Quantum theory of optical stochastic cooling

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Quantum theory of optical stochastic cooling is presented. Results include a full quantum analysis of the interaction of the beam with radiation in the undulators and in the quantum amplifier. A density matrix of the whole system is constructed and the cooling rate is evaluated. It is shown that quantum fluctuations change classical results of stochastic cooling at low bunch population and set a limit on the cooling rate.

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I. INTRODUCTION

Recently, optical stochastic cooling (OSC) was proposed [1,2] as a method for the fast cooling of short bunches in storage rings. In OSC, a wave of radiation is generated by a particle in an undulator. After amplification in the optical amplifier, the wave is sent to another undulator where its interaction with the parent particle provides the desired cooling. The first and second undulators work as a pickup-kicker pair in conventional rf stochastic cooling. The phase shift of the wave, with respect to the particle, is controlled in a dispersion section between the two undulators.

Particle radiation affects other particles in the bunch leading to diffusion that limits the damping rate. In this respect, optical stochastic cooling is not different from rf stochastic cooling. The number of interacting particles (the number of "particles per slice") in the bunch with rms length $\sigma_b = c\tau_b$ and bunch population N_B is

$$N_s = \frac{\pi N_b}{\Delta\omega\sqrt{2\pi}\tau_b}, \quad (1)$$

and is defined by the bandwidth of the amplifier $\Delta\omega$. This interaction of particles changes the momentum of the jth particle [3],

$$\overline{p_j} = p_j - \Lambda p_j - \Lambda \sum_{i\neq j} p_i, \quad (2)$$

where Λ is the parameter of interaction between particles proportional to the electronic gain of the amplifier. The rms energy spread $\sigma_p^2 = (1/N_b)\Sigma_j[\langle (p_j)^2\rangle - \langle p_j\rangle^2]$ for initially uncorrelated particles is changed by

$$\Delta[\sigma_p^2] = -2\Lambda\sigma_p^2 + \Lambda^2 N_s \sigma_p^2. \quad (3)$$

The cooling rate is

$$\frac{1}{N_{Turn}} = \frac{\Delta\sigma_p^2}{\sigma_p^2} = -2\Lambda + \Lambda^2 N_s. \quad (4)$$

The maximum cooling rate of $1/N_{turn} = 1/N_s$ is determined by the number of particles per slice, and is achieved at $\Lambda = 1/N_s$.

Cooling for short bunches requires the amplifier to have a large bandwidth. For example, the bunches in the PEP-II B factory have $\tau_b = 33$ ps. To affect different slices in the bunch, N_s has to be small, $N_s < N_b$, which requires $\Delta f = \Delta\omega/2\pi > 10$ GHz. The actual PEP-II bandwidth is only 250 MHz.

The advantage of optical stochastic cooling is large Δf, allowing for fast cooling. In this method, the bandwidth $\Delta f \simeq \gamma_0^2(c/L_u)$ where $L_u = N_u\lambda_u$ is the undulator length and γ_0 is relativistic factor of the beam in the laboratory frame. Parameters of the undulator have to be chosen to match the undulator mode to the central frequency and bandwidth Δf of the amplifier. For the typical solid state Ti:sapphire amplifier ($\lambda = 0.8$ μm, $\Delta f/f \simeq 1/5$).

Given bandwidth, fast cooling (for example, in the muon collider) can be achieved by reducing the number of particles per slice N_s. For one particle per slice, cooling would be achieved just in one turn. However, with a small N_s, classical and quantum fluctuations could become dangerous. Indeed, the average number of photons radiated in the main undulator mode is $\alpha_0 = 1/137$. Therefore, one can expect that a photon is radiated once per 137 turns, and that fluctuations may be large. Concern with quantum fluctuations is the primary motivation for the study presented here.

In our consideration we determine the evolution of the quantum-mechanical density matrix of the system (bunch and radiated mode) through the undulators and the quantum amplifier. The paper starts by defining the initial density matrix of the beam and then by describing the beam/radiation interaction in the undulator. Beam dynamics in the undulators is described in the rest frame of a bunch (cf. to Renieri [4,5] and co-workers where other references can be found). The formalism we use [7] reproduces Becker and McIver [6] results but differs from their formalism. In both cases, the number of particles per bunch N_s can be arbitrary but the effects of bunching are neglected. In this sense the interaction of particles with radiation is weak. This assumption substantially simplifies consideration being quite adequate for describing optical stochastic cooling.

Next, the evolution of the density matrix in the quantum amplifier is described, taking into account the nondiagonal components of the density matrix. Interaction between amplified radiation with the bunch in the kicker is then considered in the same way as it was for the pickup undulator.

Finally, the full density matrix is constructed and is used to calculate the one-pass variation of the rms energy spread of the bunch. Optimum cooling rate of OSC is determined and compared with the cooling rate of classical theory. It is shown that quantum fluctuations set a limit for the damping rate of this method.

II. INITIAL DENSITY MATRIX OF A BUNCH

We consider only longitudinal motion in a bunch, assuming that the N_b particle bunch is described by the Gaussian normalized distribution function (DF) $f(p,z)$,

$$\int f(p,z)dpdz=1. \tag{5}$$

The classical DF f is related to Wigner's density matrix $\hat{\rho}$. In the momentum representation, the relation is

$$f(p,z)=\int \frac{Ldq}{(2\pi)^2}\rho(p+q/2,p-q/2)e^{iqz/h}, \tag{6}$$

where the density matrix is normalized,

$$\hat{\rho}=|p'\langle\rho(p',p)\rangle p|, \quad \int \frac{Ldp}{2\pi}\rho(p,p)=1. \tag{7}$$

For a Gaussian DF localized around the point z_0, p_0 in the phase space,

$$f(p,z)=\frac{1}{2\pi\sigma\Delta}\exp\left[-\frac{(p-p_0)^2}{2\Delta^2}-\frac{(z-z_0)^2}{2\sigma^2}\right]. \tag{8}$$

The corresponding density matrix ρ_0 is the wave packet

$$\rho^0(p',p)=\frac{h\sqrt{2\pi}}{L\Delta}\exp\left[-\frac{i}{h}(p'-p)z_0-\frac{1}{2}\left(\frac{\sigma}{h}\right)^2(p'-p)^2 -\frac{1}{2}\left(\frac{1}{\Delta}\right)^2\left(\frac{p+p'}{2}-p_0\right)^2\right], \tag{9}$$

where L is the normalization length. The rms values σ and Δ of the wave packet may be small compared to the rms energy spread Δ_B and rms length σ_B of a bunch.

III. PICKUP

We assume that there are N_B relativistic particles per bunch, there is no initial z,p correlation, and correlations that may be generated in the undulators are wiped out in one turn. The pickup and the kicker undulators are helical with the undulator parameter K_0 and period $\lambda_u=2\pi/k_u$. The bunch is described [5] in the frame moving with the relativistic factor $\gamma=\gamma_0/\sqrt{1+K_0^2}$ where the resonance frequency of the mode is $k=\gamma k_u$. The bunch centroid has zero initial velocity and an energy spread of Δp,

$$\Delta p=\frac{1}{\sqrt{1+K_0^2}}\left(\frac{\Delta p_{lab}}{\gamma_0}\right), \tag{10}$$

corresponding to the energy spread Δp_{lab} in the laboratory frame.

At the entrance to the pickup, each particle is described by the density matrix Eq. (9), $\rho^0(p_i',p_i)$, $i=1,2,\ldots,N_b$. The density matrix of the whole bunch $\hat{\rho}=\Pi_{i=1}^{N_B}|p'\rangle\rho^0(p_i',p_i)\langle p|$.

In the moving frame, interaction of particles with the mode $k=\omega/c$ is described by the Hamiltonian

$$H=\sum_{i=1}^{N_B}\frac{\hat{p}_i^2}{2m}+h\omega(a^+a+1/2)-ihg[ae^{2ik\hat{z}+i\omega t}-\text{c.c.}], \tag{11}$$

where $m=m_e\sqrt{1+K_0^2}$.

If the vector potential of the radiation is normalized to one photon per volume V [6,7]

$$\vec{A}=\sqrt{\frac{2\pi hc^2}{V\omega}}\vec{y}[\hat{a}e^{ik\hat{z}}+\text{c.c.}], \quad |\vec{y}|=1, \tag{12}$$

then the parameter of the interaction $g=g_k$, where

$$g_k=c\frac{K_0}{\sqrt{1+K_0^2}}\sqrt{\frac{e^2}{hc}\frac{2\pi}{kV}}. \tag{13}$$

In the one-dimensional model, the beam interacts with one radiated mode. In this case, operators a and a^+ change the number of coherent photons in the mode, and vector potential must be normalized within the phase volume Ω of the mode.

