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1 Optical Cooling

Consider optical stochastic cooling using dependence on transit time through the bypass to couple transverse and longitudinal phase space in the pickup to phase in the kicker. The packet emits radiation in the pickup undulator that will arrive in the kicker with some relative phase $\phi = k\Delta s$, where k is the wavenumber of the characteristic undulator radiation and $\Delta s = s - s_0$ is the change in path length through the bypass. The interaction of the packet with the radiation in the kicker shifts its energy by

$$\Delta p/p = \xi \sin(\phi) = \xi \sin(k\Delta s). \tag{1}$$

In order to effect cooling, the phase is necessarily correlated with the phase space coordinate of the packet in the kicker, $\phi(\vec{x}_p)$. That is, the phase depends $\vec{x}_p$. The linear dependence of Δs on $\vec{x}_p$ is written

$$\Delta s = M_{51}x_p + M_{52}x'_p + M_{56}z'_p \tag{2}$$

where M is the 6X6 transfer matrix from the center of the pickup undulator to the center of the kicker. Since $x = x_\beta + x_e$ and $x' = x'_\beta + x'_e$ equation 2 becomes

$$\begin{aligned} \Delta s &= M_{51}(x_\beta + x_e) + M_{52}(x'_\beta + e'_e) + M_{56}z'_p \\ \Delta s &= M_{51}x_\beta + +M_{52}x'_\beta + (M_{51}\eta + M_{52}\eta' + M_{56})z'_p \end{aligned} \tag{3}$$

Next we write phase space coordinates at the pickup in terms of betatron amplitude and phase

$$\begin{aligned} x_{p\beta} &= a\sqrt{\beta_p}\cos\theta \\ x'_{p\beta} &= \frac{1}{2}\frac{a\beta'_p}{\sqrt{\beta_p}}\cos\theta - \frac{a}{\sqrt{\beta_p}}\sin\theta & (4) \\ &= -\frac{a}{\sqrt{\beta_p}}(\alpha_p\cos\theta + \sin\theta) & (5) \end{aligned}$$

and likewise at the kicker for future reference

$$x_{k\beta} = a\sqrt{\beta_k}\cos(\theta + \phi) \tag{6}$$

$$x'_{k\beta} = -\frac{a}{\sqrt{\beta_k}}(\alpha_k(\cos(\theta + \phi) + \sin(\theta + \phi)) \tag{7}$$

Then

$$\Delta s = a(M_{51}\sqrt{\beta_p}\cos\theta) - M_{52}\frac{(\alpha_p\cos\theta + \sin\theta)}{\sqrt{\beta_p}}) - a_z(M_{51}\eta + M_{52}\eta' + M_{56})\frac{(\alpha_p\cos\theta_z + \sin\theta_z)}{\sqrt{\beta_z}}) \tag{8}$$

$$\Delta s = A_x\sin(\theta_x + \theta_{xt}) + A_z\sin(\theta_z + \theta_{zt}) \tag{9}$$

where

$$A_x = a_x\left[M_{51}^2\beta_x + M_{52}^2\gamma_x - 2M_{51}M_{52}\alpha_x\right]^{1/2} \tag{10}$$

$$\theta_{xt} = \tan^{-1}\frac{M_{51}\beta_p - M_{52}\alpha_p}{M_{52}} \tag{11}$$

$$A_z = a_z(M_{51}\eta + M_{52}\eta' + M_{56}){\gamma_z}^{1/2} \tag{12}$$

$$\theta_{zt} = \tan^{-1}\alpha_p \tag{13}$$

2 Cooling

The cooling is quantified as the change in the invariant amplitude due to interaction of packet with radiation in the kicker undulator. At the kicker $\Delta x_{k\beta} = -\eta_k \Delta p/p$ and $\Delta x'_{k\beta} = -\eta'_k \Delta p/p$. And $\Delta z_k = 0, \Delta z'_k = \Delta p/p$. If $x = a_x\sqrt{\beta_x}\cos\phi_x$, or $z = a_x\sqrt{\beta_z}\cos\phi_z$ then the amplitude

$$a_x^2 = \beta {x'}^2 + \gamma x^2 + 2\alpha x x'$$

The change in the amplitude

$$\begin{aligned}
\Delta a_x^2 &= -2(\Delta p/p)(\beta_x x'_{k\beta}\eta'_x + \gamma_x x_{k\beta}\eta_x + \alpha_x(x_{k\beta}\eta'_k + x'_{k\beta}\eta_k)) && (14)\\
\Delta a_x^2 &= -2(\Delta p/p)((\gamma_x\eta_x + \alpha_x\eta'_k)a_x\sqrt{\beta_x}\cos\theta - a_x(\beta_x\eta'_x + \alpha_x\eta_k)(\frac{\alpha_x\cos\theta + \sin\theta}{\sqrt{\beta_x}})\\
\Delta a_x^2 &= -2(\Delta p/p)((\gamma_x\eta_x + \alpha_x\eta'_k - \alpha_x\eta' - \frac{\alpha_x^2\eta_k}{\beta})a_x\sqrt{\beta_x}\cos\theta - a_x(\beta_x\eta'_x + \alpha_x\eta_k)(\frac{\sin\theta}{\sqrt{\beta_x}})\\
\Delta a_x^2 &= -2(\Delta p/p)a_x(\frac{\eta}{\sqrt{\beta_x}}\cos\theta - (\beta_x\eta'_x + \alpha_x\eta_k)\frac{\sin\theta}{\sqrt{\beta_x}})\\
&= -2(\Delta p/p)E_x\sin(\theta_{xk} + \theta_{xc}) && (15)
\end{aligned}$$

where

$$\begin{aligned}
E_x &= a_x(\frac{\eta^2}{\beta} + \frac{\beta^2(\eta')^2 + \alpha^2\eta^2 + 2\alpha\beta\eta'\eta}{\beta})^{1/2}\\
&= a_x(\eta^2\gamma + \beta{\eta'}^2 + 2\alpha\eta'\eta)^{1/2} && (16)\\
\theta_{xc} &= -\tan^{-1}\frac{\eta}{\beta_x\eta'_x + \alpha\eta_x} && (17)
\end{aligned}$$

