H = \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right)$$

for simplicity: ~~$m = \omega = 1$~~ (dimensions ...)
$m = 1/2 \quad \omega = 2$

$$H = p^2 + q^2$$

looks like a circle!

WANT: $p = f(P)\cos Q$
$q = f(P)\sin Q$

$$K = f^2(P)\left(\cos^2 Q + \sin^2 Q\right) = f^2(P)$$

aha! Q is a cyclic (ignorable) coordinate!

How to get this?

eg: $F_1 = q^2 \cot Q$

$$p = \frac{\partial F_1}{\partial q} = 2q \cot Q$$

$$P = -\frac{\partial F_1}{\partial Q} = q^2 \frac{1}{\sin^2 Q}$$

$$\boxed{\begin{aligned} q &= \sqrt{P} \sin Q \\ p &= \sqrt{P} \cos Q \end{aligned}}$$

$$H = p^2 + q^2 = P$$

restoring m & ω: $P = E/\omega$

$\dot{Q} = \omega \rightarrow Q = \omega t + Q_0$

$$q = \sqrt{\frac{2E}{m\omega^2}} \sin(\omega t + Q_0)$$

$$p = \sqrt{2mE} \cos(\omega t + Q_0)$$

↗ { REVIEW!

An example of Poisson Brackets :

$$\{f,g\} = \sum_i \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial f}{\partial q_i}$$

why is it useful?

$$\dot{A} = \frac{\partial A}{\partial t} + \frac{\partial A}{\partial q}\underbrace{\dot{q}}_{\frac{\partial H}{\partial p}} + \frac{\partial A}{\partial p}\underbrace{\dot{p}}_{-\frac{\partial H}{\partial q}}$$

$$= \frac{\partial A}{\partial t} + \{A, H\}$$

← LOOKS A LOT LIKE QM! H "GENERATES" TIME TRANSLATIONS.

<u>CONSIDER</u> : $V(r) = -\frac{k}{r}$ KEPLER PROBLEM

~~CLAIM: RUNGE LENZ VECTOR IS CONSERVED~~

$$\cancel{\vec{A} \equiv \vec{p} \times \vec{l} + mk\frac{\vec{r}}{r}}$$

① WHAT IS THE HAMILTONIAN?

$$L = \frac{1}{2}m(\dot{r})^2 + \frac{1}{2}m(r^2\dot{\theta})^2 + k/r$$

$$p_r = m\dot{r} \qquad l_z = mr^2\dot{\theta}$$

$$H = \frac{p_r^2}{2m} + \frac{l_z^2}{2mr^2} - \frac{k}{r}$$

Poisson Bracket facts

- $\{ab, c\} = a\{b,c\} + \{a,c\}b$

$$\frac{\partial(ab)}{\partial q}\frac{\partial c}{\partial p} - (q \leftrightarrow p) = \left(a\frac{\partial b}{\partial q}\frac{\partial c}{\partial p} + b\frac{\partial a}{\partial q}\frac{\partial c}{\partial p}\right) - (q \leftrightarrow p)$$

$$= a\left(\frac{\partial b}{\partial q}\frac{\partial c}{\partial p} - (q \leftrightarrow p)\right) + \left(\frac{\partial a}{\partial q}\frac{\partial c}{\partial p} - (q \leftrightarrow p)\right) b$$

✓

- $\vec{l} = \vec{r} \times \vec{p} \rightarrow l_i = \varepsilon_{ijk} r_j p_k$

$$\{l_i, r_a\} = \varepsilon_{ijk} \{r_j p_k, r_a\}$$

$$= \varepsilon_{ijk}\left(r_j \{p_k, r_a\} + \underbrace{\{r_j, r_a\}}_{=0} p_k \right)$$

