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Beam Damping in Optical Stochastic Cooling

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An important necessary condition for transverse phase space damping in the optical stochastic cooling (also applicable in the microwave stochastic cooling) with transit-time method is derived. The longitudinal and transverse damping dynamics for the optical stochastic cooling is studied. An optimal laser focusing condition for laser-beam interaction in the correction undulator was also obtained. The amplification factor and the output peak power of the laser amplifier are found to differ substantially from earlier publications. The required laser amplification power can be large for hadron colliders at very high energies.

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I. INTRODUCTION

The stochastic cooling, invented by S. van der Meer in 1968, has been first experimentally demonstrated at the Intersecting Storage Ring (ISR) at CERN, and used for anti-proton cooling and collection facilities. The success of the stochastic cooling leads to many new discoveries in particle physics [1, 2].

Applications of the stochastic cooling to high energy storage rings encounter a few difficulties. First, the phase space areas of beams in a high energy accelerators are adiabatically damped, thus the stochastic cooling method becomes less efficient. Furthermore, the bunch length (σ_τ) is shorter in high energy storage colliders, the cut-off frequency ($\sim \frac{1}{\sigma_\tau}$) for the coherent signal is extended upward to the GHz region. The Schottky signal has often been contaminated by the coherent beam signals [3]. Without a good Schottky signal, it would be difficult to carry out the stochastic cooling.

High energy charged particles emit photons in dipoles. The photon emission is a random process. Using the photons instead of microwave signals in beam cooling would solve the problem of coherent signal contamination, and may dramatically enhance the cooling rate. The optical stochastic cooling (OSC) was proposed by Mikhailichenko and Zolotorev [4], in which a quadrupole wiggler and a longitudinal kicker system at a high dispersion location were applied to damp betatron and synchrotron motions via the synchro-betatron coupling. Subsequently, Zolotorev and Zholents applied transit-time method, which is a traditional method in stochastic cooling, to optical stochastic cooling [5]. The scheme of a typical optical stochastic cooling and formula related to damping were also derived in Ref. [5].

In general, a high energy charge particle emits synchrotron radiation in a synchrotron. The critical frequency is $\omega_c = \frac{3}{2}\gamma^3 c/\rho$, where γ is the relativistic Lorentz factor, c is the speed of light, and ρ is the bending radius. The number of photons emitted per revolution is $\mathcal{N}_\gamma = 5\pi\alpha\gamma/\sqrt{3}$, where $\alpha = q^2/(4\pi\epsilon_0\hbar c)$, q is the charge of the particle, ϵ_0 is the electric permittivity of free space, and $\hbar$ is the Planck constant divided by 2π. In an undulator with planar magnetic field, the wavelength of the undulator radiation is $\lambda = \lambda_{\rm u}(2+K^2)/(4\gamma^2)$ with a bandwidth of $\Delta\omega|_{\rm FWHM} = \omega/N_{\rm u}$, where $N_{\rm u}$ is the number of undulator period, $K = qB_{\rm u}\lambda_{\rm u}/(2\pi mc)$ is the undulator strength parameter, $B_{\rm u}$ is the undulator field strength, and $\lambda_{\rm u}$ is the undulator wavelength. The number of photons, emitted within the solid angle $\lambda/(N_{\rm u}\lambda_{\rm u})$ and bandwidth $\Delta\omega|_{\rm FWHM}$, is $\mathcal{N}_\gamma = \pi\xi\alpha[{\rm JJ}]^2$ where $\xi = K^2/(2+K^2)$, and the factor $[{\rm JJ}] = J_0(\frac{1}{2}\xi) - J_1(\frac{1}{2}\xi)$. The emitted photon can be amplified and used to give proper kick in energy and betatron coordinates. With a proper choice of the beam parameters, the phase space volume of the beam can be damped.

Although the basic principle of the optical stochastic cooling has been published in 1994, the requirements of the beam cooling section have not been fully analyzed. In particular, there are deficiencies in an earlier paper [5] on the beam transport properties in the derivation of the optical stochastic cooling. This paper is intended to derive the necessary conditions for the beam transport system for the optical stochastic cooling with transit-time method, and study the OSC cooling dynamics. In section II, the principle of transit-time method of optical stochastic cooling is briefly introduced. In section III, the damping rates and the transfer matrix condition of equal decrements are derived. In section IV, the amplification factor, an optimal optical focusing condition for the inverse free electron laser, and the peak output power requirement are discussed. The conclusion is given in Sec. V.

II. TRANSIT-TIME METHOD OF OPTICAL STOCHASTIC COOLING

A typical stochastic cooling system consists of a pickup, an amplifier, and a kicker [1, 2]. The optical stochastic cooling includes two undulators, a particle beam bypass, and an optical amplifier. A schematic drawing of this insertion can be found in Fig. 1 of Ref. [5]

The electromagnetic (EM) wave radiated by a charged particle in the first undulator is amplified by the optical amplifier, while the particle travels in the beam-bypass. The amplified EM wave and the particle are brought together to interact in the second undulator. This will change the particle's energy. The amount of energy change depends on the magnitude of the EM wave and the relative phase between the particle's and the EM wave's transit times from the first to the second undulators.

We consider a test particle with a momentum deviation $\delta_i = \Delta P_i/P$, and the betatron phase space coordinates (x_i, x_i'). In the Frenet-Serret coordinate system, the path length of the test particle in the bypass section is [6]

$$\ell_i = \int_{s_1}^{s_2} \sqrt{\tilde{x}'^2 + \tilde{z}'^2 + (1+\frac{\tilde{x}}{\rho})^2}\, ds \approx \int_{s_1}^{s_2} (1+\frac{\tilde{x}}{\rho}) ds, \tag{1}$$

where $\hat{x}, \hat{s}$ and $\hat{z}$ form a curvilinear coordinate system with a horizontal bending radius ρ, the coordinates $\tilde{x}$ and $\tilde{z}$ are the deviation from a reference orbit, and s is the longitudinal coordinate along the reference orbit. We have also assumed $\tilde{x}' \ll 1$ and $\tilde{z}' \ll 1$ to obtain the last approximate equality. For a bypass with planar geometry, the transverse displacement of an orbiting particle is given by $\tilde{x} = x_{\rm co}(s) + M_{11}(s,s_1)x_1 + M_{12}(s,s_1)x_1' + D(s)\delta$, where x_1 and x_1' is the betatron phase space coordinates at s_1, $M_{11}(s,s_1)$ and $M_{12}(s,s_1)$ are transport matrix elements of the Hill's equation, $x_{\rm co}(s)$ is the closed orbit around the reference orbit, and $D(s)$ is the dispersion function. The path length for an i-th particle in the bypass region becomes

$$\ell_i = \ell_0 + x_{i1}I_1 + x_{i1}'I_2 + \delta_i I_D, \tag{2}$$

where x_{i1}, x_{i1}' are the conjugate phase space coordinates for the i-th particle at the location s_1, and the integrals I_1, I_2, and I_D are

$$I_1 = \int_{s_1}^{s_2} \frac{M_{11}(s,s_1)\,ds}{\rho(s)}, \qquad I_2 = \int_{s_1}^{s_2} \frac{M_{12}(s,s_1)\,ds}{\rho(s)}, \qquad I_D = \int_{s_1}^{s_2} \frac{D(s)\,ds}{\rho(s)}. \tag{3}$$

where the integrals are carried out from the first undulator at s_1 to the second undulator at s_2 via the particle beam bypass.

In the first undulator, a test particle radiates an EM wave propagating in the s-direction: $\mathcal{E}_i = \mathcal{E}_0 \sin(ks - \omega t + \phi_i)$ with electric field amplitude $\mathcal{E}_0$ and phase ϕ_i. The wave number and frequency are $k = 2\pi/\lambda$ and $\omega = kc$. This radiation propagates to the optical amplifier, while the particle follows the bypass and traverses it in a time $\Delta t_i = \ell_i/\beta c$, where βc is the speed of the particle.

