ANNOUNCEMENTS

REMINDER: Mathematica TUTORIAL Rock B03 @ 3:45
suggested: BYO LAPTOP, review tutorial

↳ we'll focus on the kinds of things you'll need for the HW

APP B.1 IN BOOK

REVIEW: FOURIER SERIES

IF YOU KNOW THIS, GREAT — JUST REVIEW
IF NOT: YOU'LL SEE IT IN MATH METHODS OR $\mathbb{C}$ ANALYSIS, PDE, ...

IDEA: FUNCTIONS ARE A VECTOR SPACE

CAN PROJECT ONTO A NICE BASIS & USE INNER PRODUCT W/ BASIS VECTORS TO PICK OUT COMPONENTS.

eg: VECTOR $\underline{V}$. WHAT IS THE x-COMPONENT?

$$V_x = \underline{V}\cdot\underline{\hat{e}}_x = \langle \hat{e}_x, \underline{V}\rangle$$

MORE GENERALLY, ARBITRARY DIRECTION $\hat{e}_a$; WHAT IS $\hat{a}$-COMPONENT of $\underline{V}$? (ASSUMING ORTHONORMAL COORD)

$$V_a = \langle \hat{e}_a, \underline{V}\rangle$$

for FUNCTIONS, USUALLY USE L_2 NORM:

$$\langle f, g\rangle = \int f^*(x)\, g(x)\, dx$$

OVER RELEVANT DOMAIN

for $\mathbb{C}$ func, eg in QUANTUM
[not relevant for this class]

& for FOURIER SERIES, USE sin (or cos, or exp) AS BASIS.

ORTHOGONALITY: $\int_{-a}^{a} \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}x\right)dx = a\,\delta_{mn}$

$$\hat{e}_n = \sin\left(\frac{n\pi}{a}x\right)$$

THEN: $\langle \hat{e}_n, f \rangle \equiv f_n = \frac{1}{a}\int_{-a}^{a} f(x) \sin\left(\frac{m\pi}{a}x\right) dx$

↳ CAN WRITE $f = (\cdots, f_n, f_{n+1}, \cdots)$ ← ∞ DIM VECTOR SPACE

↳ IN FACT, $f(x)$ IS JUST A REPRESENTATION OF f ON A BASIS OF δ FUNCTIONS:

$$f(x) = \sum_y f(y)\,\delta(x-y) \longrightarrow \int f(y)\,\delta(x-y)\,dy$$

Why is this useful? IT WILL GIVE US A HANDLE FOR FINDING SOLUTIONS TO THE LAPLACE EQN W/ BOUNDARY COND!

↳ gives a way to match a general func w/ unknown coefficients to a known BC.

<u>REMARK</u>: GRIFFITHS §3.3 IS <u>MUCH</u> EASIER TO READ.

MAIN IDEA: $\phi(x,y,z) = X(x)\,Y(y)\,Z(z)$

LAPLACE EQN $\longrightarrow$

$$\left.\begin{aligned} \frac{1}{X}X'' &= \alpha^2 \\ \frac{1}{Y}Y'' &= \beta^2 \\ \frac{1}{Z}Z'' &= \gamma^2 \end{aligned}\right\} \text{ w/ } \alpha^2+\beta^2+\gamma^2 = 0$$

@ LEAST ONE IS NEG & AT LEAST ONE IS POS!

(OR ALL ZERO!)

↳ REDUCES TO 3 SEPARATE ODE BUT HAVE TO IMPOSE BC CAREFULLY

↳ EASY TO MAKE LIFE UNNECESSARILY HARD.

HOW TO PICK WHICH ARE POS / NEG? ⟶ ULTIMATELY DOESN'T MATTER: BC WILL FORCE YOU INTO RIGHT CHOICE (but α may be imaginary)

$\frac{1}{X}X'' = +\alpha^2$ EXPONENTIAL $X \sim e^{\alpha x}$

$\frac{1}{X}X'' = -\alpha^2$ SINUSOID $X \sim \sin(\alpha x)$

eg 2 DIRICHLET BC

HOW TO PICK $e^{\alpha x}$ vs $\sinh(\alpha x)$
$e^{i\alpha x}$ vs. $\sin(\alpha x)$?

USUALLY ONE IS MORE NATURAL. (eg in HW, STICK TO THE [HYPERBOLIC] TRIG FUNCTIONS)

Strategy:

- CLEARLY WRITE OUT BC
- ~~[illegible]~~ WRITE OUT APPROPRIATE GENERAL SOLUTIONS

$$\text{eg} \quad X(x) = \sum_n A_n \sin(k_n x) \qquad (+\cos\text{ term?})$$

$k_n \sim \alpha$ ON PVS. PAGE

similar w/ Y, Z.

WHY SUM? HAVEN'T IDENTIFIED k_n YET. THERE MIGHT BE MANY ALLOWED k_n, EACH SOLUTION $\sin(k_n x)$ MUST BE SUMMED w/ RELATIVE WEIGHTS DETERMINED BY BC.