θ_{xk} is the horizontal betatron phase at the kicker. The corresponding change in the longitudinal amplitude

$$\begin{aligned}
\Delta a_z^2 &= 2(\Delta p/p)(\beta_z z'_k + \alpha_z z) && (18)\\
&= 2(\Delta p/p)a_z(-\sqrt{\beta_z}(\alpha_z\cos\theta_z + \sin\theta_z) + \alpha_z\sqrt{\beta}\cos\theta_z)\\
&= -2(\Delta p/p)a_z\sqrt{\beta_z}\sin\theta_z\\
&= -2(\Delta p/p)E_z\sin(\theta_{zk}) && (19)
\end{aligned}$$

where θ_{zk} is the longitudinal betatron phase at the kicker. Combining equations 1 and 14 we find

$$\begin{aligned}
\Delta a_x^2 &= -2(\xi\sin(k\Delta s))\left((\beta_x x'_{k\beta}\eta'_x + \gamma_x x_{k\beta}\eta_x + \alpha_x(x_{k\beta}\eta'_k + x'_{k\beta}\eta_k)) + (\beta_z z'_k + \alpha_z z)\right) && (20)\\
&= -2\xi\sin(k\Delta s))(E_x\sin(\theta_{xk} + \theta_{xc})) && (21)\\
&= -2\xi\sin(k(A_x\sin(\theta_{xp} + \theta_{xt}) + A_z\sin(\theta_{zp} + \theta_{zt}))(E_x\sin(\theta_{xk} + \theta_{xc})) && (22)
\end{aligned}$$

Now let's average over all betatron phases

$$\begin{aligned}
\int_0^{2\pi}\Delta a_x^2 d\theta_x d\theta_z &= -2\xi\int\sin(k(A_x\sin(\theta_{xp} + \theta_{xt}) + A_z\sin(\theta_{zp} + \theta_{zt}))(E_x\sin(\theta_{xk} + \theta_{xc}))d\theta_x d\theta_z && (23)\\
&= -2\xi\int\sin(k(A_x\sin(\theta_x) + A_z\sin(\theta_z + \theta_{zt}))(E_x\sin(\theta_x + \theta_0 + \theta_{xc} - \theta_{xt}))d\theta_x d\theta_z && (24)
\end{aligned}$$

where we use the fact that the betatron phase advance from pickup to kicker is θ_0, that is $\theta_{xk} = \theta_{xp} + \theta_0$ Typically $\theta_{zt} = \tan -1\alpha_p$ is small, (α_p is for longitudinal motion in the pickup undulator) and we assume that it is zero.

Then

$$\begin{aligned}
\langle \Delta a_x^2 \rangle &= -2\xi E_x \int [\sin(kA_x \sin(\theta_x)) \cos(kA_z \sin(\theta_z)) + \\
&\quad \cos(kA_x \sin(\theta_x)) \sin(kA_z \sin(\theta_z))] (\sin(\theta_x + \theta_0 + \theta_{xc} - \theta_{xt})) d\theta_x d\theta_z && (25) \\
&= -2\xi E_x J_0(kA_z) \int \sin(k(A_x \sin\theta_x) [\sin\theta_x \cos\phi + \cos\theta_x \sin\phi] \, d\theta_x && (26) \\
&= -2\xi E_x J_0(kA_z) J_1(kA_x) \cos(\theta_0 + \theta_{xc} - \theta_{xt}) && (27)
\end{aligned}$$

We used the Bessel integral

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(n\tau - x\sin(\tau))d\tau = \frac{1}{\pi}\int_0^\pi (\cos(n\tau)\cos(x\sin\tau) + \sin(n\tau)\sin(x\sin\tau))\,d\tau$$

Optimum cooling is realized when $\theta_0 + \theta_{xc} - \theta_{xt} = m\pi$. For example if the phase advance from pickup to kicker $\theta_0 = \pi$ and $\theta_{xc} = \theta_{xt}$ is small. Then

$$\langle \Delta a_x^2 \rangle = -2\xi E_x J_1(kA_x) J_0(kA_z) \quad (28)$$

There is cooling as long as $J_1(kA_x) > 0$ and $J_0(kA_z) > 0$, or if $kA_x < \mu_1$ where $\mu_1 = 3.8$ is the first zero of J_1 and $kA_z < \mu_0$ the first zero of J_0. Therefore

$$\begin{aligned}
kA_x &< \mu_1 \rightarrow a_x < \frac{\mu_1}{[M_{51}^2\beta_x + M_{52}^2\gamma_x - 2M_{51}M_{52}\alpha_x]^{1/2}} \\
kA_z &< \mu_0 \rightarrow a_z < \frac{\mu_0}{(M_{51}\eta + M_{52}\eta' + M_6)\gamma_z{}^{1/2}}
\end{aligned}$$

thus determining the maximum transverse and longitudinal betatron amplitudes that can be cooled. Or we can write that

$$\left[M_{51}^2\beta_x + M_{52}^2\gamma_x - 2M_{51}M_{52}\alpha_x\right]^{1/2} < \frac{\mu_1}{ka_x^{max}} \quad (29)$$

For small x, $J_1(x) \sim \frac{x}{2}$ and $J_0(x) \sim 1$. In that limit Equation 28 becomes

$$\begin{aligned}
\Delta a_x^2 &\sim -2\xi a_x(\eta^2\gamma + \beta\eta'^2 + 2\alpha\beta\eta'\eta)^{1/2}\frac{1}{2}\left(\mu_1 \frac{a_x}{a_x^{max}}\right) \\
\rightarrow \frac{\Delta a_x^2}{a_x^2} &\sim -\xi(\eta^2\gamma + \beta\eta'^2 + 2\alpha\beta\eta'\eta)^{1/2}\frac{\mu_1}{a_{max}}
\end{aligned}$$