The time Δt_0 required for radiation to pass all the way between undulators, including the amplifier delay, must be constrained and maintained by a feedback system to yield the condition $\ell_0 - c\Delta t_0 = (n \pm \frac{1}{4})\lambda$, where $n = 0, 1, 2, \cdots$, and the $\pm$ sign depends on the beam transport property in the bypass. The test particle arrives at the second undulator with a time delay $\delta(\Delta t) = \Delta t_i - \Delta t_0$ and with a phase shift

$$\Delta\phi_i = k(\ell_i - \ell_0) = k\left[x_i I_1 + x_i' I_2 + \delta_i I_D\right], \tag{4}$$

relative to the phase of the electric field at zero crossing. For simplicity, hereafter, we use x_i, x_i', and δ_i as the betatron phase-space coordinates and fractional off-momentum variable of the i-th particle at the first undulator location. In the second undulator, the particle interacts with the electric field of its own radiation. The fractional change of its momentum is [7]:

$$\delta P_i/P = -\left[\mathrm{sgn}(I_D)\right] G \sin(\Delta\phi_i), \tag{5}$$

where $\mathrm{sgn}(I_D)$ is the sign of I_D, $G = gq\,\mathcal{E}_0 N_{\rm u}\lambda_{\rm u} K[\mathrm{JJ}]/(2c\gamma P)$ is the amplitude of the fractional momentum gain-factor, q is the magnitude of the particle charge, $N_{\rm u}$ is the number of undulator periods, g is the amplification factor of the optical amplifier, and δP_i is the amount of the momentum change related to the coherent longitudinal kick $\Delta\delta_i = \delta P_i/P$.

Let D_2 and D_2' be the dispersion function and its derivative at the second undulator. The changes of the particle betatron coordinates at the exit of the second undulator are $\Delta x_{i2} = -D_2(\delta P_i/P)$ and $\Delta x_{i2}' = -D_2'(\delta P_i/P)$, where x_{i2} and x_{i2}' are the phase space coordinates of the i-th particle at the second undulator location.

Thus, after passing the entire cooling insertion, the test particle has received coherent longitudinal and transverse kicks that are proportional to a linear combination of the particle's momentum deviation and betatron deviations. We will see in the next section that a proper choice of the parameters of the bypass lattice makes it possible to use these kicks to simultaneously damp transverse and longitudinal oscillations.

III. COOLING RATES

We have so far considered the interaction of a test particle with the EM wave of its own radiation. However, each particle also interacts with the EM waves emitted by other particles in a sample within a distance less than $N_u\lambda$. These interactions constitute the incoherent component of the kick received by the particle. Assume that a test particle interacts with N_s electromagnetic waves (including its own wave) in a sample. The change of the particle's momentum at the exit of the cooling insertion becomes

$$\delta_{ic} = \delta_i - [\mathrm{sgn}(I_D)]\, G \sum_j^{N_s} \sin(\Delta\phi_i + \psi_{ij}) \tag{6}$$

where δ_{ic} is the relative momentum of the i-th particle after the longitudinal kick, N_s is the number of particles in the sample, $\psi_{ij} = \Delta\phi_j - \Delta\phi_i$, and $\mathrm{sgn}(I_D)$ is the sign of the integral I_D.

A. Longitudinal effects

Hereafter, we assume that $I_D > 0$ with a proper phase shift. A test particle interacts with the electromagnetic waves radiated from the sample of N_s particles. We have to evaluate the ensemble average of the quadratic change:

$$\Delta(\delta_i^2) = \delta_{ic}^2 - \delta_i^2 = -2\delta_i G \sum_j^{N_s} \sin(\Delta\phi_i + \psi_{ij}) + G^2 \left[\sum_j^{N_s} \sin(\Delta\phi_i + \psi_{ij})\right]^2 .$$

Using the fact that $\langle \sin^2(\Delta\phi_i + \psi_{ij})\rangle = \frac{1}{2}$ for a random sample of N_s particles, the ensemble average of the second term is

$$\langle G^2 [\sum_j^{N_s} \sin(\Delta\phi_i + \psi_{ij})]^2 \rangle = \frac{1}{2} G^2 N_s,$$

which contributes to heating. The ensemble average of the coherent kick term is $\langle -2\delta_i G \sum_j^{N_s} \sin(\Delta\phi_i + \psi_{ij})\rangle \approx \langle -2\delta_i G \sin(\Delta\phi_i)\rangle$, i.e.

$$\langle -2\delta_i G \sin(\Delta\phi_i)\rangle = -\Im\left\{2G \int \delta e^{ik(xI_1 + x' I_2 + \delta I_D)} \rho(x, P_x, \delta) dx dP_x d\delta\right\} \tag{7}$$

where $\Im\{\cdots\}$ stands for the imaginary part. The distribution function is given by

$$\rho(x, P_x, \delta) = \frac{1}{(2\pi)^{3/2}\sigma_x^2\sigma_\delta} \exp\left[-\frac{x^2 + P_x^2}{2\sigma_x^2} - \frac{\delta^2}{2\sigma_\delta^2}\right] \tag{8}$$

with x and $P_x = \beta x' + \alpha x$ as the normalized betatron phase space coordinates at the first undulator location s_1, and δ as the fractional off-momentum coordinate. The integral can be carried out easily, and the longitudinal damping decrement becomes

$$\alpha_\delta \equiv -\frac{\langle \delta_{ic}^2 - \delta_i^2 \rangle}{\sigma_\delta^2} = 2GkI_D e^{-u} - \frac{G^2 N_s}{2\sigma_\delta^2} \tag{9}$$

where

$$u = \frac{1}{2} k^2 [(\beta_1 I_1^2 - 2\alpha_1 I_1 I_2 + \gamma_1 I_2^2)\epsilon_x + I_D^2 \sigma_\delta^2], \tag{10}$$

is a measure of the total thermal energy of the beam. The optimal momentum gain-factor and the maximum damping decrement are

$$G_\delta = \frac{2kI_D\sigma_\delta^2}{N_s} e^{-u}; \qquad \alpha_\delta|_{\max} = \frac{2k^2 I_D^2 \sigma_\delta^2}{N_s} e^{-2u}. \tag{11}$$

B. Transverse effects

As the particle gains or loses energy by its interaction with the electric field of itself and its sampling partners, the corresponding momentum closed orbit is also modified. Thus the betatron phase space coordinates are changed as well. This may generate heating and cooling effect to the beam. The change of transverse betatron coordinates are (for $I_D > 0$)

$$x_{i2c} = x_{i2} + D_2 G \sum_j^{N_s} \sin(\Delta\phi_i + \psi_{ij}), \tag{12}$$

$$x'_{i2c} = x'_{i2} + D'_2 G \sum_j^{N_s} \sin(\Delta\phi_i + \psi_{ij}), \tag{13}$$

where (x_{i2}, x'_{i2}) and (x_{i2c}, x'_{i2c}) are the betatron phase space coordinates of the i-th particle before and after correction at the second undulator location, and D_2, D'_2 are the value of the dispersion function at the second undulator location.

Now we transform the phase space coordinates into the normalized phase space coordinates $(x, P_x = \beta_x x' + \alpha_x x)$. The change of the invariant action of the betatron phase-space coordinates is $\beta_2\Delta\epsilon_i = P^2_{xi2c} + x^2_{i2c} - (P^2_{xi2} + x^2_{i2})$, where β_2 is the betatron amplitude at the second undulator location and ϵ_i is twice the action of the i-th particle. We find

$$\begin{aligned}\beta_2\Delta\epsilon_i &= +2P_{xi2}P_{D2}G\sum_j^{N_s}\sin(\Delta\phi_i + \psi_{ij}) + 2x_{i2}D_2G\sum_j^{N_s}\sin(\Delta\phi_i + \psi_{ij}) \\ &\quad +\beta_2\mathcal{H}_2G^2\left[\sum_j^{N_s}\sin(\Delta\phi_i + \psi_{ij})\right]^2\end{aligned} \tag{14}$$

where $P_{xi2} = \beta_2 x'_{i2} + \alpha_2 x_{i2}$ is the normalized betatron coordinate for the i-th particle at the second undulator location, $P_{D2} = \beta_2 D'_2 + \alpha_2 D_2$ is the normalized dispersion phase space coordinate at the second undulator location, β_2 and α_2 are the value of the β_x and α_x at the second undulator location, and $\mathcal{H}_2 = \frac{1}{\beta_2}(D_2^2 + P_{D2}^2)$ is the value of the $\mathcal{H}$-function at s_2.