In the laboratory frame [11], $\Omega=(V/(2\pi)^3)(\pi k^3/N_u^2)$. This can be obtained from the constrain $|2\pi N_u-\psi|\leqslant\pi$ on the phase slippage $\psi=|\omega t-k_z z|$ along the undulator and from the requirement that the frequency spread $|(\omega-\omega_r)/\omega_r|<1/(2N_u)$, where ω_r is resonance frequency of radiation at zero angle. The result for the phase volume in the moving frame follows from the relativistic invariance of d^3k/ω.

Normalized vector potential is obtained by multiplying Eq. (12) by $\sqrt{\Omega}$. The parameter of interaction with the mode then becomes $g=g_k\sqrt{\Omega}$ and, using the time of the interaction in the moving frame $t=2\pi N_u/(ck)$, we get $gt=(K_0/\sqrt{1+K_0^2})\sqrt{\pi e^2/hc}$. Hence, $(gt)^2$ is of the order of $\alpha_0=e^2/hc$ and is always small.

Initial state $|p_i,n\rangle=|p_1,p_2,\ldots,p_{N_B},n\rangle$ of the system with n photons and particles with momentums p_i, $i=1,\ldots,N_B$ is transformed in the undulator by the interaction with the mode $k=\omega/c=\gamma k_u$ to the vector $|\Psi(t)\rangle$. As it is shown in Appendix A, Eq. (A21), (for more details, see [7]),

$$\left|\Psi(t)\right>=\sum_{l_i,p_i}\left|p_i-2hkl_i,n+l_\Sigma\right\rangle\sqrt{\frac{n!}{(n+l_\Sigma)!}}\int\frac{d\psi}{2\pi}e^{-il_\Sigma\psi} \times\exp[-i\omega t(n+l_\Sigma)]\prod_{i=1}^{N_B}F_n(t,p_i,l_i). \tag{14}$$

Here $l_\Sigma=\Sigma_i l_i$ is the total number of radiated photons, $E(p_i,l_i)=(p_i-2hkl_i)^2/(2m_0)$,

$$F_n(t,p,l)=\int_0^\infty d\lambda \frac{\lambda^n}{n!}e^{-\lambda}\hat{O}_{\lambda\kappa}\left(\frac{\lambda a_i}{\kappa a_i^*}\right)^{l_i/2} J_{l_i}(2g|a_i|\sqrt{\lambda\kappa}) \times\exp[-(it/h)E(p_i,l_i)]e^{il_i\psi}|_{\kappa=1}. \quad (15)$$

The operator $\hat{O}_{\lambda\kappa}=\exp[-(1/2)(\partial^2/\partial\lambda\partial\kappa)]$, J_l is the Bessel function, and

$$a_i(t)=\frac{\sin(\epsilon_i t/2)}{(\epsilon_i/2)}e^{-i\epsilon_i t/2},\quad \dot{a}_i(t)=e^{-i\epsilon_i t},\quad \epsilon_i=\frac{2kp_i}{m_e}. \quad (16)$$

The integration over ψ in Eq. (14) is introduced to separate the global parameters n and l_Σ of radiation and parameters p_i and l_i of individual particles.

Expression Eq. (15) is derived neglecting terms of the order of hk^2/m_e,

$$\frac{hk^2t}{2m_0}\simeq\pi N_u k\lambda_{Compt}\ll 1. \quad (17)$$

Doing this we ignore bunching due to radiation, which is irrelevant for our purposes.

For short undulators, $kpt/m_e\ll 1$, $[\sin(\epsilon_i t/2)]/(\epsilon_i/2)\simeq t$. The function F_n depends on parameter gt, where t is time of flight in the undulator, which is $t=N_u\lambda_u/(c\gamma)$ in the moving frame.

Using Eq. (14), the initial density matrix

$$\hat{\rho}(0)=|p',l'_\Sigma\rangle\rho(p',p,l_\Sigma,l'_\Sigma)\langle p,l_\Sigma| \quad (18)$$

is then transformed into

$$\hat{\rho}(t)=|\Psi'(t)\rangle\rho(p',p,l_\Sigma,l'_\Sigma)\langle\Psi(t)|. \quad (19)$$

IV. DENSITY MATRIX OF THE UNDULATOR

Let us obtain the explicit form of the density matrix Eq. (19).

We assume that at the entrance to the pickup there is no radiation, $n=0$. In this case, initial density matrix $\hat{\rho}=\Pi_{i=1}^{N_B}|p'\rangle\rho^0(p'_i,p_i)\langle p|$ is transformed according to Eqs. (14) and (19) to $\hat{\rho}(t)=|q',l'_\Sigma\rangle\rho(q',q,l_\Sigma,l'_\Sigma)\langle q,l_\Sigma$, where

$$\rho(q',q,l'_\Sigma,l_\Sigma)=\frac{1}{\sqrt{l_\Sigma!l'_\Sigma!}}\int\frac{d\psi d\psi'}{(2\pi)^2}\exp[-i(l'_\Sigma\psi'-l_\Sigma\psi)] \times\exp[i\omega t(l_\Sigma-l'_\Sigma)]\int d\lambda d\lambda' \times e^{-\lambda-\lambda'}\hat{O}_{\lambda\kappa}\hat{O}_{\lambda'\kappa'}F_{loc}(q',q). \quad (20)$$

Here $|q\rangle$ stands for the set $|q_1,\ldots,q_{N_B}\rangle$,

$$F_{loc}(q',q)=\prod_{i=1}^{N_B}F^i_{loc},$$

$$F^i_{loc}(q'_i,q_i)=\sum_{l,l'}f'_i f_i^*\rho^0(q'_i+2hkl'_i,q_i+2hkl_i) \times\exp\left\{-i\frac{[(q'_i)^2-q_i^2]t}{2m_eh}\right\}, \quad (21)$$

$f_i=f(q_i,l_i,\psi)$, $f'_i=f(q'_i,l'_i,\psi')$, where

$$f(q,l,\psi)=\left(\frac{\lambda a}{\kappa a^*}\right)^{l/2}J_l[2g|a(t)|\sqrt{\lambda\kappa}]e^{il\psi}. \quad (22)$$

It is convenient to consider Fourier transform

$$F^i_{loc}(p,z)=\int\frac{Ldq}{2\pi h}e^{iqz/h}F^i_{loc}(p+q/2,p-q/2). \quad (23)$$

For a short undulator, parameter $\epsilon_i t\ll 1$. In this case, $a(t)\simeq te^{-i\epsilon_i t/2}$. Parameter $(gt)^2$ is the average number of photons radiated in the main mode of the undulator per particle and is always small. This justifies the expansion of f_i in series over gt. Neglecting terms of the order of $(gt)^3$, we write for the ith particle

$$F^i_{loc}(p,z)=F^0_i(p,z)[1+gtF^{(1)}_i+(gt)^2F^{(2)}_i], \quad (24)$$

where

$$F^0_i(p,z)=\frac{h}{\sigma\Delta}\exp\left[-\frac{(p-p_0)^2}{2\Delta^2}-\frac{(z-z^0-pt/m_e)^2}{2\sigma^2}\right], \quad (25)$$

and

$$F^{(1)}_i=\exp[-(1/2)(hk/\Delta)^2]\left\{-\kappa\exp\left[\frac{hk(p-p^0)}{\Delta^2}+i\psi-2ik\left(z-\frac{pt}{2m_e}\right)\right]-\kappa'\exp\left[\frac{hk(p-p^0)}{\Delta^2}-i\psi'+2ik\left(z-\frac{pt}{2m_e}\right)\right]+\lambda\exp\left[-\frac{hk(p-p^0)}{\Delta^2}-i\psi+2ik\left(z-\frac{pt}{2m_e}\right)\right]+\lambda'\exp\left[-\frac{hk(p-p^0)}{\Delta^2}+i\psi'-2ik\left(z-\frac{pt}{2m_e}\right)\right]\right\}. \quad (26)$$

$F^{(2)}$ has a similar structure.

With the same accuracy,

$$F_{loc}(p,z)=\left\{\prod_{i=1}^{N_B}F^0_i(p_i,z_i)\right\}\exp[gt\Sigma_iF^{(1)}_i+(gt)^2\Sigma_iF^{corr}_i], \quad (27)$$

where $F^{corr}_i=F^{(2)}_i-(1/2)[F^{(1)}_i]^2$. Equation (27) takes into account all terms of the order of N_bgt and $N_b(gt)^2$ neglecting terms $N_b(gt)^3$.