Some numbers: $a_x^2 \sim \epsilon_{max} \sim 1$nm, and $(\eta^2\gamma + \beta\eta'^2 + 2\alpha\beta\eta'\eta)^{1/2} \sim 1$, and $|\frac{\Delta a_x^2}{a_x^2}| < 1$ then

$$\xi = \frac{3 \times 10^{-5}}{7.6} \sim 10^{-5}$$

Recall

$$\frac{\Delta p}{p} = \xi \sin(k\Delta s).$$

The most effective damping requires that the power in the kicker undulator be sufficient to change the fractional electron energy by 1 part in 10^5 or 3 keV for a 300 MeV electron beam. Constraints on the design of the optics of the bypass and lattice are:

1. Minimize equilibrium emittance. $a_x^{max} = \sqrt{\epsilon_x^{max}}$ is the maximum transverse amplitude that will be cooled. The equilibrium emittance from radiation damping is necessarily less than ϵ_x^{max} if most of the particles are to be cooled. Ideally $\epsilon_x^{max} > n\epsilon_x$ where n 2. (Equation 29)

2. Maximize $\left[M_{51}^2\beta_x + M_{52}^2\gamma_x - 2M_{51}M_{52}\alpha_x\right]^{1/2}$ where M_{5i} are the elements of the transfer matrix from pickup to kicker and η, η' are dispersion in the pickup consistent with requirement 1 and Equation 29.
3. Maximize $(\eta^2\gamma + \beta{\eta'}^2 + 2\alpha\eta'\eta)^{1/2}$ (Equation 16)
4. Maximize $|\cos(\theta_0 + \theta_{xc} - \theta_{xt})|$ (see Equations 11 and 17. θ_0 is the horizontal phase advance from pickup to kicker.

3 Longitudinal motion

Evidently longitudinal cooling requires $J_0(kA_z) > 0$ and therefore $kA_z < \mu_0$ where μ_0 is the first zero of J_0. Then

$$ka_z < \frac{\mu_0}{(M_{51}\eta + M_{52}\eta' + M_6)\gamma_z} \tag{30}$$

Combine Equations 1, 10-12 and 30 to determine the change in longitudinal amplitude in the kicker.

$$\Delta a_z^2 = -2(\xi \sin(k(A_x \sin(\theta_{xp} + \theta_{xt}) + A_z \sin(\theta_{zp} + \theta_{zt}))(E_z \sin\theta_{zk}) \tag{31}$$

As for transverse motion

$$\begin{aligned} \langle \Delta a_z^2 \rangle &= \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi} -2(\xi \sin(k(A_x \sin(\theta_{xp} + \theta_{xt}) + A_z \sin(\theta_{zp} + \theta_{zt}))(E_z \sin\theta_{zk}) d\theta_{xp} d\theta_{zp} &(32)\\ &= -2\xi E_z J_0(kA_x)\sqrt{2}\sin(\theta_{xt} + \pi/4) J_1(kA_z)\cos(\theta_{z0} - \theta_{zt}) &(33)\end{aligned}$$

4 Summary

If a_x, a_z are the invariant horizontal and longitudinal betatron amplitudes, for $\alpha, \beta, \gamma, \eta, \eta'$ in the pickup and M_{5i} transport from pickup to kicker then

$$\begin{aligned} A_x &= a_x\left[M_{51}^2\beta + M_{52}^2\gamma - 2M_{51}M_{52}\alpha\right]^{1/2}, \quad \theta_{x0} = \tan^{-1}\frac{M_{51}\beta - M_{52}\alpha}{M_{52}} \\ A_z &= a_z(M_{51}\eta + M_{52}\eta' + M_{56})\gamma_z, \quad \theta_{z0} = \tan^{-1}\alpha_z \sim 0 \end{aligned}$$

then

$$\Delta s = A_x \sin(\theta_x + \theta_{x0}) + A_z \sin(\theta_z) \tag{34}$$

The change in the square of the invariant amplitude due to the change in energy in the kicker

$$\Delta a_x^2 = -2(\Delta p/p)E_x \sin(\theta_{xk} + \theta_{xc})$$

where for $\eta, \gamma, \eta', \alpha, \beta$ in the kicker

$$E_x = a_x(\eta^2\gamma + \beta{\eta'}^2 + 2\alpha\eta'\eta)^{1/2}, \quad \theta_{xc} = -\tan^{-1}\frac{\eta}{\beta\eta' + \alpha\eta}$$

Then averaging over betatron phase

$$\langle(\Delta a_x^2)\rangle = -2\xi E_x J_1(kA_x) J_0(kA_z)\cos\theta_{z0pk}$$

For a particular choice of twiss parameters and phase advance $\cos\theta_{x0pk} \sim 1$. As above we write $\left[M_{51}^2\beta_x + M_{52}^2\gamma_x - 2M_{51}M_{52}\alpha_x\right]^{1/2} \sim \frac{\mu_1}{ka_x^{max}}$ so that

$$\frac{\Delta\epsilon}{\epsilon} \sim -\xi(\eta^2\gamma + \beta{\eta'}^2 + 2\alpha\beta\eta'\eta)^{1/2}\frac{\mu_1}{\sqrt{\epsilon_{max}}} \tag{35}$$