The ensemble average of the quadratic terms is $\frac{1}{2}G^2 N_s \beta_2 \mathcal{H}_2$, which contributes to quantum fluctuation like that of synchrotron radiation damping. The ensemble average of the coherent kick is given by $\langle K_{c1}\rangle = \langle 2P_{xi2}P_{D2}G\sum_j^{N_s}\sin(\Delta\phi_i + \psi_{ij})\rangle \approx \langle 2P_{xi2}P_D G\sin(\Delta\phi_i)\rangle$, and $\langle K_{c2}\rangle = \langle 2x_{i2}D_2G\sum_j^{N_s}\sin(\Delta\phi_i + \psi_{ij})\rangle \approx \langle 2x_{i2}D_2G\sin(\Delta\phi_i)\rangle$, i.e.

$$\langle K_{c1}\rangle = \Im\left\{2G\int P_{D2}P_{xi2}\; e^{ik(xI_1 + x'I_2 + \delta I_D)}\rho(x, P_x, \delta)dxdP_xd\delta\right\}, \tag{15}$$

$$\langle K_{c2}\rangle = \Im\left\{2G\int D_2 x_{i2}\; e^{ik(xI_1 + x'I_2 + \delta I_D)}\rho(x, P_x, \delta)dxdP_xd\delta\right\}. \tag{16}$$

The distribution function, shown in Eq. (8), is a function of the phase space coordinates at the first undulator. The ensemble average is equivalent to integrating over the phase space coordinates of the ensemble at the location of the first undulator, while x_{i2} and P_{xi2} are the phase space coordinates of the particle at the second undulator location. Expressing x_{i2} and P_{xi2} in terms of x and P_x at the first undulator location, we obtain the relative transverse cooling

$$\alpha_x = -\frac{\langle P^2_{x2c} + x^2_{2c} - (P^2_{x2} + x^2_2)\rangle}{\sigma^2_{x2}} = 2GkI_\perp e^{-u} - \frac{G^2 N_s \mathcal{H}_2}{2\epsilon_x}, \tag{17}$$

where

$$\begin{aligned}I_\perp &= -\frac{\beta_1}{\beta_2}\left\{P_{D2}\left[\left((\beta_2 M_{21} + \alpha_2 M_{11}) - \frac{\alpha_1}{\beta_1}(\beta_2 M_{22} + \alpha_2 M_{12})\right)(I_1 - \frac{\alpha_1}{\beta_1}I_2)\right.\right. \\ &\quad \left.\left. + \frac{1}{\beta_1^2}(\beta_2 M_{22} + \alpha_2 M_{12})I_2\right] + D_2\left[(M_{11} - \frac{\alpha_1}{\beta_1}M_{12})(I_1 - \frac{\alpha_1}{\beta_1}I_2) + \frac{1}{\beta_1^2}M_{12}I_2\right]\right\}.\end{aligned} \tag{18}$$

The transverse cooling requires the condition $I_\perp > 0$ (for $I_D > 0$). This is an important condition for the design of the bypass optics.

If the betatron phase space coordinates are properly chosen, the coherent kicks will also produce coherent cooling to the transverse emittance. The necessary condition is $I_\perp > 0$. The optimal gain factor and the maximum damping decrement for the transverse cooling are

$$G_x = \frac{2kI_\perp \epsilon_x}{N_s \mathcal{H}_2} e^{-u}, \qquad \alpha_x|_{\max} = \frac{2k^2 I_\perp^2 \epsilon_x}{N_s \mathcal{H}_2} e^{-2u}. \tag{19}$$

Making a constraint of $-I$ with reflection symmetry for the bypass insert and having the undulators placed at the betatron waists ($\alpha_1 = \alpha_2 = 0 = D_2' = 0$), as that of Ref. [5], we obtain $I_\perp = D_2 I_1$. Our resulting damping decrements does not agree with Eq. (6) of Ref. [5].

C. Stochastic Cooling Dynamics

The cooling process can be expressed as

$$\frac{d\epsilon_x}{dt} = -\frac{2GkI_\perp \epsilon_x}{T_0} e^{-u} + \frac{G^2 N_s \mathcal{H}_2}{2T_0}, \tag{20}$$

$$\frac{d\sigma_\delta^2}{dt} = -\frac{2GkI_D \sigma_\delta^2}{T_0} e^{-u} + \frac{G^2 N_s}{2T_0}, \tag{21}$$

where T_0 is the revolution period. The momentum gain-factor G is set by the laser amplifier [8]. If the optimal gain factors for the momentum and transverse cooling are the same, we can set the laser gain factor to obtain an optimal momentum gain-factor. The condition for equal optimal gain-factors is $I_D \mathcal{H}_2 \sigma_\delta^2 = I_\perp \epsilon_x$. In this case, the ratio of the damping decrements becomes $\alpha_\delta/\alpha_\perp = I_D/I_\perp$. However, if $I_D \neq I_\perp$, the equal gain condition can not be fulfilled at all time.

1. *Cooling Dynamics for equal decrements*

For the equal decrement condition, the particle bypass line should be designed with the condition: $I_\perp = I_D$. The beam will maintain the equilibrium condition with $\epsilon_x = \mathcal{H}_2 \sigma_\delta^2$. Let G_0 be an initial gain factor. The equation of damping dynamics becomes

$$\frac{du}{dt} = -\frac{2G_0 k I_D}{T_0} u e^{-u} + \frac{G_0^2 N_s v}{2T_0}, \tag{22}$$

where

$$v = \frac{1}{2} k^2 [(\beta_1 I_1^2 - 2\alpha_1 I_1 I_2 + \gamma_1 I_2^2)\mathcal{H}_2 + I_D^2]. \tag{23}$$

The equilibrium emittance is reached when $du/dt = 0$. Figure 1 shows the right hand side of Eq. (22). Note that cooling is possible when $u_{\rm eq} \le u \le u_{\rm th}$, where $u_{\rm eq}$ is the equilibrium thermal energy and $u_{\rm th}$ is the cooling threshold energy. The initial laser power gain should be adjusted so that the beam condition falls within the cooling limit.

2. *The optimal gain factor*

The optimal gain factor G for the cooling equation with equal decrement is

$$G_{\rm opt} = \frac{2kI_D}{vN_s} u e^{-u}. \tag{24}$$

With this optimal gain factor, that depends on the lattice and beam conditions, the cooling equation becomes

$$\frac{du}{dt} = -\frac{2k^2 I_D^2}{vN_s T_0} u^2 e^{-2u}. \tag{25}$$

The solution of the damping equation is

$$\int_u^{u_0} \frac{e^{2u}}{u^2} du = \frac{2k^2 I_D^2}{vN_s T_0} t, \tag{26}$$

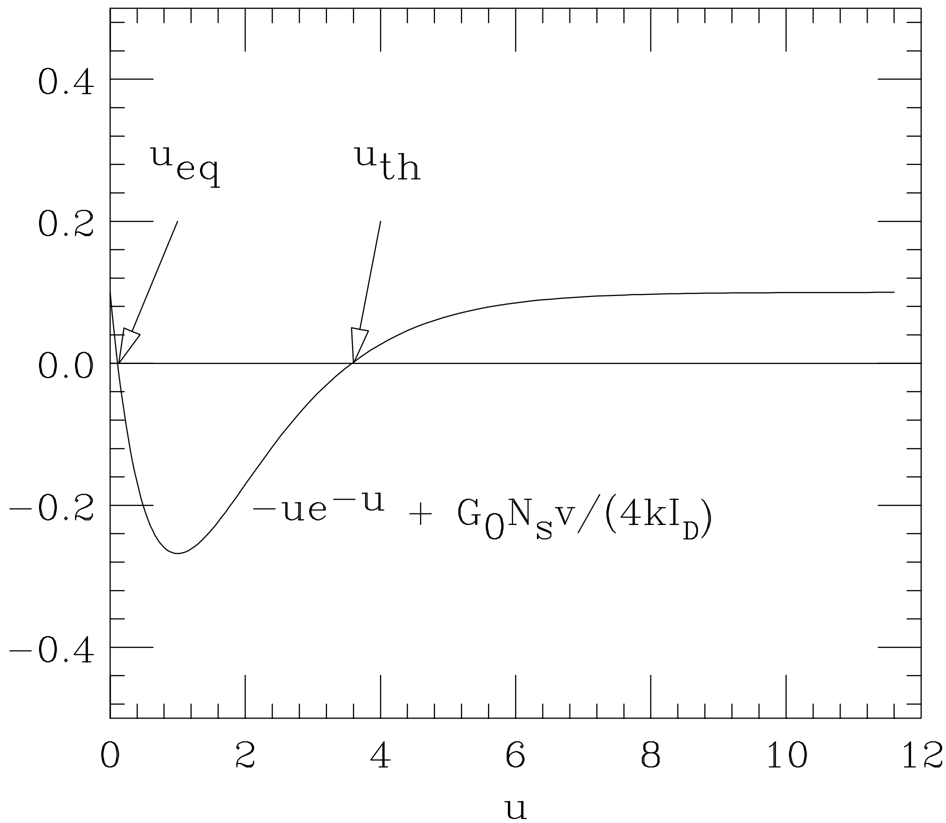

FIG. 1: The cooling dynamical function for a fixed gain G_0 with $G_0 N_s v/4kI_D = 0.1$ (used only for an illustrative example), is shown as a function the parameter u. Note that beam cooling occurs only when $u \leq u_{\rm th}$, and the cooling stop when $u = u_{\rm eq}$ is reached. In this case, cooling appears to be possible for $u \leq u_{\rm th} \approx 3.6$. However, the sinusoidal nature of the momentum kick in Eq. (5) renders this parametric region not applicable.

where u_0 is the initial value of u in Eq. (10), and $u \leq u_0$ during the beam cooling process. Figure 2 shows the integral of the left-hand side of Eq. (26) assuming that $u_0 = 3.0$. As time increases, the corresponding emittance function u can be obtained from the graph. Note that when the beam is sufficiently cold with $u \leq 1$, the cooling process will behave like $u \sim \frac{1}{t}$ at the optimized gain factor.