The sum

$$f_0 \equiv gt \sum_i F_i^{(1)} \tag{28}$$

in the exponent of Eq. (27) is defined by parameters

$$\sigma_\pm(p,z) = gt \sum_{i=1}^{N_B} \exp\left[-2ik\left(z_i - \frac{p_i t}{2m_e} \right) \pm \frac{hk(p_i - p_i^0)}{\Delta^2} \right] \times \exp\left[-\frac{1}{2}\left(\frac{hk}{\Delta} \right)^2 \right]. \tag{29}$$

This expression must be averaged over the frequency spread in the mode around $\overline{k} = \gamma k_u$,

$$\sigma_\pm(p,z) = gt \sum_{i=1}^{N_B} \exp\left[-2i\overline{k}\left(z_i - \frac{p_i t}{2m_e} \right) \pm \frac{h\overline{k}(p_i - p_i^0)}{\Delta^2} \right] \times \exp\left[-\frac{1}{2}\left(\frac{h\overline{k}}{\Delta} \right)^2 \right] s_i, \tag{30}$$

where

$$s_i = \int \frac{dk}{\pi} \exp[-2i(k-\overline{k}) \times (z_i - p_i t/2m_e)] \frac{\sin^2(\pi N_u (k-\overline{k})/\overline{k})}{(\pi N_u/\overline{k})(k-\overline{k})^2}. \tag{31}$$

Factors s_i restrict summation in Eq. (30) over particles within the length proportional to $2\pi N_u/(2\overline{k})$ (the length of a "slice"), or, in the laboratory system, within $l_s = N_u \lambda_{lab}$. Parameter $N_s = \langle\langle \sigma_- \sigma_-^* \rangle\rangle/(gt)^2$ is the fundamental parameter of the theory defining the number of interacting particles within the bandwidth of the mode (number of particles per slice). Here double averaging means averaging with the density matrix of the wave packet Eq. (25) and over z^0, p^0 within the Gaussian bunch $\rho_B(z_0,p_0) = (1/2\pi\sigma_B\Delta_B)\exp[-p_0^2/2\Delta_B^2 - z_0^2/2\sigma_B^2]$. If the width of packet σ in Eq. (25) is of the order of the length of a slice, and $N_u \gg 1$, then $\overline{k}\sigma \gg 1$, and

$$N_s = \sum_i \int \frac{dx}{\pi} \frac{\sin^2 x}{x^2} \frac{dy}{\pi} \frac{\sin^2 y}{y^2} \times \left\langle \left\langle \exp\left[-\frac{2ik}{\pi N_u}(x-y)\left(z_i - \frac{p_i t}{2m_e} \right) \right] \right\rangle \right\rangle. \tag{32}$$

Neglecting terms of the order of h, we get

$$N_s = N_b \frac{N_u \sqrt{2\pi}}{3k\sigma_B}, \tag{33}$$

where σ_B is the rms bunch length in the moving frame and we use $\int (dx/\pi)(\sin x/x)^4 = 0.6666$. In terms of the wavelength of the mode and the bunch length in the laboratory frame,

$$N_s = N_b \left(\frac{N_u \lambda_L}{3\sqrt{2\pi}\sigma_B^0} \right). \tag{34}$$

In terms of averaged $\sigma_\pm$ introduced in Eq. (30), Eq. (28) takes the form of

$$f_0(\psi,\psi') = -\kappa\sigma_+ e^{i\psi} - \kappa'\sigma_+^* e^{-i\psi'} + \lambda\sigma_-^* e^{-i\psi} + \lambda'\sigma_- e^{i\psi'}. \tag{35}$$

The second terms $(gt)^2 \Sigma_i F_i^{corr}$ in the exponent of Eq. (27) can be expanded over h. Expansion starts with the term proportional to h^2. It can be split in two parts: the first of which,

$$f_{cor}^{(1)} = -N_s (gt)^2 \left(\frac{hk}{\Delta} \right)^2 (\kappa e^{i\psi} + \lambda' e^{i\psi'})(\kappa' e^{-i\psi'} + \lambda e^{-i\psi}), \tag{36}$$

is proportional to the number of particles N_s, and the second one $f_{cor}^{(2)}$, is proportional to the sum over oscillating factors. Introducing $r_\pm = \Sigma_i \exp[\pm 4ik(z_i - p_i t/2m_e)]$, we can write

$$f_{cor}^{(2)} = -\frac{(gt)^2}{2}\left(\frac{hk}{\Delta} \right)^2 [(\kappa e^{i\psi} + \lambda' e^{i\psi'})^2 r_- + (\kappa' e^{-i\psi'} + \lambda e^{-i\psi})^2 r_+]. \tag{37}$$

In these notations,

$$F_{loc}(p,z) = \left\{ \prod_{i=1}^{N_B} F_i^0(p_i, z_i) \right\} e^{f_0(\psi,\psi') + f_{cor}^{(1)} + f_{cor}^{(2)}}. \tag{38}$$

The first factor is the product of unperturbed single particle distribution functions while the exponential factor describes particle interaction. The last term, $f_{cor}^{(2)}$, is small. Equation (38) can be simplified by writing $e^{f_{cor}^{(2)}} = (1 + f_{cor}^{(2)})$ and replacing $-gt\kappa' e^{-i\psi'}$, $gt\lambda e^{-i\psi}$, $-gt\kappa e^{i\psi}$, and $gt\lambda' e^{i\psi'}$ by the derivatives over σ_+^*, σ_-^*, σ_+, and σ_-, respectively. The result is the differential operator $\hat{P}(\sigma_\pm)$. The factor $e^{f_{cor}^{(1)}}$ can be written as

$$e^{f_{cor}^{(1)}} = \hat{O}_{\mu,\nu} \exp[-\nu(\kappa e^{i\psi} + \lambda' e^{i\psi'}) - \mu(\kappa' e^{-i\psi'} + \lambda e^{-i\psi})]|_{\mu=\nu=0}, \tag{39}$$

where $\hat{O}_{\mu,\nu} = \exp[-\zeta^2(\partial^2/\partial\mu\partial\nu)]$, and $\zeta^2 = N_s(gt)^2(hk/\Delta)^2$. Then,

$$F_{loc}(p,z) = \left\{ \prod_{i=1}^{N_B} F_i^0(p_i, z_i) \right\} (1+\hat{P})\hat{O}_{\mu,\nu} \times \exp[-\kappa(\sigma_+ + \nu)e^{i\psi} - \kappa'(\sigma_+^* + \mu)e^{-i\psi'} + \lambda(\sigma_-^* - \mu)e^{-i\psi} + \lambda'(\sigma_- - \nu)e^{i\psi'}]. \tag{40}$$

Now it is easy to calculate

$$\hat{O}_{\kappa,\lambda}\hat{O}_{\kappa',\lambda'}\exp[-\kappa(\sigma_+ +\nu)e^{i\psi}+\lambda(\sigma_-^*-\mu)e^{-i\psi} -\kappa'(\sigma_+^*+\mu)e^{-i\psi'}+\lambda'(\sigma_- -\nu)e^{i\psi'}]|_{\kappa=\kappa'=1}$$
$$=\exp[(1/2)(\sigma_+ +\nu)(\sigma_-^*-\mu) +(1/2)(\sigma_+^*+\mu)(\sigma_- -\nu)]\exp[-(\sigma_+ +\nu)e^{i\psi} +\lambda(\sigma_-^*-\mu)e^{-i\psi}-(\sigma_+^*+\mu)e^{-i\psi'} +\lambda'(\sigma_- -\nu)e^{i\psi'}]. \quad (41)$$

Integration over ψ and ψ' can be carried out using

$$\int \frac{d\psi}{2\pi}e^{il\psi}\exp[\lambda e^{-i\psi}-\kappa e^{i\psi}]=\left(\frac{\lambda}{\kappa}\right)^{l/2}J_l(2\sqrt{\lambda\kappa}). \quad (42)$$

After that, integrals over λ and λ' are [9]

$$\int_0^\infty d\lambda e^{-\lambda}\lambda^{l/2}J_l(2\sqrt{\lambda a})=a^{l/2}e^{-a}. \quad (43)$$