5 Power

Recalled that $\Delta p/p = \xi \sin(k\Delta x)$. $\Delta p/p$ is the fractional energy change on passage of the electrons through the kicker undulator. Evidently the amplitude of the energy shift is ξ. Solve 35 for

$$\xi = \frac{\Delta\epsilon_x}{\epsilon_x}\frac{\sqrt{\epsilon_{max}}}{\mu_1\mathcal{M}}$$

where $\epsilon_x = a^2$ where $\mathcal{M} = (\eta^2\gamma + \beta\eta'^2 + 2\alpha\beta\eta'\eta)^{1/2}$. If we aim to correct the offset measured in the pickup in a single pass through the kicker then

$$\xi = \frac{\sqrt{\epsilon_{max}}}{\mu_1\mathcal{M}} \tag{36}$$

If $\mathcal{M} \sim 1$, and $\epsilon_{max} \sim 1$ nm, then the required fractional energy change $\xi \sim 10^{-5}$. For $E_{beam} = 300$MeV, and the number of electrons in a slice $N_s = 10^5$ then $\Delta E = \xi E_{beam} N_s \sim 300MeV = 4.8 \times 10^{-11}$ J. The total power for the 0.1mA bunch is $P = I\xi E_{beam} = 0.3$ W
How to think about this. Suppose the accelerating fields are contained in a pulse of radiation that co-propagates with the electrons. From above we conclude that the peak accelerating field is $\hat{E} = 3$ keV. The energy density is $u = \frac{1}{2}\hat{E}^2 = \frac{1}{2}\epsilon_0\hat{E}^2 \sim \frac{1}{2}8.8 \times 10^{-12} \times 9 \times 10^6 = 4 \times 10^{-5}$ Joules/m^3. If the volume is 1 cm X 1 mm^2 then the total energy is $U = 4 \times 10^{-13}$ Joules.

6 Limits

In that limit where $k\Delta s \ll \pi/2$, and with substitution of equation 2 into 20 we have

$$\Delta\epsilon_x = -2(\xi k(M_{51}x_p + M_{52}x'_p + M_{56}z'_p)(\beta_x x'_{k\beta}\eta'_x + \gamma_x x_{k\beta}\eta_x + \alpha_x(x_{k\beta}\eta'_k + x'_{k\beta}\eta_k)) \tag{37}$$

We compute the average change in the emittance $\langle\Delta\epsilon_x\rangle$ where the average is over betatron phase. Substituting Equations 4-7 into 37 and averaging over betatron phase (see Appendix for details)

$$\begin{aligned}\langle\Delta\epsilon_x\rangle &= -2\pi\xi k\frac{a^2}{2}(M_{51}\left(-\sqrt{\beta_p\beta_k}\sin\phi\eta'_k + \sqrt{\frac{\beta_p}{\beta_k}}\eta_k(\cos\phi - \alpha_k\sin\phi)\right)\\ &\quad + M_{52}\left(\sqrt{\frac{\beta_k}{\beta_p}}\eta'_k(\cos\phi + \alpha_p\sin\phi) + \sqrt{\frac{1}{\beta_k\beta_p}}\eta_k(\sin\phi(1+\alpha_k\alpha_p) + \cos\phi(\alpha_k - \alpha_p))\right) \quad (38)\\ &= -\pi\xi k a^2\mathcal{M} \quad (39)\end{aligned}$$

Consider a couple of special cases. If the phase advance ϕ from pickup to kicker is $\phi = \pi$ then

$$\langle\Delta\epsilon_x\rangle = -2\pi\xi k\frac{a^2}{2}(M_{51}\left(-\sqrt{\frac{\beta_p}{\beta_k}}\eta_k\right) + M_{52}\left(-\sqrt{\frac{\beta_k}{\beta_p}}\eta'_k - \sqrt{\frac{1}{\beta_k\beta_p}}\eta_k\cos\phi(\alpha_k - \alpha_p)\right)$$

and if the optics are symmetric so that $\beta_k = \beta_p, \alpha_k = -\alpha_p$, $\eta_k = \eta_p$,$\eta'_k = -\eta'_p$ then

$$\langle\Delta\epsilon_x\rangle = 2\pi\xi k\frac{a^2}{2}(M_{51}\eta + M_{52}\left(\eta'_k + \frac{\eta}{\beta}\cos\phi(2\alpha_k)\right)$$

7 Sample Lengthening

As noted above, cooling requires that the change in path length be less than the optical wavelength, $\Delta s < \lambda$. Substitution of Equations 4 and 5 into the expression for the change in path length 3

The average change in path length is of course $\langle \Delta s \rangle = 0$. The mean square change in path length is

$$\langle (\Delta s)^2 \rangle = \frac{\pi}{2} \left(a^2 (M_{51}^2 \beta_p + M_{52}^2 \gamma - 2M_{51}M_{52}\alpha) + a_z^2 (M_{51}\eta + M_{52}\eta' + M_{56})^2 \gamma_z) \right) \tag{40}$$

a^2 and a_z^2 are the horizontal and longitudinal emittances respectively. Particles with amplitudes within one standard deviation of the emittance will be cooled if $\sqrt{\langle (\Delta s)^2 \rangle} < \lambda$.

8 Damping

The matrix that maps from kicker to pickup is M_{kp} and from pickup to kicker M_{pk}. At the kicker

$$\Delta \vec{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \Delta p/p \end{pmatrix} = M_e M_l \vec{x}_p = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \xi k & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ M_{51} & M_{52} & 0 & M_{56} \\ 0 & 0 & 0 & 0 \end{pmatrix} \vec{x}_p$$

where $\vec{x}_p$ is the phase space vector in the pickup. Then the effect of a single turn is

$$\vec{x}_{k,n+1} = M_{pk} M_{kp} \vec{x}_n + \Delta \vec{x} = (M_e M_l + M_{pk}) M_{kp} \vec{x}_{k,n} = T \vec{x}_{k,n} \tag{41}$$

The full turn matrix at the kicker is

$$T = \Delta M + M$$

where

$$\begin{aligned} \Delta M &= M_e M_l M_{kp} \\ M &= M_{pk} M_{kp} \end{aligned}$$

Compute the eigenvectors ($\vec{v}_i$) and eigenvalues of M. We know how to do this since we have standard methods for diagonalizing a symplectic matrix. (The eigenvalues are $\lambda_x^{\pm} = e^{\pm i\mu_x}$ and $\lambda_z^{\pm} = e^{\pm i\mu_z}$) where μ_x and μ_z are the horizontal and longitudinal tunes.) Then in the limit where ΔM is small, (it clearly scales with $\xi k M_{5j}^{pk}$) the shift in the eigenvalues (tunes) is given by

$$\Delta \lambda_i \sim \vec{v}_i^T (\Delta M) \vec{v}_i$$

An imaginary component will correspond to damping.