It appears that at the optimized gain factor, a hot beam could be very efficiently cooled. However, the OSC takes place through Eq. (5), which is proportional to $\sin(\Delta\phi_i)$, and the correction will be in the *wrong direction* if the phase shift $|\Delta\phi_i| > \pi/2$. Thus, for a large thermal energy, like $u_0 = 3$, only the part of the beam sufficiently close to the on-momentum particle will be cooled while the rest will be heated instead. To ensure OSC, we must make sure that all the particles in the beam (usually 95% is assumed) be within the $\pi/2$ phase shift. Since a bypass can be designed with very small I_1 and I_2, this phase shift requirement translates into

$$u = u_0 \approx \frac{1}{2}(kI_D\sigma_\delta)^2 \leq \frac{\pi^2}{48}. \tag{27}$$

As a result, OSC at optimum gain factor is rather inefficient because the emittance of a cold beam decreases inversely with the cooling time. As will be seen below, OSC at small gain turns out to be more efficient. Although the cooling represented by Eq. (25) is not exponential, an initial cooling time can nevertheless be defined by

$$\tau_{\rm cool} = -\left.\frac{u}{du/dt}\right|_{u=u_0} \approx \frac{N_s T_0}{4}\frac{e^{2u_0}}{u_0} \tag{28}$$

for an optimum gain factor.

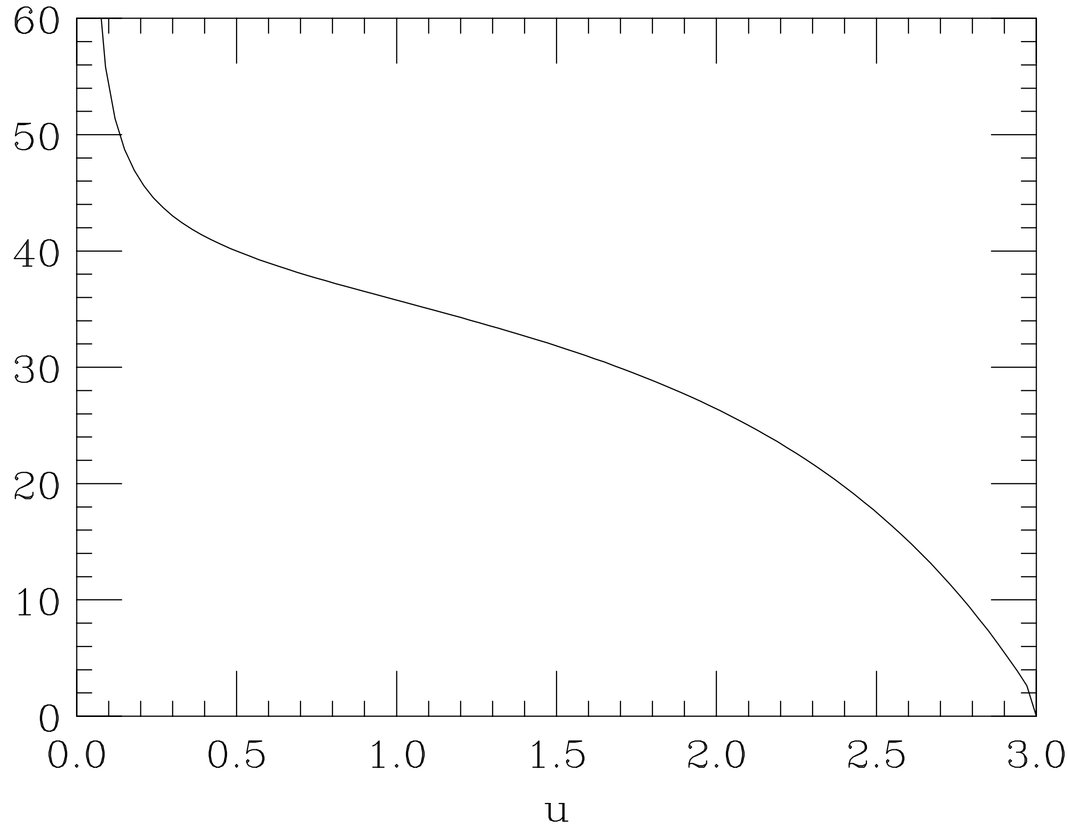

FIG. 2: The integral of Eq. (26) with $u_0 = 3$ is shown as a function of u with optimal gain factor. The resulting thermal energy parameter u can be obtained from this graph with the value given by the right hand side of Eq. (26).

3. One dimensional optical stochastic cooling dynamics

If the second undulator location is designed to be non-dispersive, i.e. $D_2 = P_{D2} = 0$, the betatron cooling and heating vanish. The optical stochastic cooling is a one-dimensional momentum cooling device. This will simplify the cooling bypass design. Let $u_x = \frac{1}{2}k^2(\beta_1 I_1^2 - 2\alpha_1 I_1 I_2 + \gamma_1 I_2^2)\epsilon_x$, and $u_\delta = \frac{1}{2}k^2 I_D^2 \sigma_\delta^2$. The damping equation becomes

$$\frac{du_\delta}{dt} = -\frac{2GkI_D}{T_0}e^{-u_x}u_\delta e^{-u_\delta} + \frac{G^2 N_s k^2 I_D^2}{4T_0}. \tag{29}$$

The optimal gain is

$$G_{\rm opt} = \frac{4}{N_s k I_D}e^{-u_x}u_\delta e^{-u_\delta}. \tag{30}$$

At the optimal gain, the cooling dynamics equation becomes

$$\frac{du_\delta}{dt} = -\frac{4}{N_s T_0}e^{-2u_x}u_\delta^2 e^{-2u_\delta}. \tag{31}$$

Note that the longitudinal damping rate is reduced by the factor e^{-2u_x} of the thermal energy of the transverse plane. The longitudinal cooling rate can be increased by a reduction of the transverse thermal energy u_x, which can be made zero by the additional design constraints of $I_1 = 0$ and $I_2 = 0$. The dynamical evolution of the one-dimensional OSC is similar to that of the equal-decrement cooling dynamics discussed in earlier sections.

When the bypass optics is designed such that $I_1 = I_2 = 0$, i.e. $u_x = 0$, we obtain a one-dimensional optical stochastic cooling with

$$\frac{du_\delta}{dt} = -\frac{2GkI_D}{T_0}u_\delta e^{-u_\delta} + \frac{G^2N_sk^2I_D^2}{4T_0}. \tag{32}$$

The optimal gain is

$$G_{\rm opt} = \frac{4}{N_skI_D}u_\delta e^{-u_\delta}. \tag{33}$$

Since the phase shift condition requires $2\sqrt{6}kI_D\sigma_\delta \leq \pi$, or $u_\delta \leq \pi^2/48$, the condition for a maximum optimal gain with $u_\delta = 1$ assumed in Ref. [5] is incorrect.

IV. AMPLIFICATION FACTOR

The total energy of the photon emission in the first undulator is

$$W_1 = \frac{1}{2}\epsilon_0\mathcal{E}_1^2A_1c\Delta t_R = \frac{1}{4\pi\epsilon_0}\pi\xi kq^2[\mathrm{JJ}]^2, \tag{34}$$

where $\mathcal{E}_1$ is the peak electric field amplitude produced in the first undulator, A_1 is the cross-section area of the coherent radiation [9], and $\Delta t_R = N_{\rm u}\lambda/c$ is the duration of the radiation pulse. We also use the fact that a particle with a charge q emits about $\frac{\pi q^2\xi}{4\pi\epsilon_0\hbar c}[\mathrm{JJ}]^2$ coherent photons at the energy $\hbar\omega$ during one pass of the undulator.

The input and output peak powers of the laser amplifier are

$$\begin{aligned} \hat{P}_1 &= \frac{W_1}{\Delta t_R}N_s = \frac{1}{2}\epsilon_0\mathcal{E}_1^2A_1cN_s, \\ \hat{P}_2 &= g^2\hat{P}_1, \end{aligned} \tag{35}$$

where g^2 is the power gain from the laser amplifier,

$$N_s = N_{\rm B}\frac{N_{\rm u}\lambda}{2\sqrt{6}c\sigma_\tau} \tag{36}$$

is the number of particles in a sample within a bandwidth of $\Delta\omega|_{\rm FWHM} = \omega/N_{\rm u}$. Here, we have assumed 100% photon transmission in the optical amplifier, and assume that the bandwidth of the laser amplifier is larger than that of the undulator radiation.