The distribution function at the end of the pickup

$$\rho(p,z,l'_\Sigma,l_\Sigma)=\int \frac{Ldq}{2\pi h}e^{iqz/h}\rho(p+q/2,p-q/2,l'_\Sigma,l_\Sigma) \quad (44)$$

takes the form of

$$\rho(p,z,l'_\Sigma,l_\Sigma)=\frac{1}{\sqrt{l_\Sigma!l'_\Sigma!}}\exp[i\omega t(l_\Sigma-l'_\Sigma)]\left\{\prod_{i=1}^{N_B}F_i^0(p_i,z_i)\right\} \times(1+\hat{P})R(p,z), \quad (45)$$

where

$$R(p,z)=\hat{O}_{\mu,\nu}(\sigma_-^*-\mu)^{l_\Sigma}(\sigma_- -\nu)^{l'_\Sigma}\exp[-(1/2)(\sigma_-^*-\mu) \times(\sigma_+ +\nu)-(1/2)(\sigma_- -\nu)(\sigma_+^*+\mu)]. \quad (46)$$

For small ζ, the density matrix at the end of the pickup is

$$\rho(p,z,l'_\Sigma,l_\Sigma)=\frac{1}{\sqrt{l_\Sigma!l'_\Sigma!}}\left\{\prod_{i=1}^{N_B}F_i^0(p_i,z_i)\right\} \times(1+\hat{P})R(p,z,N,\mu)\exp[i\omega t(l_\Sigma-l'_\Sigma)], \quad (47)$$

where $R=\exp[-(1/2)(\sigma_-^*\sigma_+ +\text{c.c.})]\tilde{R}(p,z,N,\mu)$, and

$$\tilde{R}(p,z,N,\mu)=\left(\frac{\sigma_-^*}{\sigma_-}\right)^\mu|\sigma_-|^{2N}, \quad N=\frac{l_\Sigma+l'_\Sigma}{2}, \quad \mu=\frac{l_\Sigma-l'_\Sigma}{2}. \quad (48)$$

Correction $\zeta^2|\sigma_-|^2$ is of the order of $[N_s(gt)^2(hk/\Delta)]^2$ and is always negligible.

V. SOME RESULTS FOR THE UNDULATOR

First, let us show that the total density matrix Eq. (47) of the system (particles and radiation) allows us to reproduce results obtained by different methods.

A trace of the density matrix Eq. (47) defines energy loss

$$\langle p\rangle=\text{Tr}(\hat{p}\hat{\rho})=-2hk(gt)^2. \quad (49)$$

The density matrix of radiation, ρ_{rad}, can be obtained as the trace of the full density matrix Eq. (47) over particle indexes,

$$\rho_{rad}(l'_\Sigma,l_\Sigma)=\sum_{p_i}\rho(p'_i,p_i,l'_\Sigma,l_\Sigma). \quad (50)$$

Equations (49) and (50) reproduce results [10] obtained by the operator formalism method.

The density matrix of radiation for a single electron, $N_b=1$, with no initial photons,

$$w_{rad}(l)=\rho_{rad}(l,l)=\frac{(gt)^{2l}}{l!}e^{-(gt)^2}, \quad (51)$$

is a density matrix of a coherent state. The average number of radiated photons in a single mode is small, $\langle l\rangle=(gt)^2=\alpha_0=1/137$. Fluctuations are large $\langle(\Delta l)^2\rangle=\langle l^2\rangle-\langle l\rangle^2=\langle l\rangle$, and

$$\frac{\langle(\Delta l)^2\rangle}{\langle l\rangle^2}=\frac{1}{(gt)^2}=137. \quad (52)$$

Radiation of a bunch corresponds to thermal statistics [6]

$$\rho_{rad}(l'_\Sigma,l_\Sigma)=\delta_{l_\Sigma,l'_\Sigma}\frac{(\alpha)^{l_\Sigma}}{(1+\alpha)^{l_\Sigma+1}}, \quad \alpha=N_b(gt)^2. \quad (53)$$

For the following, it is convenient to transform the density matrix $\rho(p,z,l'_\Sigma,l_\Sigma)$ back to the momentum representation,

$$\rho(p+q/2,p-q/2)=\left\{\prod_i\int(dz_i/L)\exp[-i(q'-q)z_i]\right\} \times\rho\left(\frac{q'_i+q_i}{2},z_i\right).$$

The result then becomes

$$\hat{\rho}=|q',l'_\Sigma\rangle\rho(q'q)\langle q,l_\Sigma|, \quad (54)$$

where

$$\rho(q',q)=\frac{1}{\sqrt{l_\Sigma!l'_\Sigma!}}\left\{\prod_i\int\frac{dz_i}{L}F^i(q',q,z)\right\}(1+\hat{P}) \times R\left(\frac{q'+q}{2},z,N,\mu\right)e^{2i\mu\omega t}, \quad (55)$$

and

$$F^i(q',q,z)=\frac{h}{\sigma\Delta}\exp\left[-i(q'-q)z/h-\frac{1}{2\Delta^2}\left(\frac{q'+q}{2}\right)^2-\frac{1}{2\sigma^2}\left(z-\frac{q'+q}{2m_e}t\right)^2\right]. \tag{56}$$

Note that $\sigma_\pm$ are functions of the coordinates $(z_i,[q_i'+q_i]/2)$ of all particles.

VI. OPTICAL AMPLIFIER AND DISPERSION SECTION

The density matrix Eqs. (47) and (48) at the exit of the pickup undulator is the superposition of coherent states. Transformation of such a state in the optical amplifier can be obtained in the following way [8]:

Let us consider the two level model of the amplifier with inverse population $N_u > N_d$. The equation describing the time evolution of the density matrix is well known [12],

$$\dot{\rho}=-gN_u[aa^+\rho+\rho aa^+-2a^+\rho a]-gN_d[\rho a^+a+a^+a\rho-2a\rho a^+]. \tag{57}$$

Let us use this representation with a fixed number of photons, $\rho(t)=|n'\rangle\rho(n',n,t)\langle n|$, and define $F(N,\mu,t)$,

$$\rho(n',n,t)=\frac{F(N,\mu,t)}{\sqrt{n!n'!}},\quad N=(n+n')/2,\quad \mu=(n-n')/2. \tag{58}$$

The function F is a solution of Eq. (57). Parameter μ is the integral of motion. Dependence on N can be obtained using a Mellin transform

$$F(N,\mu,t)=\int_0^\infty dz' G_m(N,z',\tau)f_0(z',m), \tag{59}$$

where $\tau=gN_ut$, t is the amplification time and f_0 is given by the initial condition,

$$f_0(z,\mu)=\int_{-i\infty}^{i\infty}\frac{dN}{2\pi i}z^{-N}F(N,\mu,0). \tag{60}$$

The kernel G_m can be obtained [8] for an arbitrary ratio of N_u/N_d. In the case of a fully inverted population, $N_d=0$,

$$G_m(N,z',\tau)=(N-\mu)!\frac{\xi}{z'}(1-\xi)^N b^\mu L^{2\mu}_{N-\mu}(-b), \tag{61}$$

where $b=\xi z'/(1-\xi)$, $\xi=e^{-2N_ugt}$, and t is the the time of amplification.

Parameter ξ is related to a gain of the amplifier. Consider, for example, radiation from the undulator of a single electron described by initial coherent state

$$\rho(n',n,0)=\frac{\alpha^{n'}(\alpha^*)^n}{\sqrt{n!n'!}}e^{-|\alpha|^2}. \tag{62}$$

The result of amplification is described by the density matrix

$$\rho(n,n',t)=\frac{(N-\mu)!}{\sqrt{n!n'!}}\left(\frac{\alpha^*}{\alpha}\right)^\mu|\alpha|^{2\mu}e^{-|\alpha|^2}\times\xi^{\mu+1}(1-\xi)^{N-\mu}L^{2\mu}_{N-\mu}\left(-\frac{\xi}{1-\xi}\left|\alpha\right|^2\right). \tag{63}$$

Then

$$\langle a(t)\rangle=\frac{\alpha}{\sqrt{\xi}}, \tag{64}$$

which shows that parameter ξ defines the gain G of the amplifier, $G=1/\xi$.

The lowest moments are

$$\langle a(t)\rangle=\sqrt{G}\alpha,\quad \langle a^+a\rangle=G|\alpha|^2+(G-1), \tag{65}$$

and the signal-to-noise ratio is independent of G for $G\gg 1$

$$\frac{\langle(a^+a)^2\rangle-\langle a^+a\rangle^2}{\langle a^+a\rangle^2}=\frac{1+2|\alpha|^2}{(1+|\alpha|^2)^2}\simeq 1. \tag{66}$$

Let us use these results to transform the density matrix Eq. (48) in the amplifier.