8.1 Pickup to Kicker matrix

Next to work out the matrix M_{pk} that maps pickup to kicker. We can write

$$M_{pk} = \begin{pmatrix} A_{pk} & B_{pk} \\ C_{pk} & D_{pk} \end{pmatrix}$$

And

$$C = \begin{pmatrix} M_{51} & M_{52} \\ 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & M_{56} \\ 0 & 1 \end{pmatrix}$$

The symplectic condition requires that

$$\begin{aligned} ASA^T + BSB^T &= S \\ ASC^T + BSD^T &= 0 \\ CSA^T + DSB^T &= 0 \\ CSC^T + DSD^T &= S \end{aligned}$$

from which we can conclude that

$$B = ASC^T(D^T)^{-1}S$$

For simplicity we suppose $\alpha_p = \alpha_k = 0$. Then

$$\begin{aligned}
A_{pk} &= \begin{pmatrix} \cos\mu_x^{pk} & \beta_x \sin\mu_x^{pk} \\ -\frac{\sin\mu_x^{pk}}{\beta_x} & \cos\mu_x^{pk} \end{pmatrix} \\
D_{pk} &= \begin{pmatrix} 1 & M_{56} \\ 0 & 1 \end{pmatrix} \\
B_{pk} &= \begin{pmatrix} \cos\mu_x^{pk} & \beta_x \sin\mu_x^{pk} \\ -\frac{\sin\mu_x^{pk}}{\beta_x} & \cos\mu_x^{pk} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} M_{51} & 0 \\ M_{52} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -M_{56} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \cos\mu_x^{pk} & \beta_x \sin\mu_x^{pk} \\ -\frac{\sin\mu_x^{pk}}{\beta_x} & \cos\mu_x^{pk} \end{pmatrix} \begin{pmatrix} M_{52} & 0 \\ -M_{51} & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & -M_{56} \end{pmatrix} = \begin{pmatrix} \cos\mu_x^{pk} & \beta_x \sin\mu_x^{pk} \\ -\frac{\sin\mu_x^{pk}}{\beta_x} & \cos\mu_x^{pk} \end{pmatrix} \begin{pmatrix} 0 & M_{52} \\ 0 & -M_{51} \end{pmatrix} \qquad (42)
\end{aligned}$$

where μ_x^{pk} is the phase advance from pickup to kicker. We assume $\beta_p = \beta_k$. If we also suppose that the phase advance from pickup to kicker is 180 degrees, then

$$B_{pk} = \begin{pmatrix} 0 & -M_{52} \\ 0 & M_{51} \end{pmatrix}$$

And

$$M_{pk} = \begin{pmatrix} -1 & 0 & 0 & -M_{52} \\ 0 & -1 & 0 & M_{51} \\ M_{51} & M_{52} & 1 & M_{56} \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (43)$$

8.2 Full turn at pickup

Construct the full turn at the pickup

$$T_p = \begin{pmatrix} A & \begin{pmatrix} 0 & (I-A)\vec{\eta} \end{pmatrix} \\ C & D \end{pmatrix}$$

$$\begin{aligned}
C &= DSB^T(A^T)^{-1}S \\
&= \begin{pmatrix} 1 & T_{56} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \eta - A_{1i}\eta_i & \eta' - A_{2i}\eta_i \end{pmatrix} (A^T)^{-1}S \\
&= \begin{pmatrix} 1 & T_{56} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \eta - A_{1i}\eta_i & \eta' - A_{2i}\eta_i \end{pmatrix} \begin{pmatrix} \cos\mu_x & \frac{\sin\mu_x}{\beta_x} \\ -\beta_x \sin\mu_x & \cos\mu_x \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & T_{56} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \eta - A_{1i}\eta_i & \eta' - A_{2i}\eta_i \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{\sin\mu_x}{\beta_x} & \cos\mu_x \\ -\cos\mu_x & -\beta_x \sin\mu_x \end{pmatrix} \\
&= \begin{pmatrix} \eta - A_{1i}\eta_i & \eta' - A_{2i}\eta_i \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{\sin\mu_x}{\beta_x} & \cos\mu_x \\ -\cos\mu_x & -\beta_x \sin\mu_x \end{pmatrix} \\
&= \begin{pmatrix} (\eta - A_{1i}\eta_i)A_{21} - (\eta' - A_{2i}\eta_i)A_{11} & (\eta - A_{1i}\eta_i)A_{11} - (\eta' - A_{2i}\eta_i)A_{12} \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \eta(A_{21} - A_{11}A_{21} + A_{21}A_{11}) - \eta'(A_{11} + A_{12}A_{21} - A_{22}A_{11}) & \eta(A_{11} - A_{11}^2 + A_{21}A_{12}) - \eta'(A_{12} + A_{12}A_{11} - A_{22}A_{12}) \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} -\frac{\eta}{\beta_x}\sin\mu_x + \eta'(1-\cos\mu_x) & \eta(\cos\mu_x - 1) - \eta'\beta_x \sin\mu_x \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

Finally, (assuming no RF) the full turn matrix at the pickup is

$$T_P = \begin{pmatrix} \cos\mu_x & \beta_x \sin\mu_x & 0 & \eta(1-\cos\mu_x) - \eta'\beta_x \sin\mu_x \\ -\frac{-\sin\mu_x}{\beta_x} & \cos\mu_x & 0 & \frac{\eta}{\beta_x}\sin\mu_x + \eta'(1-\cos\mu_x) \\ -\frac{\eta}{\beta_x}\sin\mu_x + \eta'(1-\cos\mu_x) & \eta(\cos\mu_x - 1) - \eta'\beta_x\sin\mu_x & 1 & T_{56} \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{44}$$

