REMARKS ON SECTION $\Rightarrow$ DISCUSSION

main goal is for us to talk to each other about material.

$\rightsquigarrow$ SOME KIND OF SMALL CONTRIBUTION TO YOUR COURSE GRADE

READ MATERIAL AHEAD OF TIME, BRING YOUR QUESTIONS

RULE: ITS FRI AFTERNOON, YOU ALL HAVE BUSY LIVES.
I WON'T BE PERSONALLY OFFENDED IF YOU FALL ASLEEP

BUT: I RESERVE THE RIGHT TO TAKE A PICTURE of YOU & POST IT ON MY FACEBOOK WALL.

A few remarks about VECTORS & TENSORS → for CULTURE, WILL BE USEFUL LATER

IN LEC 1 I MADE A BIG DEAL ABOUT INDICES.
INDICES ARE KIND of A CRUTCH THAT PHYSICISTS USE & THAT MATHEMATICIANS KIND of LOOK DOWN UPON.

THE ACTUAL THINGS WE CARE ABOUT ARE USUALLY SCALARS.

eg: STRESS TENSOR: HOW MUCH "P_z" EMITTED IN $\hat{x}$ DIR?

$$T_{zx} = \hat{\underline{e}}_z \cdot T \cdot \hat{e}_x$$

SO MORE GENERALLY: HOW MUCH MOMENTUM IN DIR $\hat{u}_1$ IS EMITTED IN THE $\hat{u}_2$ DIR?

$$= \hat{u}_1 \cdot T \cdot \hat{u}_2$$

NOW LET'S BE MORE GROWN UP W/ INDICES.

CONVENTION CHOICE: COLUMN VECTOR HAS ~~LOWER~~ UPPER INDEX: V^i
ROW VECTOR HAS ~~UPPER~~ INDEX: V_i

DOT PRODUCTS: $(\underline{W}^T) \cdot \underline{V} = \underbrace{\sum_i W_i V^i}$ allowed contractions of indices only go from top to bot.

So: $W_i V_i$ is NOT ALLOWED AS A SCALAR!
(btw: $W_i V_j$ is A VALID TENSOR)

note index structure!

<u>ROTATIONS</u> $\quad V^i \rightarrow V'^i = R^i{}_j V^j$

$$W_i \rightarrow W'_i = W_j R^j{}_i = R^j{}_i W_j$$

$\Rightarrow$ UPPER & LOWER INDICES TRANSFORM DIFFERENTLY!

$\hookrightarrow$ THIS IS PRECISELY WHAT'S REQUIRED TO KEEP INNER PRODUCT INVARIANT

... why?

<u>DOT/INNER PRODUCTS</u> $\longleftrightarrow$ <u>"THE METRIC"</u>

START w/ VECTORS: W^i, V^i. HOW DO WE FORM A SCALAR? NEED TO LOWER THE INDEX SOMEHOW

$\hookrightarrow$ GO FROM VECTOR $\longrightarrow$ DUAL VECTOR

QM: BRA $\longrightarrow$ KET

GR: VECTOR $\longrightarrow$ one-form

THIS IS DONE w/ THE METRIC TENSOR: g_{ij}

$$\underline{V} \cdot \underline{W} = \left(\underline{V}^T \underline{W} =\right) \underbrace{g_{ij} V^j}_{V_i} W^i$$

<u>NOTE</u>: METRIC DOES NOT TRANSFORM (DESPITE HAVING INDICES) (BY DEFINITION)

$\hookrightarrow R^k{}_i R^l{}_j g_{kl} = g_{ij}$, also: $g_{ij} = g_{ji}$

$$\underline{V} \cdot \underline{W} = g_{ij} V^j W^i \rightarrow g_{ij} (R^j{}_k V^k)(R^i{}_l W^l)$$

$$= V^k \underbrace{R^j{}_k g_{ij}}_{R_{ik}} R^i{}_l W^l$$

$$= \underbrace{R_{ik} R^i{}_l}$$

$$\sim R^T R \quad \checkmark$$

also inverse metric: $g^{ij} = g^{-1}$ s.t. $g^{ij} g_{jk} = \delta^i{}_k$

IN 3D EUCLIDEAN SPACE w/ CARTESIAN COORDINATES: $g_{ij} = \delta^{ij}$

4D = RELATIVITY

NON-EUCLIDEAN = GENERAL RELATIVITY

also in this class: 3D NON-CARTESIAN COORDS FOR EUCLIDEAN SPACE.