The peak electric field at the second undulator depends on the amplifier gain factor and focusing property through conservation of energy, i.e.

$$\mathcal{E}_2^2A_2 = g^2\mathcal{E}_1^2A_1, \tag{37}$$

where $\mathcal{E}_2$ and A_2 are the peak electric field amplitude and the photon beam area at the waist [9], presumably at the mid-point, of the second undulator. The momentum gain-factor G is given by

$$G = \frac{q\langle\mathcal{E}\rangle_2N_{\rm u}\lambda_{\rm u}K[\mathrm{JJ}]}{2c\gamma P}, \tag{38}$$

where the average electric field that the charged particle sees in the second undulator is

$$\langle\mathcal{E}\rangle_2 = \frac{2\mathcal{E}_2}{L}\int_0^{L/2}\frac{ds}{\sqrt{1+(s/\beta_*)^2}}, \tag{39}$$

where $L = N_{\rm u}\lambda_{\rm u}$ is the length of the second undulator and β_* is the betatron amplitude function for the photon beam at the waist.

For a given momentum gain factor, the peak power becomes

$$\hat{P}_2 = G^2\frac{N_s(E_b/q)^2}{Z_0\xi N_{\rm u}[\mathrm{JJ}]^2}\mathcal{F}_2, \tag{40}$$

where E_b is the beam energy, Z_0 is the impedance in vacuum,

$$\mathcal{F}_2 = \frac{A_0/A_2}{8[\ln(A_0/A_2 + \sqrt{1+(A_0/A_2)^2})]^2}, \tag{41}$$

$A_2 = 2\pi\sigma_*^2$ is the rms photon beam area at the waist of the second undulator [9], and $A_0 = N_\mathrm{u}\lambda_\mathrm{u}\lambda/4$. Minimum laser amplifier occurs when $A_2 = 0.3012A_0$, where $\mathcal{F}_2 = 0.1132$ [10].

The average laser power is equal to the peak power multiply the duty factor, i.e.

$$\langle P\rangle_2 = \hat{P}_2 \frac{n_b 2\sqrt{6}\sigma_\tau}{T_0} = G^2 \frac{(E_b/q)^2}{Z_0\xi[\mathrm{JJ}]^2} \frac{N_\mathrm{B} n_b \lambda}{C} \mathcal{F}_2, \tag{42}$$

where n_b is the number of bunches, σ_τ is the rms bunch length in time, and C is the circumference of the storage ring. Note that the average power is proportional to the total number of particles in the storage ring.

A. Laser power for optimal gain

Substituting the optimal gain of Eqs. (24) or (30) into Eq. (40), we obtain the output peak power of the laser amplifier given by

$$\hat{P}_2 = \frac{N_s\,(E_b/q)^2}{Z_0\xi[\mathrm{JJ}]^2 N_\mathrm{u}} \left(\frac{2kI_D}{v\,N_s} u e^{-u}\right)^2 \mathcal{F}_2, \tag{43}$$

Since the cooling rate is inversely proportional to N_s, the peak power for an optimized cooling of N_s particle is also inversely proportional to N_s. Because of the stability condition of $u \le \pi^2/48$, the peak power is highly reduced.

Figure 3 shows the peak power versus γ (beam energy) for proton storage rings at optimal gain. The laser wavelength is taken to be $\lambda = 1$ μm and each undulator has $N_\mathrm{u} = 10$ periods. Most parameters correspond to the Tevatron: $N_\mathrm{B} = 2.7\times10^{11}$ particles, $\sigma_\ell = 0.37$ m, and $\sigma_\delta = 1.3\times10^{-4}$. With the Tevatron revolution period of $T_0 = 20.1$ μs, the initial cooling time is 57 s given by Eq. (28). The magnetic field of the undulator varies from 1 to 10 T.

For a fixed laser wavelength and the undulator magnetic field, the undulator parameter is obtained by solving the cubic equation:

$$\lambda = \frac{\pi mc}{2qB_\mathrm{u}\gamma^2} K(2+K^2)\ , \tag{44}$$

from which the undulator period λ_u can be solved and plotted in Fig. 3. The self consistent solution gives $K \sim \gamma^2$ at low energies and $\hat{P}_2 \sim \frac{(E_b/q)^2}{\xi} \sim \frac{(E_b/q)^2}{K^2} \sim \frac{1}{\gamma^2}$, i.e., it requires a large laser power to compensate the small coherent radiation flux for hadron beams at low energies. At high beam energies, $\xi \to 1$ and the output power increases as γ^2 instead. The position of the minimum laser power can be easily calculated to be

$$\gamma_\mathrm{min} = \sqrt{\frac{4\sqrt{2}\pi}{3\sqrt{3}}} \sqrt{\frac{mc}{qB_\mathrm{u}\lambda}}\ . \tag{45}$$

The Tevatron at 1 TeV happens to be near the minimum of the power-vs-gamma curve and is therefore favored by OSC [11]. The undulator period of $\lambda_\mathrm{u} = 1.93$ m ($B_\mathrm{u} = 6$ T) is long enough for superconducting undulators. RHIC lies on the left side of the minimum and has its output amplifier power scale as $\gamma^{-2}(m/q)^4$. VLHC lies on the right side of the minimum and has its output power scale as $(m\gamma/q)^2$.

Figure 4 is a similar plot for electron rings. Because of the small electron mass, there is no need to consider high magnetic field undulators and we set the magnetic field at $B_\mathrm{u} = 1$ T. The bunch parameters are $N_\mathrm{B} = 1.0\times10^{11}$, $\sigma_\ell = 1$ cm, and $\sigma_\delta = 1.3\times10^{-4}$. Besides laser wavelength $\lambda = 1$ μm, we also include $\lambda = 5$, 20, and 100 μm, where the corresponding numbers of sampling particle are $N_s = 2.0\times10^7$, 1.0×10^8, 4.1×10^8, and 2.0×10^9 respectively. The initial cooling time for the optimal gain is given by Eq. (28) $\tau_\mathrm{cool} = 1.8N_sT_0$, which depends on the revolution period T_0.

When $\lambda = 1$ μm, the minimum peak power occurs at $\gamma_\mathrm{min} = 76.3$ or $E_b = 39.0$ MeV, i.e. nearly all electron storage rings lie on the right side of the minimum. However, because of the $(m/q)^2$ factor, the output power of the amplifier is very much reduced. That does not implies that OSC favors electron rings of high energies because the radiation damping rate increases rapidly with energy. To be effective, the OSC cooling rate, discussed in the last paragraph, has to be faster than the radiation damping rate of the electron ring.