8.3 Full turn at kicker

$$T_k = M_{pk} T_p M_{pk}^{-1} \tag{45}$$

$$\begin{aligned} M_{pk}^{-1} &= -\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & M_{51} & 0 \\ 0 & -1 & M_{52} & 0 \\ 0 & 0 & 1 & 0 \\ -M_{52} & M_{51} & M_{56} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \\ &= -\begin{pmatrix} 0 & -1 & M_{52} & 0 \\ 1 & 0 & -M_{51} & 0 \\ -M_{52} & M_{51} & M_{56} & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \\ &= -\begin{pmatrix} 1 & 0 & 0 & M_{52} \\ 0 & 1 & 0 & -M_{51} \\ -M_{51} & -M_{52} & -1 & M_{56} \\ 0 & 0 & 0 & -1 \end{pmatrix} = M_{pk} \end{aligned}$$

$$\begin{aligned} T_k &= \begin{pmatrix} -1 & 0 & 0 & -M_{52} \\ 0 & -1 & 0 & M_{51} \\ M_{51} & M_{52} & 1 & -M_{56} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\mu_x & \beta_x \sin\mu_x & 0 & \eta(1-\cos\mu_x) - \eta'\beta_x \sin\mu_x \\ -\frac{\sin\mu_x}{\beta_x} & \cos\mu_x & 0 & \frac{\eta}{\beta_x}\sin\mu_x + \eta'(1-\cos\mu_x) \\ -\frac{\eta}{\beta_x}\sin\mu_x + \eta'(1-\cos\mu_x) & \eta(\cos\mu_x - 1) - \eta'\beta_x\sin\mu_x & 1 & T_{56} \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &\quad \times \begin{pmatrix} -1 & 0 & 0 & -M_{52} \\ 0 & -1 & 0 & M_{51} \\ M_{51} & M_{52} & 1 & -M_{56} \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} \\ &= \begin{pmatrix} m_1 A + m_2 C & m_1 B + m_2 D \\ m_3 A + m_4 C & m_3 B + m_4 D \end{pmatrix} \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} \\ &= \begin{pmatrix} m_1 A m_1 + m_2 C m_1 + m_1 B m_3 + m_2 D m_3 & m_1 A m_2 + m_2 C m_2 + m_1 B m_4 + m_2 D m_4 \\ m_3 A m_1 + m_4 C m_1 + m_3 B m_3 + m_4 D m_3 & m_3 A m_2 + m_4 C m_2 + m_3 B m_4 + m_4 D m_4 \end{pmatrix} \end{aligned}$$

Write the submatrices

$$m_1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad m_2 = \begin{pmatrix} 0 & -M_{52} \\ 0 & M_{51} \end{pmatrix}, \quad m_3 = \begin{pmatrix} M_{51} & M_{52} \\ 0 & 0 \end{pmatrix}, \quad m_4 = \begin{pmatrix} 1 & -M_{56} \\ 0 & 1 \end{pmatrix} \tag{46}$$

The 2X2 components are

$$
\begin{aligned}
T_{11} &= A - 0 + 0 + \begin{pmatrix} 0 & -M_{52} \\ 0 & M_{51} \end{pmatrix} \begin{pmatrix} M_{51} & M_{52} \\ 0 & 0 \end{pmatrix} = A && (47) \\
T_{12} &= -\begin{pmatrix} 0 & -M_{52}\cos\mu_x + M_{51}\beta_x \sin\mu_x \\ 0 & M_{52}\frac{\sin\mu_x}{\beta_x} + M_{51}\cos\mu_x \end{pmatrix} - \begin{pmatrix} 0 & \eta(1-\cos\mu_x) - \eta'\beta_x \sin\mu_x \\ 0 & \frac{\eta}{\beta_x}\sin\mu_x + \eta'(1-\cos\mu_x) \end{pmatrix} + \begin{pmatrix} 0 & -M_{52} \\ 0 & M_{51} \end{pmatrix} \\
&= \begin{pmatrix} 0 & -M_{52}(1-\cos\mu_x) - M_{51}\beta_x\sin\mu_x \\ 0 & -M_{52}\frac{\sin\mu_x}{\beta_x} + M_{51}(1-\cos\mu_x) \end{pmatrix} - \begin{pmatrix} 0 & \eta(1-\cos\mu_x) - \eta'\beta_x\sin\mu_x \\ 0 & \frac{\eta}{\beta_x}\sin\mu_x + \eta'(1-\cos\mu_x) \end{pmatrix} && (48) \\
T_{21} &= -\begin{pmatrix} M_{51}\cos\mu_x - M_{52}\gamma\sin\mu_x & M_{51}\beta_x\sin\mu_x + M_{52}\cos\mu_x \\ 0 & 0 \end{pmatrix} \\
&\quad - \begin{pmatrix} -\frac{\eta}{\beta_x}\sin\mu_x + \eta'(1-\cos\mu_x) & \eta(\cos\mu_x - 1) - \eta'\beta_x\sin\mu_x \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} M_{51} & M_{52} \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} M_{51}(1-\cos\mu_x) + M_{52}\gamma\sin\mu_x & -M_{51}\beta_x\sin\mu_x + M_{52}(1-\cos\mu_x) \\ 0 & 0 \end{pmatrix} \\
&\quad - \begin{pmatrix} -\frac{\eta}{\beta_x}\sin\mu_x + \eta'(1-\cos\mu_x) & \eta(\cos\mu_x - 1) - \eta'\beta_x\sin\mu_x \\ 0 & 0 \end{pmatrix} && (49)
\end{aligned}
$$

If $M_{51} = 2\eta', M_{52} = -2\eta$. Then

$$T_k = T_p$$

But mirror symmetry requires not that $T_k = T_p$ but that $T_k = \tilde{T}_p$, that is where $\eta'_k = -\eta'_p,\ \alpha_k = -\alpha_p$ and $\eta_k = \eta_p, \beta_k = \beta_p$. $\eta'_k = -\eta'_p$ implies $M_{51} = 0$.