Why is it called a METRIC? IT MEASURES DISTANCE. AN ALTERNATE WAY of WRITING IT IS:

$$d\ell^2 = d\underline{\ell} \cdot d\underline{\ell} = g_{ij}\, dx^i dx^j = dx^2 + dy^2 + dz^2$$

BUT WE KNOW THIS IS DIFFERENT FOR DIFF COORDS!

$$d\ell^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

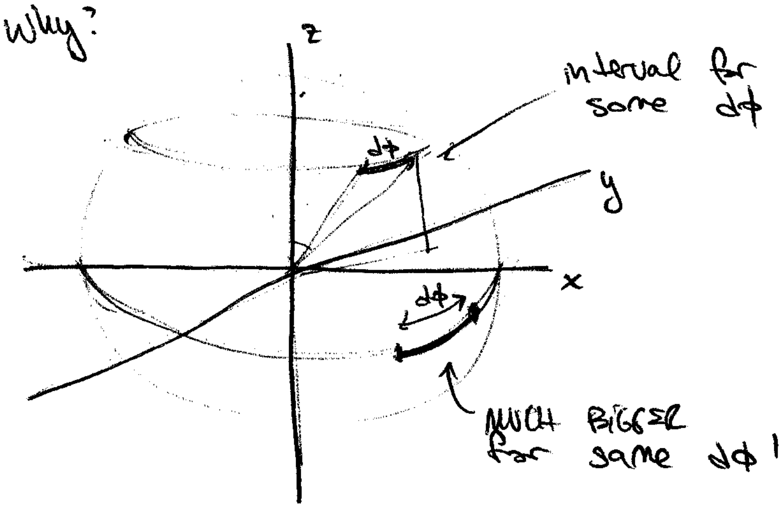
$$= d\rho^2 + \rho^2 d\phi^2 + dz^2$$

SO, eg, IN SPHERICAL COORDINATES:

ORTHOGONAL BASIS $\Rightarrow g$ is DIAGONAL

$$g_{ij} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}}_{\text{depends on position!}}$$

why?



[NOW I APOLOGIZE THAT I HAVE TO BE A LITTLE SKETCHY; for MORE BACKGROUND REFER TO YOUR FAVORITE DIFFERENTIAL GEOMETRY TEXT.]

GRADIENT: WHAT KIND of INDEX?

$$\nabla \sim \frac{\partial}{\partial x^i} \Rightarrow \nabla_i$$

this kind of heuristic logic drives mathematicians crazy

$\forall$ x, not just cartesian

BUT USUALLY WHEN WE TAKE GRADIENT, WE WANT A <u>COLUMN</u> VECTOR

so: $$(\nabla f)^i = \underbrace{g^{ij}}\nabla_j f$$

this is where all the weird coefficients come from!

$$\text{eg } \nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi}$$

THE OTHER DERIVATIVES ARE MORE COMPLICATED BECAUSE THEY ARE RELATED TO DIFFERENTIAL FORMS

see Frankel 2.9c
Boas 14

the full expression for div:

$$\nabla \cdot V = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x_i}\left(\sqrt{\det g}\, V^i\right)$$

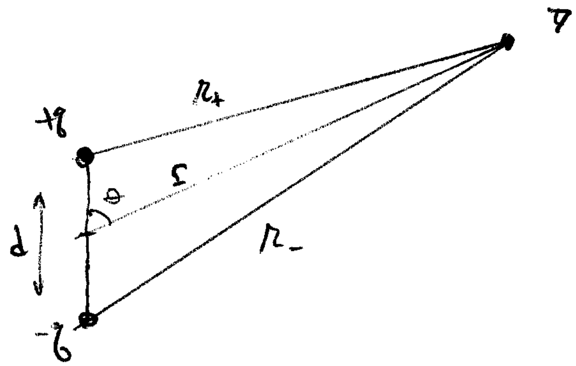
why those determinants? VOLUME FORM
→ THESE COME FROM JACOBIANS of THE VOLUME ELEM.

YOU DON'T NEED TO UNDERSTAND THE DERIVATION of THIS
↳ BUT DO APPRECIATE THE GEOMETRIC FOUNDATION of EVERYTHING WE'RE DOING!

by the way: $dF = j \qquad d{*}F = 0$

OK. NOW WE MOVE ON: MULTIPOLES.

RECALL ELECTRIC DIPOLE (Griffiths 3.4)



INTUITION: FAR AWAY, LOOKS LIKE NO NET CHARGE
BUT THERE'S A CHARGE "MOMENT"
(cf "MOMENT" of INERTIA, etc ...)

$\Phi_{\text{POINT CH}} \sim 1/r$

$\Phi_{\text{DIPOLE}} \sim \cos\theta / r^2$

because $1/r \to$ net source
(b/c $\nabla\, 1/r \sim \delta$)
so must be weaker than this

angular distribution
eg @ $\theta = \pi/2$ STILL DONT SEE
Φ BECAUS $+q$ & $-q$ CONT CANCEL.

still $\underline{E} = \nabla\Phi$, of course!