The Mellin transform $\tilde{R}_M(N,\mu)$ of $\tilde{R}([q'+q]/2,z,N,\mu)$,

$$\tilde{R}_M(\zeta,\mu)=\int_{-i\infty}^{i\infty}\frac{dN}{2\pi i}\zeta^{-N}\tilde{R}\left(\frac{q'+q}{2},z,N,\mu\right), \tag{67}$$

is proportional to $\delta(\zeta-\zeta_0)$, $\zeta_0=|\sigma_-|^2$,

$$\tilde{R}_M(\zeta,\mu)=\zeta_0\left[\frac{\sigma_-^*}{\sigma_-}\right]^\mu\delta(\zeta-\zeta_0). \tag{68}$$

After the amplifier, $\tilde{R}([q'+q]/2,z,N,\mu)$ should be replaced [8] by F_{ampl},

$$F_{ampl}(N,\mu)=(N-|\mu|)!\frac{1}{G}\left[\frac{\sigma_-^*}{\sigma_-}\right]^\mu\left[\frac{\sigma_-^*\sigma_-}{G-1}\right]^{|\mu|}\left(\frac{G-1}{G}\right)^N\times L^{2|\mu|}_{N-|\mu|}\left(-\frac{|\sigma_-|^2}{G-1}\right). \tag{69}$$

Here G is the power gain of the amplifier, L_N^m are Laguerre polynomials, and $N=(l_\Sigma+l_\Sigma')/2$, $\mu=(l_\Sigma-l_\Sigma')/2$.

Equation (69) describes the amplification of the main term in Eq. (47). Calculation of the derivatives in the correction term, $\hat{P}R(p,z,N,\mu)$ where $\hat{P}$ is a differential operator of the second order in $\sigma_\pm$, gives a polynomial of the second order in N multiplied by $R(p,z,N,\mu)$. The result can be written as $\hat{P}(y[\partial/\partial y])x^\mu y^N$, where $\hat{P}$ is now a differential operator of the second order in y, independent of N, and $y=|\sigma_-|^2$, $x=\sigma_-^*/\sigma_-$. This expression can be transformed in the amplifier in the same way as the main term above.

VII. DISPERSION SECTION

The dispersion section with momentum compaction α_{MC} and length L_{ds} introduces (z,p) correlation by changing the path length in the lab frame by $\Delta z=\alpha_{MC}L_{ds}(p-p^0)/q_0$. In the moving frame, such a correlation is described by the classical distribution function

$$f(p,z)=\frac{1}{2\pi\Delta\sigma}\exp\left[-\frac{(p-p_0)^2}{2\Delta^2}-\frac{(z-z_0-\eta p)^2}{2\sigma^2}\right], \tag{70}$$

where parameter $\eta=\gamma_0\alpha_{MC}L_{ds}/m_e c$. The corresponding density matrix is different from Eq. (9) by the factor $\exp[-(i/h)\eta(q'^2-q^2)/2]$.

Hence, the dispersion section modifies $F^i(q',q,z)$ in Eq. (55). Equation (56) has to be replaced with

$$F^i(q',q,z)\exp[-(i/h)\eta(q'^2-q^2)/2]e^{i\theta}. \tag{71}$$

Here, a phase slip θ of a bunch centroid is added and should be controlled in the experiment.

VIII. KICKER

The density matrix at the entrance to the kicker is obtained by combining Eqs. (47), (48), (69), and (71),

$$\hat{\rho}_{in}(t)=|q',l'_\Sigma\rangle\frac{F_{in}}{\sqrt{l_\Sigma!\,l'_\Sigma!}}\langle q,l_\Sigma|, \tag{72}$$

where

$$F_{in}=F_{ds}(q',q)(1+\hat{P})F_{ampl}(N,\mu)e^{2i\mu\omega t}\times\exp\left(-\frac{1}{2}[\sigma_-^*\sigma_+ +\text{c.c.}]\right)$$

and

$$F_{ds}=\prod_i\int\frac{dz}{L}F^i(q',q,z)\exp\left\{-i\frac{\eta}{2h}[(q')^2-q^2]\right\}. \tag{73}$$

The transform of the density matrix at the end of the kicker is given by Eq. (14) where n has to be replaced by the number of photons l_Σ. We use notation m_i for the number of photons radiated by the ith electron in the kicker and $m_\Sigma=\Sigma_i m_i$ for the total number of photons. We also assume that the parameters of both undulators are the same.

Given the above, the density matrix at the exit of the kicker

$$\hat{\rho}_{out}(t)=|q'-2hkm',l'_\Sigma+m'_\Sigma\rangle\Phi_{loc}(q,q',\psi,\psi')\times F^*_{out}(q,l_\Sigma,m_\Sigma)F^*_{out}(q',l'_{\Sigma'},m'_\Sigma)(1+\hat{P})\times F_{ampl}(N,\mu)e^{2i\mu\omega t}\exp\left(-\frac{1}{2}[\sigma_-^*\sigma_+ +\text{c.c.}]\right)\times\langle q-2hkm,l_\Sigma+m_\Sigma|. \tag{74}$$

Here $l_\Sigma=N+\mu$, $l'_\Sigma=N-\mu$, $m_\Sigma=M-\mu$, $m'_\Sigma=M+\mu$. Because l_Σ and l'_Σ are positive, the range of summation is $0<N<\infty$, $-N<M<\infty$, and $|\mu|<N$. Functions $\sigma_\pm$ in F_{ampl} depend on coordinates of individual particles $(q'_i+q_i)/2,z_i$. The operator F_{out} is

$$F_{out}(q,l_\Sigma,m_\Sigma)=\frac{1}{\sqrt{(l_\Sigma+m_\Sigma)!}}\int\frac{d\psi}{2\pi}e^{-im_\Sigma\psi}\times\exp[i\omega t(l_\Sigma+m_\Sigma)]\int d\lambda\frac{\lambda^{l_\Sigma}}{l_\Sigma!}e^{-\lambda}\hat{O}_{\lambda\kappa}, \tag{75}$$

and

$$\Phi_{loc}(q,q',\psi,\psi')=\prod_i\frac{dz_i}{L}F^{(i)}(q',q,z)\times\exp\left\{-\frac{i\eta}{2h}[(q'_i)^2-(q_i)^2]\right\}\times S^*_{m_i}(q,\lambda,\kappa)S_i(q',\lambda',\kappa'), \tag{76}$$

where

$$S_{m_i}(q,\lambda,\kappa)=\left(\frac{\lambda a}{\kappa a^*}\right)^{m_i/2}J_{m_i}[2g|a_i(t)|\sqrt{\lambda\kappa}]e^{im_i\psi}\times\exp\left\{\frac{-i[(q-2hkm)^2]t}{2m_e h}\right\}. \tag{77}$$

To describe stochastic cooling, it is sufficient to calculate the momentum of a particle at the end of the kicker. The average moments for the jth particle after a bunch passes through the system are $\langle p_j^k\rangle=\text{Tr}[\hat{p}_j^k\hat{\rho}_{out}(t)]$, $k=0,1,\ldots,$ where $\hat{p}$ is momentum operator and brackets $\langle\cdots\rangle$ signify averaging over the wave packet. In the momentum representation, only the diagonal components, $q'_i-2hkm'_i=q_i-2hkm_i$, $i=1,2,\ldots,N_b$, and $l'_\Sigma+m'_\Sigma=l_\Sigma+m_\Sigma$, contribute in $\langle p_j^k\rangle$. We can utilize the fact that $\sigma_\pm$ are functions only of the sum $q'+q$ and introduce P, $q'_i=P_i+hk(m'_i-m_i)$, $q_i=P_i-hk(m'_i-m_i)$. This allows us to write

$$\langle p^n\rangle=[P_j-hk(m_j+m'_j)]^n\Phi_{loc}(1+\hat{P})\times F^*_{out}(q,l_\Sigma,m_\Sigma)F_{out}(q',l'_\Sigma,m'_\Sigma)F_{ampl}(N,\mu), \tag{78}$$

where

$$\Phi_{loc}=\prod_i\frac{dz_i dP_i}{2\pi\sigma\Delta}\rho_0(P_i,z_i)\sum_{m'_i,m_i}S_{m_i}(\lambda,\kappa)S_{m'_i}(\lambda',\kappa')\times\exp[-2ik(z_i+\eta P_i)(m'_i-m_i)], \tag{79}$$

and

$$\rho_0(P_i,z_i)=\exp[-(P_i-p_i^0)^2/2\Delta^2-(z_i-z_i^0-P_it/m0)^2]. \tag{80}$$

Note that F_{ampl} depends on $\sigma_\pm$ that are given now by Eq. (32) where p_i are replaced by P_i.