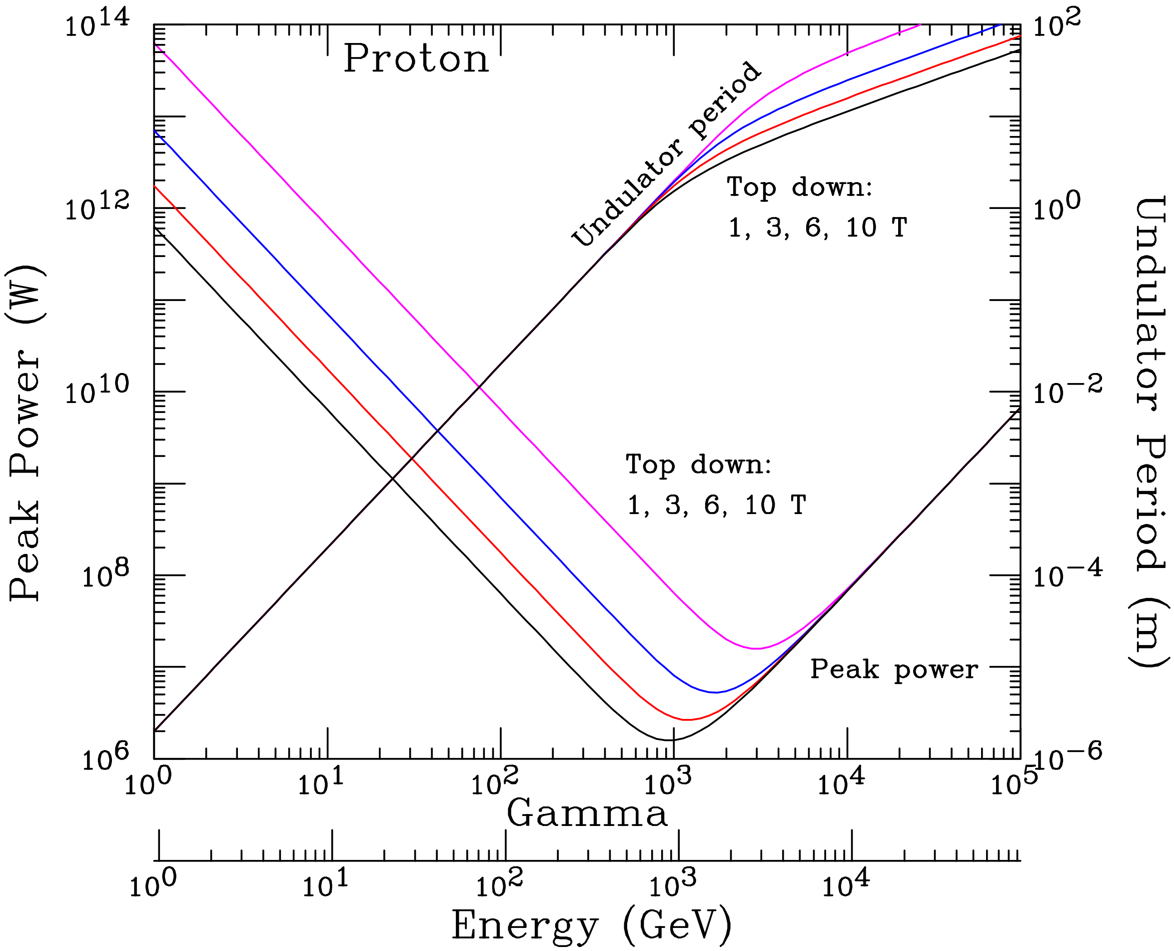

FIG. 3: The peak laser amplifier power vs γ for an optimal gain in the optical stochastic cooling for a proton storage ring (TEVATRON). The parameters for the Tevatron are $\sigma_\ell = 0.37$ m, $\sigma_\delta = 1.3 \times 10^{-4}$, $n_b = 36$ bunches, each containing $N_{\rm B} = 2.7 \times 10^{11}$ particles, $E_b = 1$ TeV, the mean radius of the TEVATRON of 1000 m, and $B_{\rm u} = 10$ T. The initial cooling time is given by Eq (28) with $u_0 = \pi^2/48$ or $\tau_{\rm cool} \approx 57$ s.

Now, we consider a possible example of converting the IUCF Cooler Ring to an electron ring and OSC is applied at the Ti-Saphire laser wavelength $\lambda = 0.78$ μm with $N_{\rm u} = 10$ and $\lambda_{\rm u} = 5$ cm. Setting an initial cooling time of 0.10 s, we find $N_s = 1.92 \times 10^5$. Since the bunch length is 3.6 cm with the rf system, we find the number of particles in a bunch is $N_{\rm B} = 4.36 \times 10^9$. At $E_b = 500$ MeV, the required laser peak power is $\hat{P} = 39$ W. The peak power is much larger than that of Fig. 4 because the number of the sampling particle is much smaller in this example. The natural horizontal emittance and the OSC-equilibrium emittance are plotted in Fig. 5 as functions of beam energy. Other parameters used in the plots are ring circumference $C = 85.03$ m, bending radius $\rho = 2.44$ m, momentum compaction $\alpha_c = 0.04938$, rf harmonic $h = 15$, and a bucket-to-bunch-height ratio of 40. We also note that the OSC damping is almost or more than an order of magnitude when the electron energy is below 500 MeV. However, at higher energies, OSC damping is completely inefficient because the rapidly increasing radiation damping rates. As a whole, applications of OSC to low energy electron storage rings can be useful for attaining high brightness electron beams.

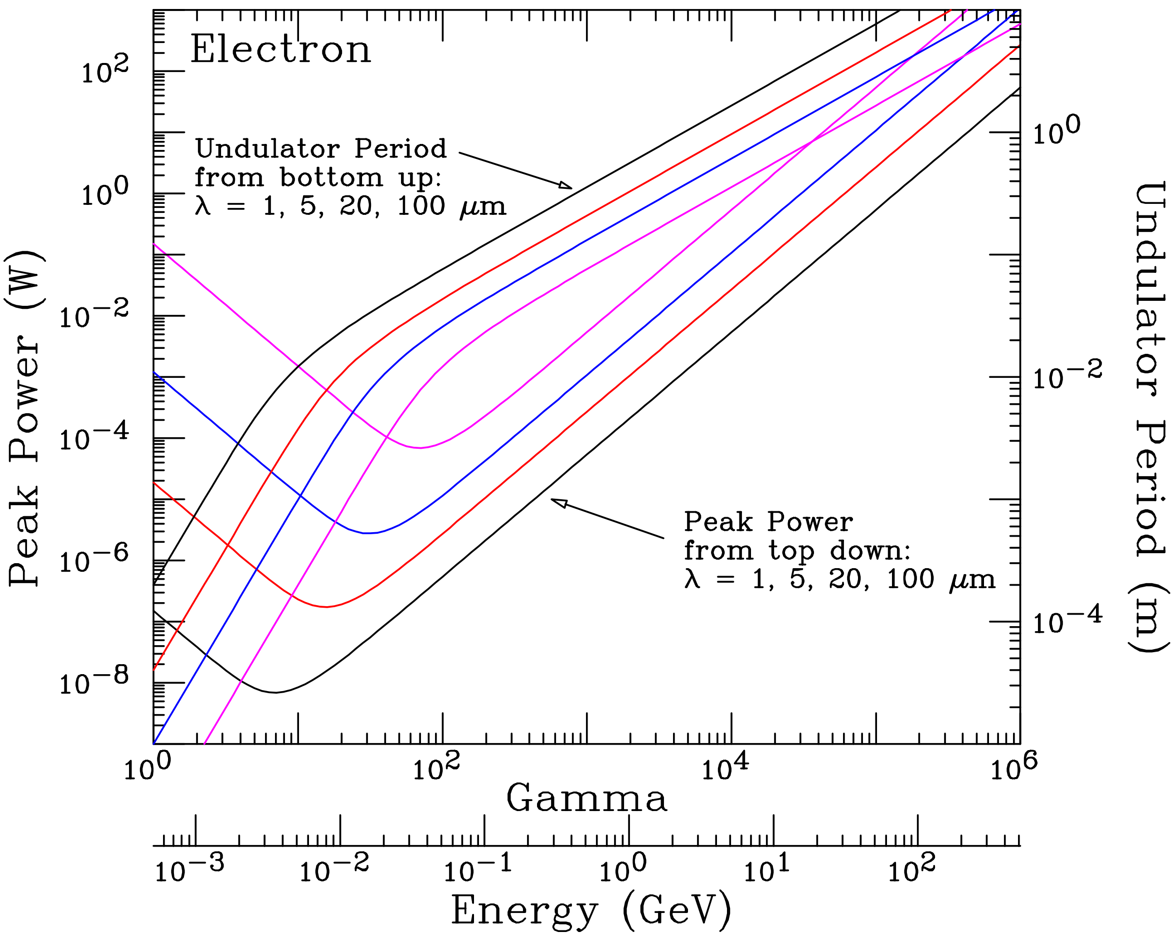

FIG. 4: The peak laser amplifier power vs γ for optimal gain in the optical stochastic cooling for electron storage rings. The parameters for the electron storage ring are $\sigma_\ell = 1$ cm, $\sigma_\delta = 1.3 \times 10^{-4}$, $N_{\rm B} = 1.0 \times 10^{11}$, and $B_{\rm u} = 1.0$ T.

B. Laser Power for Low Gain Regime

At an optimal gain, the laser power requirement is usually high (see Fig. 3), and the damping dynamics is not necessarily the most favorable for beam cooling. It would be useful to consider the OSC in the low gain regime. As an example, we consider the longitudinal cooling in the low-gain regime. The incoherent heating term is now small and can be neglected. The damping equation becomes

$$\frac{du_\delta}{dt} = -\frac{2GkI_D}{T_0} e^{-u_x} u_\delta e^{-u_\delta}. \tag{46}$$