- USE "EASIEST" BC (eg @ $x=0$, DIRICHLET, ...) TO FIX RELATIVE COEFFICIENTS.

$$\text{eg.} \quad X(x) = \sum_n A_n \sin(k_n x) + B_n \cos(k_n x)$$

w/ bc $X(0) = 0 \Leftrightarrow \phi(0,y,z) = 0$, a plane.

$\hookrightarrow \Rightarrow X(0) = \sum_n B_n \cos(0) \Rightarrow B_n = 0$

more generally: $B_n = f(A_n, \underbrace{x_0}_{\text{BC LOCATION}})$
(PICKING GOOD BASIS FUNCTIONS HELPS)

eg: TRIG: $Y(y) = \sum_n A_n \sinh(k_n y) + B_n \cosh(k_n y)$

w/ $Y(y_0) = 0$. WHAT TO DO?

$\sinh(0) = 0$
$\cosh(x) > 0 \quad \forall x$

EXPAND: $Y(y) = \sum_n A_n \sinh(k_n(y-y_0)) + B_n \cosh(k_n(y-y_0))$

$Y(y_0) = 0 \Rightarrow B_n = 0.$

- USE "OPPOSITE TO EASIEST" BC TO fix FREQUENCY

eg: $X(0)=0$, $X(a)=0$ $\longrightarrow$ $X = \sum_n A_n \sin(k_n x)$

↳ fix A_n OR k_n

A_n → GIVES TRIVIAL SOLUTION

k_n → MORE GENERAL.

where sine vanishes

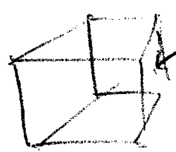
$$\Rightarrow k_n a = n\pi$$

$$\Rightarrow \boxed{k_n = \frac{n\pi}{a}} \quad \text{for } \forall\, n \in \mathbb{Z}$$

- USE REMAINING BC W/ FOURIER'S TRICK TO PROJECT OUT A CLOSED FORM EXPRESSION FOR THE COEFFICIENTS.

⊙ USUALLY YOU RESERVE THIS FOR THE "HARDEST" BC

eg.



$\Phi(x_0, y, z) = f(y,z)$ ← GIVEN

then I CAN TAKE INNER PRODUCTS

$$\langle \Phi, \hat{e}_m \rangle \sim \int_0^a f(y) \sin\left(\frac{m\pi}{a} y\right)$$

PROJECTS OUT C_m COEFFICIENT.

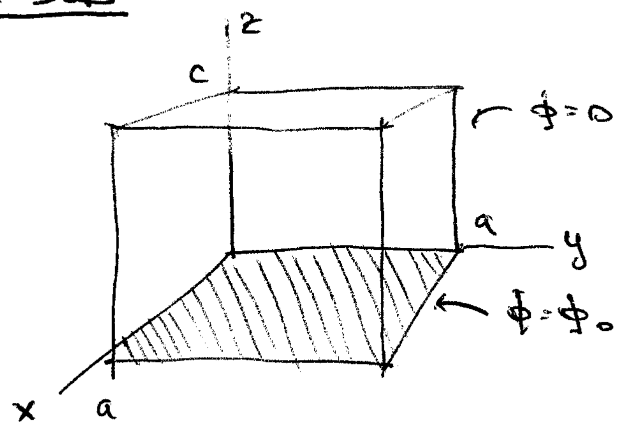
⊙ IF THE SYSTEM IS VERY COMPLICATED: TRY TO BREAK IT DOWN INTO SIMPLER SYSTEMS & USE SUPERPOSITION.

THE ONLY WAY TO GET USED TO THIS IS TO DO A BUNCH OF PRACTICE PROBLEMS. TRY GRIFFITHS IF YOU WANT MORE REVIEW.

↳ (try Jackson if you're adventurous)

HARD PART: A LOT OF CHOICES IN HOW TO SOLVE THESE PROBLEMS

H&M eg. 3.26



ACTUALLY, DO EASIER CASE WHERE $\phi = \phi_0$ @ $z = c$ & $z = 0$ WAS $\phi = 0$.

$$\frac{1}{X}X'' = -k^2$$
$$\frac{1}{Y}Y'' = -\ell^2$$
$$\frac{1}{Z}Z'' = (k^2+\ell^2)$$

why MINUS? BC STRONGLY SUGGESTS SINUSOID!

① SIMPLEST BC: $X(0) = 0$

WRITE: $X_n(x) = A_n \sin(k_n x) + B_n \cos(k_n x)$

$X_n(0) = B_n \Rightarrow \boxed{B_n = 0}$ ✓ → $X_n = A_n \sin(k_n x)$

SIMILARLY: $Y(0) = 0 \Rightarrow Y_m(y) = C_m \sin(\ell_m y)$

② next simplest BC: $X(a) = 0$ ⇒ fixes k_n

$X_n(a) = A_n \sin(k_n a) \Rightarrow k_n a = n\pi \Rightarrow \boxed{k_n = \frac{n\pi}{a}}$

SIMILARLY: $Y(b) = 0 \Rightarrow \boxed{\ell_m = \frac{m\pi}{b}}$

③ NOW DEAL w/ HARDER EVQ

DEF: $\gamma_{nm} = \sqrt{k_n^2 + l_m^2}$ for convenience

$$Z_{nm}(z) = \tilde{E}_{nm} e^{\gamma_{nm} z} + \tilde{F}_{nm} e^{-\gamma_{nm} z}$$ ← not very nice (BUT DOABLE)

TWO INDICES SEE WHY?