9 Bypass constraints revisited

Now with the assumption that $T_k = \tilde{T}_p$, and symmetry and $\alpha = 0$ then we know how the dispersion in kicker and pickup is related to M_{51} and M_{52} from pickup to kicker. Allowing us to write that If $T_k = \tilde{T}_p$ and $\eta'_k = \eta'_p = -\frac{1}{2}M_{52}$ then

$$A_x = a_x\left[M_{51}^2\beta + M_{51}^2\gamma - 2M_{51}M_{52}\alpha\right]^{1/2} \rightarrow a_x[\frac{\eta^2}{\beta}]^{1/2}$$

Also

$$
\begin{aligned}
A_z &= a_z M_{56}{\gamma_z}^{1/2} \\
E_x &= a_x(\frac{\eta^2}{\beta} + \beta{\eta'}^2)^{1/2}
\end{aligned}
$$

Recall Equation 29 where we established that

$$[M_{51}^2\beta + M_{52}^2\gamma - 2M_{51}M_{52}\alpha]^{1/2} < \frac{\mu_1}{ka_x^{max}} \rightarrow \frac{\eta}{\sqrt{\beta}} < \frac{\mu_1}{ka_x^{max}} \tag{50}$$

$\mathcal{M}$ in Equation 36 becomes $\mathcal{M} = (\frac{\eta^2}{\beta} + \beta{\eta'}^2)^{1/2}$

$$\xi = \frac{\sqrt{\epsilon_{max}}}{\mu_1(\frac{\eta^2}{\beta} + \beta{\eta'}^2)^{1/2}} \tag{51}$$

ξ is the fractional momentum change of the slice. In order to minize ξ, we want to maximize $\frac{\beta\eta'}{\eta}$. In M. Ehrlichman's symmetric bypass, November 29, 2017, $M_{52} = -0.051$, $M_{51} = 0.0069$, $\beta = 10$. Suppose $\epsilon_{max} \sim 1$nm then

$$
\begin{aligned}
\xi &\sim \frac{\sqrt{10^{-9}}}{3.2\left(\frac{M_{52}^2}{4\beta}+M_{51}^2\beta/4)^2\right)^{1/2}} \\
&\sim \frac{\sqrt{10^{-9}}}{3.2(0.01264)} \\
&\sim 7.8\times 10^{-4}
\end{aligned}
$$

For $E_{beam} = 300$ MeV, and $I = 0.1$mA. The total power is $P = I\xi E_{beam} = 23.4$ W

10 Eigenvalues and eigenvectors

This section is incomplete The coupling matrix

$$
\begin{aligned}
m+n^\dagger &= \begin{pmatrix}0 & \eta \\ 0 & \eta'\end{pmatrix} + -SA^T\begin{pmatrix}0 & -\eta' \\ 0 & \eta\end{pmatrix} \\
&= \begin{pmatrix}0 & \eta \\ 0 & \eta'\end{pmatrix} + SA^TS\begin{pmatrix}0 & \eta \\ 0 & \eta'\end{pmatrix} = (I+A^{-1})\begin{pmatrix}0 & \eta \\ 0 & \eta'\end{pmatrix} \\
C &= \frac{m+n^\dagger}{tr(A-D)+|m+n^\dagger|}
\end{aligned}
$$

The eigenvectors of the rotation matrix are $\vec{v} = \begin{pmatrix}1 \\ \pm i\end{pmatrix}$ with eigenvalues $e^{\pm i\mu}$. It appears that

$$
U = V^{-1}MV \to R(\mu_x,\mu_z) = G^{-1}V^{-1}MVG
$$

Then the eigenvalues of M are

$$
\begin{aligned}
\vec{m}_i &= VG\vec{v}_i \to \Delta\lambda_i = \vec{v}_i^T G^T V^T \Delta MVG\vec{v}_i \\
&= \vec{v}_i^T G^T \begin{pmatrix}\gamma & -(C^\dagger)^T \\ C & \gamma\end{pmatrix}\begin{pmatrix}0 & 0 \\ M_l & M_r\end{pmatrix}\begin{pmatrix}\gamma & C \\ -C^\dagger & \gamma\end{pmatrix} G\vec{v}_i \\
&= \vec{v}_i^T G^T \begin{pmatrix}\gamma & -(C^\dagger)^T \\ C & \gamma\end{pmatrix}\begin{pmatrix}0 & 0 \\ M_l\gamma - M_rC^\dagger & M_lC+\gamma M_r\end{pmatrix} G\vec{v}_i \\
&= \vec{v}_i^T G^T \begin{pmatrix}\gamma & -(C^\dagger)^T \\ C & \gamma\end{pmatrix}\begin{pmatrix}0 & 0 \\ M_l\gamma - M_rC^\dagger & M_lC+\gamma M_r\end{pmatrix} G\vec{v}_i
\end{aligned}
$$

The eignvectors of the full turn matrix are

$$
\vec{v} =
$$

11 Appendix I

11.1 Generalized kicker parameters

At the kicker $\Delta x_{k\beta} = -\eta_k\Delta p/p$ and $\Delta x'_{k\beta} = -\eta'_k\Delta p/p$.The action

$$
a^2 = \beta {x'}^2 + \gamma x^2 + 2\alpha x x'
$$

$$
2a\Delta a = -2\Delta p/p(\beta x'_{k\beta}\eta'_k + \gamma x_{k\beta}\eta_k + \alpha(x_{k\beta}\eta'_k + x'_{k\beta}\eta_k)) \tag{52}
$$

Now if the phase advance from pickup to kicker is 180 degrees, then $x_{k\beta} = -x_{p\beta}$ and $x'_{k\beta} = -x'_{p\beta}$ and

$$2a\Delta a = 2\Delta p/p(\beta x'_{p\beta}\eta'_k + \gamma x_{p\beta}\eta_k + \alpha(x_{p\beta}\eta'_k + x'_{p\beta}\eta_k))$$
$$= 2\Delta p/p\left(\eta'_k(\beta x'_{p\beta} + \alpha x_{p\beta}) + \eta_k(\gamma x_{p\beta} + \alpha x'_{p\beta})\right)$$
$$= 2(\Delta p/p)a\left(\eta'_k(-\sqrt{\beta}\sin\theta) + \eta_k(\frac{\cos\theta - \alpha\sin\theta}{\sqrt{\beta}})\right)$$