Similarly to what was done for the pickup, we expand $S(\lambda,\kappa)$ in series over gt, neglecting terms $O(gt)^3$. Here we skip over the details of the calculations and give the final result

$$\langle p_j^n\rangle=\sum_{i\neq j}\hat{K}_n(1+\hat{P})F_{out}(q,l_\Sigma,m_\Sigma)F_{out}(q',l'_\Sigma,m'_\Sigma)\times Q(b_1,b_2)F_{ampl}(P,z)|_{b_2-\rangle b_1}. \tag{81}$$

The sum stands for integrals $\Pi_i[(dz_idP_i)/(2\pi\sigma\Delta)]\rho_0(P_i,z_i)$ over all particles in a bunch, and

$$Q(b_1,b_2)=\exp(\lambda' b_1e^{i\psi'}-\kappa' b_2^*e^{-i\psi'}+\lambda b_1^*e^{-i\psi}-\kappa b_2e^{i\psi}), \tag{82}$$

where $b_1=gt\Sigma_ie^{-i\phi_i}$, and phase $\phi_j=2k[z_j+p_j\eta]$. Operators $\hat{K}_n$ for different $n=0,1,2$ are $\hat{K}_0=1$, $\hat{K}_1=q_j-hkgt(a_\lambda-a_\kappa)$, $\hat{K}_2=\hat{K}_1^2+(hk)^2gt(a_\lambda+a_\kappa)$, where

$$a_\kappa=e^{-i\phi_j}\frac{\partial}{\partial b_2}+e^{i\phi_j}\frac{\partial}{\partial b_2^*},\quad a_\lambda=e^{i\phi_j}\frac{\partial}{\partial b_1^*}+e^{-i\phi_j}\frac{\partial}{\partial b_1}. \tag{83}$$

Equation (81) after some calculations (see Appendix B) can be written as

$$\begin{aligned}\langle p_j^n\rangle=&\sum_{i\neq j}\hat{K}_n\sum_\mu(1+\hat{P})\left(\frac{\sigma_-^*}{\sigma_-}\right)^\mu\\&\times I_{2\mu}[2\sqrt{G|b_2-b_1|^2|\sigma_--\sigma_+|^2}]\left(\frac{b_1-b_2}{b_1^*-b_2^*}\right)^\mu\\&\times\exp\{(G-1)|b_2-b_1|^2+(1/2)[b_2(b_2^*-b_1^*)+\text{c.c.}]\}\\&\times\exp\{(1/2)[\sigma_-(\sigma_-^*-\sigma_+)+\text{c.c}]\}|_{b_2=b_1}.\end{aligned} \tag{84}$$

The operators $\hat{K}_n$ are not more than the second order differential operators in b_2, b_1, and the function depends on $b_{2,1}$ only through powers of b_2-b_1. Therefore, it is sufficient to take into account only terms $\mu=0$, $\mu=\pm1/2$, and $\mu=\pm1$ in the sum over μ. Additionally, we can expand the answer in series over gt and neglect terms $O(h^3)$.

To check the result, we can calculate the average $\langle p_j^n\rangle$ for $n=0$. This quantity is just the norm of the distribution function and, therefore, has to be equal to 1. Indeed, the answer is different from one by the term of the order of $N_s^2(gt)^4(hk/\Delta)^4$.

The result for the moments $n=1$ and $n=2$ were obtained with MATHEMATICA. As it will be shown below, the power gain G must be of the order of $\Delta_b/(hk)$. Because $G\gg1$, we can neglect terms that are independent of G. In this approximation, momentum $\tilde{p}_j$ of the jth particle at the end of the kicker is

$$\tilde{p}_j=p_j-2(gt)^2hk\sqrt{G}[\sigma_0^*\exp\{-2ik(z_j+\eta_{eff}p_j)+i\theta\}+\text{c.c.}], \tag{85}$$

where $\eta_{eff}=\eta+t/2m_e$, and θ is the phase slip of the bunch centroid. Calculation of $\tilde{p}_j^2$ at the end of the kicker gives

$$\begin{aligned}\tilde{p}_j^2=&p_j^2-4G(gt)^2(hk)p_j(\sigma_0^*\exp[-2ik(z_j+\eta_{eff}p_j)+i\theta]\\&+\text{c.c.})+8G(gt)^2(hk)^2[1+(gt)^2\sigma_0\sigma_0^*+\text{c.c.}]\\&+4\sqrt{G}(gt)^4(hk)^2(b_1\sigma_0^*e^{i\theta}+\text{c.c.}).\end{aligned} \tag{86}$$

Here $\sigma_0=\sigma_\pm|_{h\to0}$. Double averaging over the wave packet $\rho_0(p_j,z_j)$ and over a Gaussian distribution of particles in the bunch gives rms $\Delta^2=\langle\langle p^2\rangle\rangle-\langle\langle p\rangle\rangle^2$ at the end of the kicker

$$\begin{aligned}\frac{\tilde{\Delta}^2-\Delta^2}{\Delta^2}=&-16\sqrt{G}(gt)^2\frac{hk}{\Delta_B}\Lambda\sin\theta\\&+8G(gt)^2\left(\frac{hk}{\Delta_B}\right)^2[1+N_s(gt)^2]\\&+8\sqrt{G}(gt)^4\left(\frac{hk}{\Delta_B}\right)^2N_s\cos\theta e^{-2(k\Delta_B)^2\eta_{eff}^2}.\end{aligned} \tag{87}$$

Here $\Lambda=k\Delta_B\eta_{eff}\exp[-2(k\Delta_B\eta_{eff})^2]$.

To get damping, we have to choose $\sin\theta=1$. The damping is maximized if the power gain G of the amplifier is equal to

$$\sqrt{G}=\frac{\Lambda}{(hk/\Delta_B)[1+N_s(gt)^2]}. \tag{88}$$

Parameter Λ as a function of $x=k\Delta_B\eta_{eff}$ has a maximum value of $\Lambda_{max}\simeq0.3$ at $x\simeq2$. This defines the optimum parameter η of the dispersion section.

The optimized reduction of the rms energy spread in one pass through the system is

$$\frac{\tilde{\Delta}^2-\Delta^2}{\Delta^2}=-\frac{8\Lambda_{max}^2}{(gt)^{-2}+N_s}. \tag{89}$$

IX. CONCLUSION

One-pass reduction of the energy spread rms is derived following the evolution of the density matrix through all components of the system. The consideration is fully quantum mechanical both for the beam and for the radiation. The bunching effect is neglected and length of a slice of the order of $N_u\lambda_{lab}$ is assumed to be small compared to the bunch length in the laboratory frame σ_B^0.

It is shown that the constructed density matrix may be used to obtain results on the photon statistics and distribution of particles due to the beam-radiation interaction in the undulator. The time evolution of the density matrix of radiation in the amplifier is described, including the nondiagonal components of the matrix.

The final result Eq. (89) is equivalent to the classical equation of stochastic cooling with quantum noise $1/(gt)^2$. Equation (89) for large $N_s \gg 1/(gt)^2$ reproduces the main result of the classical theory of stochastic cooling. The damping rate is given by the number of particles N_s per slice. However, for small N_s the damping rate goes to a constant proportional to $1/(gt)^2$, where $(gt)^2 \propto [K_0^2/(1+K_0^2)]\alpha_0$. The quantum fluctuations set the limit on the damping rate, the minimum number of turns for cooling is of the order of $1/\alpha_0$. The term $1/(gt)^2$ is equivalent to the noise induced by $1/\alpha_0$ particles and is related to the quantum limit of the input noise of the amplifier equal to one photon in a mode. The other quantum-mechanical corrections are small, of the order of $(hk)/\Delta_B$ (i.e., $hk_L/\Delta p_L$ in the laboratory frame) and are noticeable only in very cold beams where the energy spread is comparable to photon energy.

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APPENDIX A: BEAM DYNAMICS IN THE UNDULATOR

Interaction of particles with the mode described by Hamiltonian Eq. (11) is just backscattering of equivalent photons. The initial state $|p_i,n\rangle = |p_1,p_2,\ldots,p_{N_B},n\rangle$ of the system with n photons and particles with momentums p_i, $i=1,\ldots,N_B$ is transformed by the interaction with the mode $k=\omega/c=\gamma k_u$ to the vector $|\Psi(t)\rangle$,

$$|\Psi(t)\rangle = \sum_{l_i,p_i} |p_i - 2hkl_i, n+l_\Sigma\rangle \sqrt{\frac{n!}{(n+l_\Sigma)!}} \times \int \frac{d\psi}{2\pi} e^{-il_\Sigma\psi} \times \exp[-i\omega t(n+l_\Sigma)] \prod_{i=1}^{N_B} F_n(t,p_i,l_i). \quad \text{(A1)}$$

Here $l_\Sigma = \Sigma_i l_i$ is the total number of radiated photons, $E(p_i,l_i) = (p_i - 2hkl_i)^2/(2m_0)$, and the amplitudes $F(t,[p_i,l_i])$ are defined by the equation

$$\dot{F}_n(t,p_j,l_j) = g\sum_j [(n+l_\Sigma)F_n(t,p_j,l_j-1) \times \exp\{-2i(k/m_0)t(p-2hkl_j)\} - F_n(t,p_j,l_j+1) \times \exp\{2i(k/m_0)t(p_j-2hkl_j)\}], \quad \text{(A2)}$$

following from the Schrödinger equation. Here, we use $\langle p_j - 2hkl_j | e^{\pm 2ik\hat{z}} | p_j' - 2hkl_j'\rangle = (2\pi h/L)\delta[p_j' - p_j + 2hk \times (l_j - l_j' \pm 1)]$, and write explicitly only those quantum numbers that are changed by the interaction, $F_n(t,p_j,l_j\pm 1) = F_n[t,(p_1,l_1),\ldots,(p_j,l_j\pm 1),\ldots,(p_{N_B},l_{N_B})]$.