$$= \varepsilon_{ijk} r_j (-\delta_{ka})$$

$$= \varepsilon_{iaj} r_j$$

Similarly $\{l_i, p_a\} = \pm \varepsilon_{ikl} p_l$

MORE USEFUL

$$\{l_i, r^2\} = \{l_i, \Sigma r_a^2\} = \sum_a 2r_a \{l_i, r_a\}$$

$$= 2r_a (\varepsilon_{iaj} r_j)$$

$$= 2\varepsilon_{iaj} r_a r_j \qquad = 0$$

SIMILARLY $\{l_i, p^2\} = 0$

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2. $\vec{l}$ IS CONSERVED $\quad H = H(p^2, r^2)$

$$\{l_i, H\} = \sum_j \frac{\partial l_i}{\partial p_j} \frac{\partial H}{\partial r_j} - (p \leftrightarrow r)$$

$$= \sum_j \frac{\partial l_i}{\partial p_j} \frac{\partial H}{\partial r^2} \underbrace{\frac{\partial r^2}{\partial r_j}}_{2r_j} - (p \leftrightarrow r)$$

but: $\partial l_i / \partial p_j = \frac{\partial}{\partial p_j} \varepsilon_{iab} r_a p_b$

$$= \varepsilon_{iaj} r_a$$

so: $\frac{\partial l_i}{\partial p_j} \frac{\partial H}{\partial r^2} (2r_j) \sim \varepsilon_{iaj} r_a r_j = 0$

SIMILAR for $\partial l_i / \partial r_j \; \partial H / \partial p^2 \; (2p_j)$

EXERCISE: FOR ~~[illegible]~~ THE KEPLER PROBLEM, SHOW THAT THE RUNGE-LENZ VECTOR IS CONSERVED:

$$\vec{A} = \vec{p} \times \vec{l} + \frac{m\vec{r}k}{r}$$

$$\frac{\partial A}{\partial t} = 0$$

WE KNOW: $\{f(r), \vec{l}\} = \{p^2, \vec{l}\} = 0$

CAN PROMOTE $\frac{p_r^2}{2m} + \frac{l_z^2}{2mr^2} \to \frac{p^2}{2m}$ SINCE $p_z = 0$.

set $m=1$, $k=1$

$$\dot{A} = \{H, A_k\} = \{\tfrac{1}{2}p^2 + \tfrac{1}{r}, \varepsilon_{kij} p_i l_j + r_k/r\}$$

$$= \tfrac{1}{2}\varepsilon_{kij}\{p^2, p_i l_j\} + \varepsilon_{ijk}\{\tfrac{1}{r}, p_i l_j\} + \tfrac{1}{2}\{p^2, r_k/r\} + \{\tfrac{1}{r}, r_k/r\}$$

$\{\tfrac{1}{r}, r_k/r\} \to 0$

$l_j\{p^2, p_i\} + p_i\{p^2, l_j\} = 0$

$$= \tfrac{1}{2}\{p^2, r_k/r\} + \varepsilon_{ijk}\{\tfrac{1}{r}, p_i l_j\}$$

$$\sum_a \frac{\partial(p^2)}{\partial r_a}\frac{\partial(r_k/r)}{\partial p_a} - \frac{\partial(p^2)}{\partial p_a}\frac{\partial(r_k/r)}{\partial r_a} = \sum_a -2p_a \cdot \left(\frac{\delta_{ka}}{r} - \frac{r_k r_a}{r^3}\right)$$

0

$$\frac{\partial_a r_k}{r} + r_k \partial_a (r_b r_b)^{-1/2}$$

for example: let $k = x$ direction:

$$\{p^2, x/r\} = -2p_x\left(\frac{1}{r} - \frac{x^2}{r^3}\right) \qquad = \frac{r^2 - x^2}{r^3}$$

$$+ 2p_y\left(+xy/r^3\right)$$

$$+ 2p_z\left(+xz/r^3\right)$$

$$= \frac{-2p_x(y^2+z^2)}{r^3} + \frac{2x(p_y y + p_z z)}{r^3}$$

claim: the $\varepsilon_{ijk}\{\frac{1}{r}, p_i l_j\}$ term cancels this.