EASIER:

$$= E_{nm} \sinh(\gamma_{nm} z) + F_{nm} \cosh(\gamma_{nm} z)$$

$\cosh(0) \neq 0$

WHY? $Z(0) = F_{nm}$

$$\Rightarrow Z_{nm}(z) = E_{nm} \sinh(\gamma_{nm} z)$$

think: ① what if we used exp expressions?

② what if $Z(c) = 0$?
STILL A WAY TO USE sinh & cosh?

↳ ANSW: shift coords so you can use $\cosh(x) > 0 \ \forall x$ trick.

④ NOW DO THE TOUGH ONE:

$$\Phi(x,y,c) = \Phi_0$$

double fourier!

$$= \underbrace{A_n C_m E_{nm}}_{\equiv G_{nm}} \sin(k_n x)\sin(l_m y) \sinh(\gamma_{nm} c)$$

$\equiv G_{nm}$ ← WHAT WE WANT

WRITE: $H_{nm} \equiv G_{nm} \sinh(\gamma_{nm} c)$ for simplicity

OUR MAIN TOOL: PROJECT OUT SPECIFIC COEFFICIENT USING:

$$\int_0^a \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}x\right)dx = \frac{a}{2}\,\delta_{mn}$$

NOTE: IN FUNCTION SPACE: $\boxed{H_{nm}}\ \sin(k_n x)\sin(l_m y)$ IS A 2-TENSOR!

WE WANT TO PROJECT OUT $(\hat{e}^{(x)}_n)\cdot H\cdot(\hat{e}^{(y)}_m)$

DOTS ARE L_2 NORM (the Fourier integral)

$$\int_0^a dx\int_0^b dy\left(\sum_{nm} H_{nm}\sin\Big(\overbrace{\tfrac{n\pi}{a}}x\Big)\sin\Big(\overbrace{\tfrac{m\pi}{b}}y\Big)\right)\sin\left(\frac{p\pi}{a}x\right)\sin\left(\frac{q\pi}{b}y\right)$$

$$= \Phi_0\ \underbrace{\int_0^a dx\,\sin\left(\frac{p\pi}{a}x\right)}\ \underbrace{\int_0^b \sin\left(\frac{q\pi}{b}y\right)dy}$$

$\frac{a}{p\pi}\left(1-\cos(p\pi)\right)$

$= \begin{cases} \frac{2a}{p\pi} & \text{if } p = \text{odd} \\ 0 & \text{otherwise}\end{cases}$

similar: $\begin{cases} \frac{2b}{q\pi} & \text{if } q = \text{odd} \\ 0 & \text{otherwise}\end{cases}$

$\boxed{\text{think: what if } \Phi_0 \to \Phi_0(x,y)\ ?}$

LHS: $\sum_{nm} H_{nm}\ \frac{a}{2}\delta_{np}\cdot\frac{b}{2}\delta_{mq} = \frac{ab}{4}H_{pq}$

$$\Rightarrow\ H_{pq} = \begin{cases}\frac{\Phi_0}{pq\pi^2} & \text{if } p,q \text{ odd} \\ 0 & \text{otherwise}\end{cases}$$

$$\Rightarrow\ G_{nm} = \frac{1}{\sinh(\gamma_{nm}c)}H_{nm} = \frac{\Phi_0}{nm\,\pi^2\sinh(\gamma_{nm}c)} \qquad (n,m\in\text{odd})$$

FULL SOLUTION:

$$\Phi(x,y,z) = \sum G_{nm} \sin(k_n x) \sin(l_m y) \sinh(\gamma_{nm} z)$$

$$= \sum_{nm} \frac{\Phi_0}{nm\pi^2 \sinh(\gamma_{nm} c)} \sin\left(\frac{n\pi}{a}x\right) \sin\left(\frac{m\pi}{b}y\right) \sinh(\gamma_{nm} z)$$

ODD n

w/ $\gamma_{nm} = \sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}$

Phew!

ALL PROBLEMS ARE VARIATIONS OF THIS.

<u>Think</u>:

- WHAT IF BC HAS NEUMANN BC?
  eg. $Z'(c) = \Phi_0$?

- WHAT IF TWO (or more) SIDES ARE "HARD"?
  ↳ eg. $Z(c) = f(x,y)$
  $Z(0) = \tilde{f}(x,y)$?

<u>HINT</u>: SUPERPOSITION OF 2 CHARGE CONFIGS EA W/ ONLY ONE HARD SIDE!