Physicist: WE MAKE THE RELEVANT TAYLOR EXPANSIONS!

[I think the dipole is usually taught w/ eg law of cosines & then taylor expansion later]

WHAT IS EXPANDABLE? (SMALL): d/r

SO WE'RE THINKING OF $d \ll r$ LIMIT.

I'M NOT GOING TO DO THE TRIG → SEE TEXTBOOK
YOU END UP WITH:

$$\Phi = \frac{2q(d/2)\cos\theta}{r^2} \equiv \frac{\underline{p}\cdot\hat{\underline{e}}_r}{r^2}$$

← up to 4π's & stuff

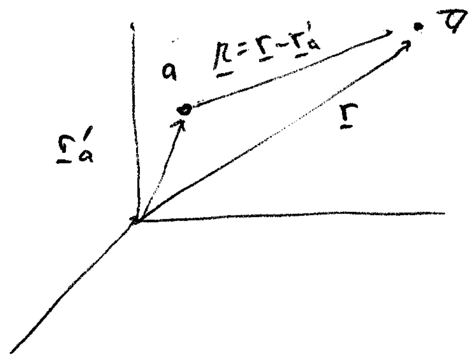
dipole moment

BUT MORE GENERALLY [see Heald + Marion 2-3]

$$\Phi_a = \underbrace{\Phi(\text{assuming } a \text{ @ origin})}_{\equiv \Phi_a^{(0)}} + \text{CORRECTIONS}$$

Φ_a: contrib from ath charge in the config

CORRECTIONS: TAYLOR EXPANSION IN $\underline{r}'_a$. (3D EXPANSION)



R IS THE DISTANCE FOR THE $\Phi \sim 1/r$ LAW, BUT a IS NOT AT THE ORIGIN SO $\underline{R} \neq \underline{r}$.

3D TAYLOR: write $\Phi_a(\underline{r})$ for $\Phi_a(\underline{r}, \underline{r}'_a)$ & evaluate @ $\underline{r}'_a = 0$

implicit; expansion point

$$\Phi_a(\underline{r}) = \Phi_a^{(0)}(r) + \underline{r}'_a \cdot \underline{\nabla'\Phi} + \frac{1}{2}(\underline{r}'_a)^i(\underline{r}'_b)^j \nabla_i\nabla_j\Phi + \cdots$$

GRAD WRT $\underline{r}'_a$

check: this is precisely "half" the dipole term in the simple system above!

SWITCH TO INDEX NOTATION... OTHERWISE HARD TO WRITE

quadrupole

$$\boxed{\Phi = \sum_a \Phi_a}$$

→ BECOMES INTEGRAL FOR CONTINUOUS DIST.

WE'LL TALK ABOUT THIS MORE NEXT WK, AFTER YOU'VE HAD MORE LECTURES (& HAVE MORE QUESTIONS) & A NEW HW. :)

NOTE THAT WE'RE TAKING DERIVATIVES OF WHAT IS BASICALLY

$$\Phi \sim \frac{1}{|\underline{r}-\underline{r}_a|}$$

there is a very useful expansion of this function in terms of LEGENDRE POLYNOMIALS:

$$\frac{1}{|\underline{r}-\underline{r}'|} = \underbrace{\frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^{l}}_{} P_l(\underbrace{\cos\theta}_{\hat{r}\cdot\hat{r}'})$$

EACH TERM IS A "NICE" FUNCTION r^{-n} RATHER THAN $1/\sqrt{\cdots}$

AS YOU KNOW FROM QUANTUM, l IS RELATED TO ANGULAR [MOMENTUM] STUFF.

$P_0 = 1$
$P_1 = \cos\theta$ } PRECISELY WHAT WE FOUND

$$P_2 = \tfrac{1}{2}(3\cos^2\theta - 1) = \tfrac{1}{2}\left[3(\hat{r}\cdot\hat{r}')^2 - 1\right]$$

$$= \tfrac{1}{2}\underbrace{\left(3\hat{r}'_i\hat{r}'_j - \delta_{ij}\right)}_{\text{traceless matrix ("IRREDUCIBLE")}}\hat{r}_i\hat{r}_j$$

SO OFTEN ITS NICE TO WRITE HIGHER POLE TERMS W/ LEGENDRE POLY.

WHAT ARE THESE P's? THEY ARE JUST A BASIS FOR THE ANGULAR PARTS OF 3D FUNCTIONS.

↳ WELL, MORE APPROPRIATELY, THE Y^{l}_{m}'s ARE THIS.

THIS IS A VECTOR SPACE (AS YOU LEARNED IN QUANTUM?!) W/ ITS OWN METRIC, SENSE OF DUAL VECTORS, etc!!