Neglecting terms $hk^2/2m_0$ in the exponent (i.e., in the laboratory frame, terms of the order of $hk_uk_L/m_0 \ll 1$), we can solve Eq. (A2) by the Fourier transform

$$F_n(t,p_j,l_j) = \left\{ \prod_{i=1}^{N_B} \int_0^{2\pi} \frac{d\phi_i}{2\pi} e^{il_i\phi_i} \right\} \times F(t,p_1,\ldots,p_{N_B},\phi_1,\ldots,\phi_{N_B}). \quad \text{(A3)}$$

Function $F(t,\phi) = F(t,p_1,\ldots,p_{N_B},\phi_1,\ldots,\phi_{N_B})$ is given by

$$\dot{F}(t,\phi) = -gv_0^* F(t,\phi) + g(n+1)v_0 F + igv_0 \sum_i \frac{\partial F}{\partial \phi_i}, \quad \text{(A4)}$$

where

$$v_0(t,\phi) = \sum_i e^{-i\epsilon_i t - i\phi_i}, \quad \epsilon_i = \frac{2kp_i}{m_0}. \quad \text{(A5)}$$

If the system is initially in the n-photon state $|\Psi(0)\rangle = |p_i,n\rangle$, then $F_n(0,p_i,l_i) = \Pi_i \delta_{l_i,0}$ and $F(0,\phi)=1$.

Characteristics $\phi(t)$ of the Eq. (A4) are defined by $\dot{\phi}_i(t) = -igv_0(t,\phi)$, with the solutions $\phi_i(t) = \phi_i^0 + V(t)$, where $V(0)=0$, $\phi_i^0 = \phi_i(0)$ are constants. The function $V(t)$ to be determined is the same for all $\phi_i(t)$.

$V(t)$ satisfies $\dot{V}(t) = -igv_0$. The substitution of $\phi(t)$ in Eq. (A5) gives

$$v_0(t,\phi) = u_0(t,\phi^0) e^{-iV(t)},$$

where

$$u_0(t,\phi^0) \equiv \sum_i \exp[-i\epsilon_i t - i\phi_i^0]. \quad \text{(A6)}$$

Hence, $(\partial/\partial t)e^{iV(t)} = gv_0 e^{iV} = gu_0(t,\phi^0)$, with the solution

$$e^{iV(t)} = 1 + g\sum a_j(t) e^{-i\phi_j^0}, \quad \text{(A7)}$$

where

$$a_i(t) = \frac{\sin(\epsilon_i t/2)}{(\epsilon_i/2)} e^{-i\epsilon_i t/2}, \quad \dot{a}_i(t) = e^{-i\epsilon_i t}. \quad \text{(A8)}$$

Equation (A7) defines characteristics $\phi_i(t)$

$$e^{i\phi_i} = e^{i\phi_i^0 + iV} = e^{i\phi_i^0}\left[1 + g\sum a_j(t) e^{-i\phi_j^0}\right], \quad e^{-i\phi_i} = [e^{i\phi_i}]^{-1}. \quad \text{(A9)}$$

Equation (A9) can be reversed to get constants of motion ϕ_i^0 in terms of ϕ_i. Defining $\Lambda(\phi_1,\ldots,\phi_{N_B})$ as $e^{i\phi_i^0} = \Lambda e^{i\phi_i(t)}$ and substituting this in the right-hand side of Eq. (A9) to get

$$\Lambda = 1 - g\sum_j a_j(t)e^{-i\phi_j}. \tag{A10}$$

Hence,

$$e^{i\phi_i^0} = e^{i\phi_i}\left[1 - g\sum a_j(t)e^{-i\phi_j}\right], \quad e^{-i\phi_i^0} = [e^{i\phi_i^0}]^{-1}. \tag{A11}$$

The general solution of the Eq. (A4) can be found as $F(t,\phi) = \Phi(t, e^{i\phi_i^0})$, where Φ is arbitrary function of the arguments $\xi_i(t,\phi) \equiv e^{i\phi_i^0}$ given by Eq. (A11). Equation (A4) in terms of t,ξ takes form

$$\frac{\partial\Phi(t,\xi)}{\partial t} = -gv_0^*\Phi(t,\xi) + g(n+1)v_0\Phi, \tag{A12}$$

where

$$v_0^*(t) = u_0^*\left[1 + g\sum_i a_i(t)/\xi_i\right]. \tag{A13}$$

Integrating Eq. (A12) over t and substituting ξ_i from Eq. (A11) gives

$$F(t,\phi) = \Phi_0(\xi)\frac{\exp\left\{-g\sum_i a_i^* e^{i\phi} + (g^2/2)\left|\sum_i a_i^* e^{i\phi}\right|^2 - (g^2/2)\sum_{i,j} e^{i(\phi_i-\phi_j)}\int_0^t d\tau[a_j\dot{a}_i^* - a_i^*\dot{a}_j]\right\}}{\left[1 - g\sum_i a_i e^{-i\phi_i}\right]^{n+1}}. \tag{A14}$$

Note, $a(0)=0$. Thus, the initial condition $F(0,\phi)=1$ allows us to choose $\Phi_0(\xi)=1$.

Parameter ϵt is of the order of $\epsilon t \simeq 2\pi 2N_u(\delta p/p)_L/\sqrt{1+K_0^2}$ where N_u is the number of periods in the undulator and $(\delta p/p)_L$ is the rms energy spread in the lab system. We assume that $\epsilon t \ll 1$ that means that the rms energy spread of the bunch is small compared to the width $\Delta\omega/\omega \simeq 1/N_u$ of the mode. In this case, the integral term in the numerator is

$$\int_0^t d\tau[a_j\dot{a}_i^* - a_i^*\dot{a}_j] \simeq \frac{it^3}{6}(\epsilon_i+\epsilon_j) - \frac{t^4}{12}(\epsilon_i^2 - \epsilon_j^2), \tag{A15}$$

i.e., ϵt times smaller than the other terms in the exponent [which are of the order of gt and $(gt)^2$] and can be neglected. The remaining factors can be written introducing additional integration over λ in terms of $b = g\Sigma_i a_i(t)e^{-i\phi_i}$

$$F(t,\phi) = \int_0^\infty d\lambda \frac{\lambda^n}{n!}e^{-\lambda}\exp[-b^* + \lambda b + (1/2)bb^*]. \tag{A16}$$

Let us factorize this expression using identity

$$\exp[-b^* + \lambda b + (1/2)bb^*] = \hat{O}_{\lambda\kappa}e^{\lambda b - \kappa b^*}|_{\kappa=1}, \tag{A17}$$

where operator $\hat{O}_{\lambda\kappa} = \exp[-(\frac{1}{2})(\partial^2/\partial\lambda\partial\kappa)]$.

Now integration in Eq. (A3) over ϕ_i can be easily carried out for each particle using

$$\int \frac{d\phi}{2\pi}\exp[il\phi - \kappa a^* e^{i\phi} + \lambda a e^{-i\phi}] = \left(\frac{\lambda a}{\kappa a^*}\right)^{l/2} J_l(2\sqrt{\lambda\kappa a a^*}). \tag{A18}$$

The replacement of $k \to -k$ on the left-hand side of Eq. (A18) changes the right-hand side of the equation by $J_l(\cdots) \to I_l(\cdots)$ leaving the rest intact.