Since $u_\delta \le \pi^2/48$ is small, the damping is almost exponential and becomes more so as the cooling proceeds and will continue until the cooling force is balanced by the heating forces coming from rf noise, intra-beam scattering, etc. This is highly in contrast with the cooling at optimum gain-factor discussed in Sec. III.C.2, where the cooling process becomes more and more inefficient as the beam is cooled. With $u_x = 0$, the cooling time is

$$\tau_{\rm cool} \approx \frac{e^u_\delta}{2GkI_D} T_0. \tag{47}$$

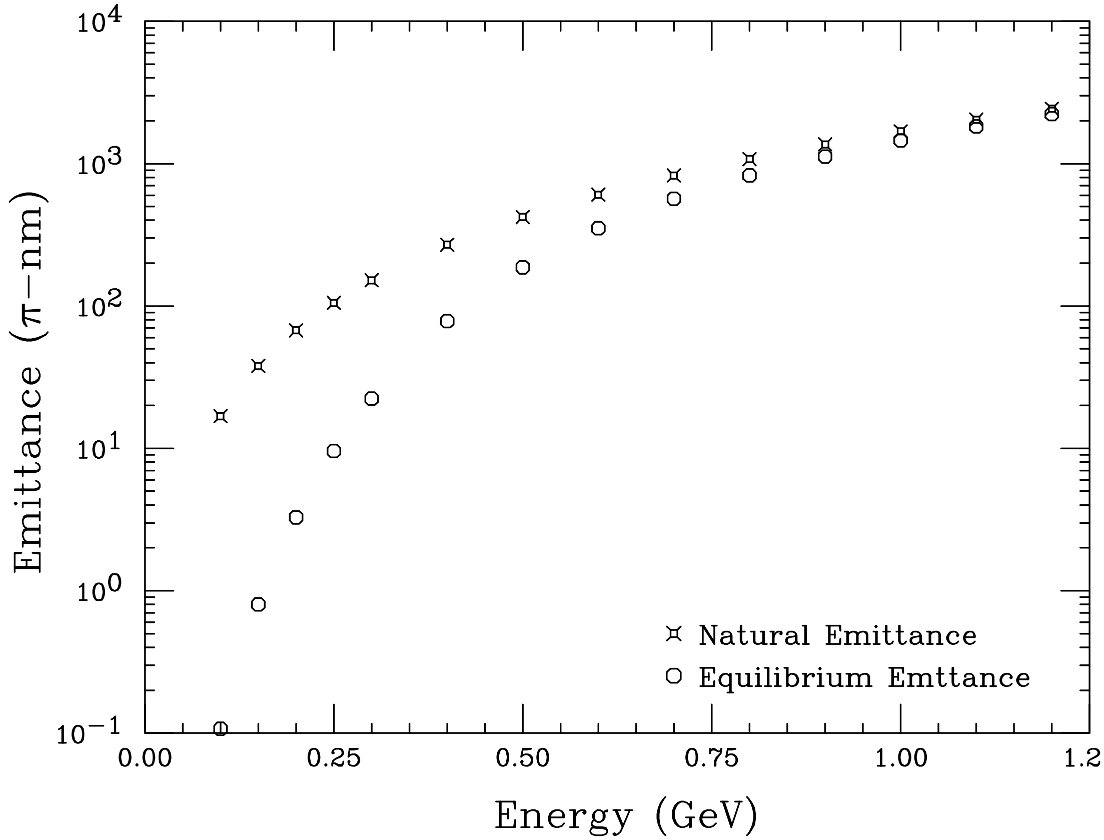

FIG. 5: The equilibrium electron emittance for a cooling time of 0.1 s is shown as a function of the electron beam energy.

The resulting peak power is

$$\hat{P}_2 = \left(\frac{T_0}{\tau_{\rm cool}}\right)^2 \frac{N_s\,(E_b/q)^2\, e^{2u_\delta}}{Z_0 N_{\rm u}\xi[{\rm JJ}]^2(2kI_D)^2}\mathcal{F}_2, \tag{48}$$

The average power of the laser amplifier is

$$\langle P\rangle_2 = \left(\frac{T_0}{\tau_{\rm cool}}\right)^2 \left(\frac{n_b N_{\rm B}\lambda}{C}\right) \frac{(E_b/q)^2\, e^{2u_\delta}}{Z_0 \xi[{\rm JJ}]^2(2kI_D)^2}\mathcal{F}_2, \tag{49}$$

where C is the circumference of the storage ring. Note that the average power depends on the total number of particles $n_b N_{\rm B}$ in the ring and the square of the energy over charge $(E_b/q)^2$.

Figure 6 shows the average power requirement versus cooling time in the low gain regime, where the undulator parameters are $\lambda = 1.0\ \mu$m, $N_{\rm u} = 10$, and the undulator magnetic field varying from 1 T to 10 T. The corresponding beam parameters are $\sigma_\ell = 0.37$ m, $\sigma_\delta = 1.3\times10^{-4}$, $n_b = 36$ bunches each containing $N_{\rm B} = 2.7\times10^{11}$ protons at $E_b = 1$ TeV for the Tevatron whose mean radius is 1 km, while $\sigma_\tau = 2.0$ ns, $\sigma_\delta = 1.0\times10^{-3}$, $n_b = 60$ bunches each containing $N_{\rm B} = 1.0\times10^9$ gold ions ($A = 197$ and $Z = 79$) at $E_b = 100$ GeV/nucleons for RHIC whose circumference is 3833.85 m. We see that for a cooling time of 1200 s which is fast enough to counteract intra beam scattering, the average output power for Tevatron is only 16 W when superconducting undulators at $B_{\rm u} = 6$ T is used. On the other hand, the average output power for RHIC is more than 1000 times larger. Because γ is one order of magnitude smaller than that of the Tevatron, the undulator period becomes $\lambda_{\rm u} = 2.3$ cm, two orders of magnitude smaller. This implies that superconducting undulators may not be used and only 1 T undulators are possible. The output power for the RHIC application is therefore increased at least one more order of magnitude.

Note that when the laser wavelength is chosen to be $\lambda = 1\ \mu$m for RHIC, the undulator period is $\lambda_{\rm u} = 2.3$ cm, which may be difficult to attain a high field undulator magnet. The wiggler number becomes very small, and the required laser amplification power becomes very large (see Fig. 6). If there is a longer wavelength high bandwidth

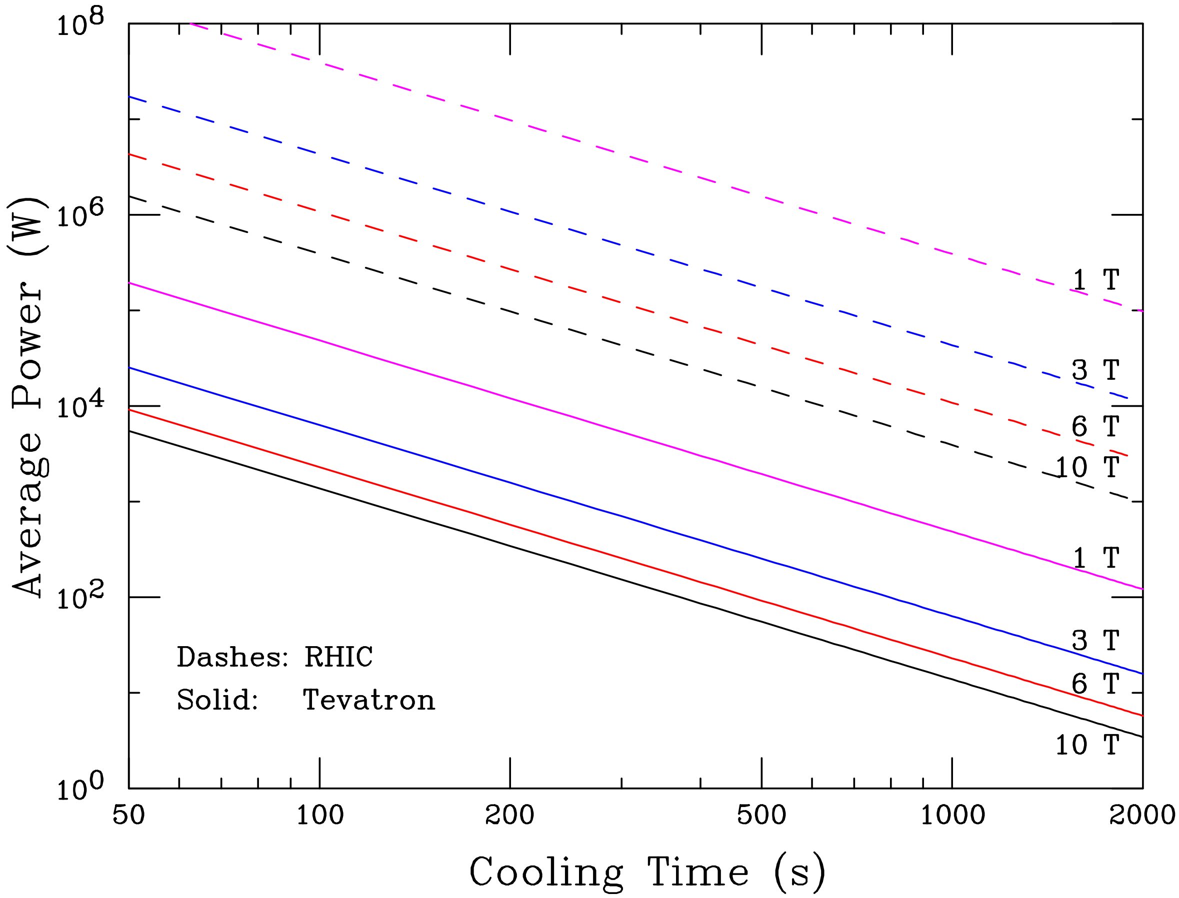

FIG. 6: The laser amplifier power in the low gain regime for Tevatron at 1 TeV and RHIC at 100 GeV/amu. The laser wavelength is $\lambda = 1\mu$, and the undulator parameters are $N_{\rm u} = 10$ with the magnetic field strength $B_{\rm u}$ listed in the graph. The corresponding beam parameters are $\sigma_\ell = 0.37$ m, $\sigma_\delta = 1.3 \times 10^{-4}$, $n_b = 36$ bunches, each containing $N_{\rm B} = 2.7 \times 10^{11}$ particles, at $E_b = 1$ TeV for the TEVATRON; and $\sigma_\tau = 2$ ns, $\sigma_\delta = 1.0 \times 10^{-3}$, $n_b = 60$ bunches, each containing $N_{\rm B} = 1.0 \times 10^9$ particles, $E_b = 100$ GeV/nucleon for gold ion, and the circumference of 3833.85 m for RHIC.

laser, e.g. $\lambda = 10\mu$m, the undulator period becomes 23 cm, and the required laser amplification power will be greatly reduced as shown in Fig. 7. Although it may still require 80 W of laser amplification power to attain a 1 hr cooling time (for $B_{\rm u} = 6$ T), this is dramatically improved in comparison with the 1000 W requirement shown in Fig. 6.

V. CONCLUSION

In this paper, we derived a necessary condition for the transverse phase space damping in the optical stochastic cooling. We have also explored the damping rates, the amplification factor, cooling dynamics, and the required peak and average output power of the laser. We derived an optimal laser focusing condition for the charged particle beam and the laser beam interaction in an undulator. With the available optical amplifiers at the present, it is rather impractical to use the optical stochastic cooling method to cool proton and heavy ion beams at *very* high energies. However, we find that the cooling method may be beneficial to low energy electron beams, and around 1 TeV proton beam energy.

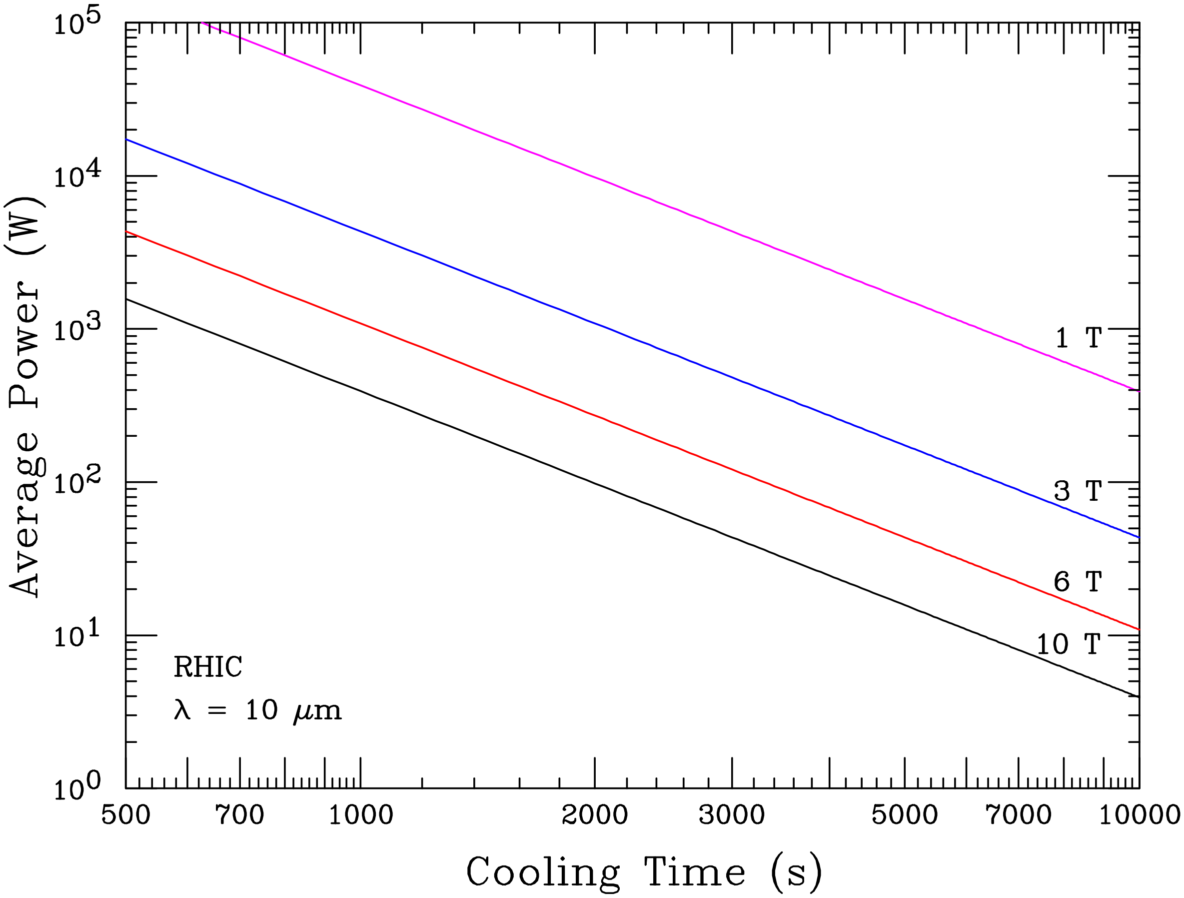

FIG. 7: The laser amplifier power in the low gain regime for RHIC at 100 GeV/amu with the laser wavelength of $\lambda = 10\mu$ and the undulator parameters are $N_{\rm u} = 10$. The magnetic field strength $B_{\rm u}$ is listed in the graph. The corresponding beam parameters are $\sigma_\tau = 2$ ns, $\sigma_\delta = 1.0 \times 10^{-3}$, $n_b = 60$ bunches, each containing $N_{\rm B} = 1.0 \times 10^9$ particles, $E_b = 100$ GeV/nucleon for gold ion, and the circumference of 3833.85 m.

We also point out the difficulties of OSC with optimal gain condition. At the optimal gain, the required laser power is usually very large. As the beam is cooled, it is difficult to change the charged particle optics for a larger kI_D to compensate the decrease in emittances. The best solution is to cool beams in the low gain regime, where the heating term may be negligible. For Tevatron, it seems to be feasible to use the Ti-Saphire $\lambda = 0.78$ μm for OSC at 1 TeV. One needs a shorter wavelength broadband laser for VLHC, and a long wavelength broadband laser for RHIC.

In actual implementation of the OSC, one should also consider the efficiency of laser pumping and optical transmission, the linearity of the laser amplification, noise, etc. These problems can be considered if there is a realistic project to carry out experimental tests.

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[9] If we assume that the photon beam be distributed as *bi-Gaussian* radially but uniformly along the longitudinal s-direction, the total energy of the photons can be written as

$$W_0 = \int \frac{W_0/\Delta s}{2\pi\sigma_x\sigma_z} \exp\left(-\frac{x^2}{2\sigma_x^2} - \frac{z^2}{2\sigma_z^2}\right) dxdz\Delta s,$$

where the energy density is $\epsilon_0\mathcal{E}^2 = (W_0/\Delta s)/(2\pi\sigma_x\sigma_z)$. Here $\mathcal{E}$ is the peak field at $r = 0$. Now, we can write $W_0 = \epsilon_0\mathcal{E}^2 A\Delta s$, i.e. the effective photon beam area is $A = 2\pi\sigma_x\sigma_z$. For a photon beam with cylindrical symmetry, we find $A = 2\pi\sigma_r^2$.
[10] The emittance of the photon beam, $\lambda/(4\pi)$, may substantially differ from the emittance of the charged particle beams, e.g. 3.3 nm for Tevatron at 1 TeV and 16 nm for RHIC beam at 100 GeV/amu. The efficiency of the cooling may be reduced by the overlap area between the charged particle and the photon beams. The optimal energy gain at the second undulator for the charged particle beams is equivalent to the minimum in the laser power.
[11] At high energy, the self-consistent solution of $\lambda = \lambda_u(2 + K^2)/(4\gamma^2)$ and $K = qB_u\lambda_u/(2\pi mc)$ leads to a conclusion that the peak power of the laser amplifier is proportional to γ^2 of Eq. (43). The minimum power requirement occurs at $\gamma_{\min} \sim 1.14\sqrt{\pi mc/(qB_u\lambda)}$.