Thus,

$$F_n(t,p,l) = \int_0^\infty d\lambda \frac{\lambda^n}{n!}e^{-\lambda}\hat{O}_{\lambda\kappa}\left\{\prod_{i=1}^{N_B}\left(\frac{\lambda a_i}{\kappa a_i^*}\right)^{l_i/2} J_{l_i}(2g|a_i|\sqrt{\lambda\kappa}) \times \exp\left[-(g^2/2)\int_0^t d\tau(a_i\dot{a}_i^* - \text{c.c.})\right]\right\}\Bigg|_{\kappa=1}, \tag{A19}$$

where J_l is the Bessel function.

For a single particle, $N_B=1$, identity

$$\hat{O}_{\lambda\kappa}\left(\frac{\lambda}{\kappa}\right)^{l/2} J_l(2g|a|\sqrt{\lambda\kappa})|_{\kappa=1} = \lambda^{l/2}\exp[(1/2)g^2|a|^2]J_l[2g|a|\sqrt{\lambda}] \tag{A20}$$

can be verified using the series for the Bessel function. This identity allows us to write

$$F_n(t,p,l)=\int_0^\infty d\lambda \frac{\lambda^{n+l/2}}{n!}e^{-\lambda}\left(\frac{a_i}{a_i^*}\right)^{l/2}J_l(2g|a|\sqrt{\lambda})$$
$$\times\exp[(1/2)g^2|a|^2]$$
$$\times\exp\left[-(g^2/2)\int_0^t d\tau(a_i\dot{a}_i^*-\text{c.c.})\right]. \tag{A21}$$

Integration over λ gives [9] the result in terms of Laguerre polynomials L_n^l

$$F_n(t,p,l)=(ga)^l\exp[-(1/2)g^2|a|^2]L_n^l(g^2|a|^2)$$
$$\times\exp\left[-(g^2/2)\int_0^t d\tau(a_i\dot{a}_i^*-\text{c.c.})\right]. \tag{A22}$$

Equation (A22) reproduces Dattoli-Renieri's [4] result for a single particle. Equation (A19) defines the evolution of the initial state of the system for a bunch with N_b particles.

APPENDIX B: DETAILS OF THE DERIVATION OF EQ. (84)

Equation (81) can be simplified, first, by integrating over ψ and ψ' and then by λ and λ' using formula

$$\int d\lambda \frac{\lambda^l}{l!}e^{-\lambda}\hat{O}_{\lambda\kappa}\left(\frac{\lambda}{\kappa}\right)^{m/2}J_m(2\sqrt{\lambda\kappa b})|_{\kappa=1}=b^{m/2}e^{-b/2}L_l^m(b). \tag{B1}$$

This can be obtained by using the series represntation of the Bessel function. The result is given in terms of Laguerre polynomials L_n^m. Equation (B1) is valid both for $m>0$ and $m<0$, where $L_l^{-|m|}(b)$ has to be understood as

$$L_l^{-|m|}(b)=(-1)^m\frac{(l-|m|)!}{l!}b^{|m|}L_{l-|m|}^{|m|}(b). \tag{B2}$$

In this way we obtain

$$F_{out}(l_\Sigma,M_\Sigma)F_{out}(l'_\Sigma,M'_\Sigma)Q(b_1,b_2)$$
$$=(b_1^*)^{M-\mu}(b_1)^{M+\mu}\exp\left[-\frac{1}{2}(b_2b_1^*+\text{c.c.})\right]$$
$$\times L_{N+\mu}^{M-\mu}(b_2b_1^*)L_{N-\mu}^{M+\mu}(b_2^*b_1), \tag{B3}$$

where $b_1=gt\Sigma\exp[-2ik(z_j+\eta p_j)]$. The average, $\langle p_j^k\rangle$ is proportional to the sum

$$S(\mu)=\sum_{M=-\infty}^{\infty}x_0^M\sum_{N=max(-M,N)}^{\infty}\frac{(N-\mu)!}{(N+M)!}\left(\frac{G-1}{G}\right)^N$$
$$\times L_{N+\mu}^{M-\mu}[x]L_{N-\mu}^{M+\mu}[x^*]L_{N-\mu}^{2\mu}\left[-\frac{y}{G-1}\right], \tag{B4}$$

where $y=|\sigma_-|^2$, $x=b_2b_1^*$, and $x_0=|b_1|^2$. Terms $\mu<0$ can be obtained by complex conjugation.

The sum $S(\mu)$ can be split into two parts, one, for $-\mu<M<\infty$, $\mu<N<\infty$, and another one for $-\infty<M<-\mu$, $-M<N<\infty$. In the first sum we may start summation from $N=-\mu$ because the maximum power of z in $L_{N+\mu}^{M-\mu}(z)$ is $N+\mu$. Therefore, derivatives over z are equal to if $N<\mu$. After this, the sum can be calculated first by expressing $L_{N-\mu}^{2\mu}[-y]$ in terms of the confluent hypergeometric function and using the integral representation of the latter,

$$L_{N-\mu}^{2\mu}[-y]=\frac{(N+\mu)!}{(N-\mu)!}y^{-2\mu}e^{-y}\int_{-i\infty}^{i\infty}\frac{ds}{2\pi i}e^{sy}\frac{s^{N-\mu}}{(s-1)^{N+\mu+1}}. \tag{B5}$$

Second, we can write $L_{N-\mu}^{M+\mu}[x^*]=(-[\partial/\partial z])^{2\mu}L_{N+\mu}^{M-\mu}[z]|_{z=x^*}$, and use [9]

$$\sum_{N=-M}^{\infty}\frac{(N+\mu)!}{(N+M)!}\xi^{N+\mu}L_{N+\mu}^{M-\mu}(x)L_{N+\mu}^{M-\mu}(z)$$
$$=\frac{(\xi xz)^{-(M-\mu)/2}}{1-\xi}\exp[-\xi(x+z)/(1-\xi)]$$
$$\times I_{|M-\mu|}\left(\frac{2\sqrt{\xi xz}}{1-\xi}\right), \tag{B6}$$

where $\xi=\{[s(G-1)/(s-1)G]\}$. In this form, the answer is valid also for the second part of the sum, $-\infty<M<-\mu$, $-M<N<\infty$.

The sum over M,

$$S(\mu)=\left(-\frac{\partial}{\partial z}\right)^{2\mu}\sum_{M=-\infty}^{\infty}\frac{x_0^M}{y^{2\mu}}$$
$$\times e^{-y}\int_{-i\infty}^{i\infty}\frac{ds}{2\pi i}\frac{(\xi s)^{-\mu}e^{sy}}{(s-1)^{\mu+1}}\frac{(\xi xz)^{-(M-\mu)/2}}{1-\xi}$$
$$\times e^{-[\xi(x+z)/(1-\xi)]}I_{|M-\mu|}\left(\frac{2\sqrt{\xi xz}}{1-\xi}\right), \tag{B7}$$

can be calculated using

$$\sum_{k=-\infty}^{\infty}\alpha^{k/2}I_{|k|}(2\beta)=e^{(\beta/\sqrt{\alpha})+\beta\sqrt{\alpha}}. \tag{B8}$$

After that, each derivative over z gives the factor $(\xi/[1-\xi])(1-[x/x_0])$. S takes the form of

$$S(\mu)=y^{-2\mu}x_0^\mu e^{-y}\int_{-i\infty}^{i\infty}\frac{ds}{2\pi i}e^{sy}$$
$$\times\exp\{[\xi x^*(x/x_0-1)+x_0-\xi x]/(1-\xi)\}$$
$$\times\left(1-\frac{x}{x_0}\right)^{2\mu}\frac{G^{\mu+1}(G-1)^\mu}{(s-G)^{2\mu+1}}. \tag{B9}$$

The integral is given here by the residues of the poles at $s = G$,

$$S(\mu)=G(G-1)^{\mu}\left(\frac{x_0}{yA}\right)^{\mu}\left(1-\frac{x}{x_0}\right)^{2\mu}I_{2\mu}(2\sqrt{GAy})\times\exp[x_0+y+(G-1)A], \tag{B10}$$

where $A=|b_2-b_1|^2$. Finally,

$$\langle p_j^n\rangle=\sum_{i\neq j}\hat{K}_n\sum_{\mu}(1+\hat{p})\left(\frac{\sigma_-^*}{\sigma_-}\right)^{\mu}\times I_{2\mu}[2\sqrt{G|b_2-b_1|^2|\sigma_--\sigma_+|^2}], \tag{B11}$$

$$\left(\frac{b_1-b_2}{b_1^*-b_2^*}\right)^{\mu}\exp\{(G-1)|b_2-b_1|^2+(1/2)[b_2(b_2^*-b_1^*)+\text{c.c.}]\}\times\exp\{(1/2)[\sigma_-(\sigma_-^*-\sigma_+)+\text{c.c.}]\}|_{b_2=b_1}. \tag{B12}$$

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