11.2 Cooling

Since $\Delta p/p = \xi\sin(k\Delta s)$ we have that

$$2a\Delta a = 2a\xi\sin(k\Delta s)\left(\eta'_k(-\sqrt{\beta_k}\sin\theta) + \eta_k(\frac{\cos\theta - \alpha_k\sin\theta}{\sqrt{\beta_k}})\right)$$

$$2a\Delta a = 2a\xi\sin\left[ka\left(M_{51}\sqrt{\beta_p}\cos\theta - M_{52}\frac{(\alpha_p\cos\theta + \sin\theta)}{\sqrt{\beta_p}}\right)\right]\left(\eta'_k(-\sqrt{\beta_k}\sin\theta) + \eta_k(\frac{\cos\theta - \alpha_k\sin\theta}{\sqrt{\beta_k}})\right)$$

In the limit where $k\Delta s \ll \pi/2$, we can write that

$$\Delta a = \xi\left[ka\left(M_{51}\sqrt{\beta_p}\cos\theta - M_{52}\frac{(\alpha_p\cos\theta + \sin\theta)}{\sqrt{\beta_p}}\right)\right]\left(\eta'_k(-\sqrt{\beta_k}\sin\theta) + \eta_k(\frac{\cos\theta - \alpha_k\sin\theta}{\sqrt{\beta_k}})\right)$$

$$\langle\Delta a\rangle = -\frac{a}{2}\xi k\left(M_{51}\eta_k\sqrt{\frac{\beta_p}{\beta_k}} + M_{52}\left(\frac{\eta_k(\alpha_p - \alpha_k)}{\sqrt{\beta_p\beta_k}} + \eta'_k\sqrt{\frac{\beta_k}{\beta_p}}\right)\right)$$

If $\alpha_k = -\alpha_p$ and $\beta_k = \beta_p$

$$\langle\Delta a\rangle = -\frac{a}{2}\xi k\left(M_{51}\eta_k + M_{52}\left(2\frac{\eta_k(\alpha_k)}{\beta_p} + \eta'_k\right)\right)$$

12 Longitudinal excitation

While the momentum shift $\Delta p/p$ is designed to damp the transverse motion, it is apparently adding noise to the longitudinal. As long as sychrotron and betatron tunes are not related the average momentum shift will be zero. Not a problem? If M_{56} is finite then

$$\Delta s = (M_{51}\eta + M_{52}\eta' + M_{56})\delta$$
$$\Delta p/p = \xi\sin(k(M_{51}\eta + M_{52}\eta' + M_{56})\delta)$$

and there will be longitudinal cooling if the sign of ξ is chosen appropriately. But this in turn will add uncorrelated noise into the transverse.

13 Appendix II

Suppose the betatron phase advance from pickup to kicker is θ_0 so that

$$x_{k\beta} = a\sqrt{\beta_k}\cos(\phi + \theta_0)$$
$$x'_{k\beta} = -\frac{a}{\sqrt{\beta_k}}(\alpha_k\cos(\phi + \theta_0) + \sin(\phi + \theta_0))$$

Since

$$
\begin{aligned}
x_{p\beta} &= a\sqrt{\beta_p}\cos(\phi) \\
x'_{p\beta} &= -\frac{a}{\sqrt{\beta_p}}\left(\alpha_p\cos(\phi)+\sin(\phi)\right)
\end{aligned}
$$

we can write

$$
\begin{aligned}
a\cos\phi &= \frac{x_{p\beta}}{\sqrt{\beta_p}} \\
a\sin\phi &= -\sqrt{\beta_{p\beta}}x'_{p\beta} - \frac{\alpha_p}{\sqrt{\beta_p}}x_{p\beta}
\end{aligned}
$$

Then

$$
\begin{aligned}
x_{k\beta} &= \sqrt{\beta_k}\left(\frac{x_{p\beta}}{\sqrt{\beta_p}}\cos\theta_0 + (\sqrt{\beta_{p\beta}}x'_{p\beta} + \frac{\alpha_p}{\sqrt{\beta_p}}x_{p\beta})\sin\theta_0\right) \\
x'_{k\beta} &= -\frac{1}{\sqrt{\beta_k}}\left(\alpha_k(\frac{x_{p\beta}}{\sqrt{\beta_p}}\cos\theta_0 + (\sqrt{\beta_{p\beta}}x'_{p\beta} + \frac{\alpha_p}{\sqrt{\beta_p}}x_{p\beta})\sin\theta_0) + \frac{x_{p\beta}}{\sqrt{\beta_p}}\sin\theta_0 - (\sqrt{\beta_{p\beta}}x'_{p\beta} + \frac{\alpha_p}{\sqrt{\beta_p}}x_{p\beta})\cos\theta_0\right)
\end{aligned}
$$

Let's write $2a\Delta a$ in terms of $x_{p\beta}, x'_{p\beta}$.

13.1 Averaging over betatron phase

$$
2a\Delta a = -2\xi k(M_{51}x_{p\beta} + M_{52}x'_{p\beta})(\beta x'_{k\beta}\eta'_k + \gamma x_{k\beta}\eta_k + \alpha(x_{k\beta}\eta'_k + x'_{k\beta}\eta_k)) \tag{53}
$$

Then we have terms like

$$
\begin{aligned}
\langle x_p x_k\rangle &= \langle\sqrt{\beta_k}\left(\frac{x^2_{p\beta}}{\sqrt{\beta_p}}\cos\theta_0 + (\sqrt{\beta_{p\beta}}x_p x'_{p\beta} + \frac{\alpha_p}{\sqrt{\beta_p}}x^2_{p\beta})\sin\theta_0\right)\rangle \\
\langle x_p x_k\rangle &= \frac{a^2}{2}\sqrt{\beta_k}\left(\frac{\beta_p}{\sqrt{\beta_p}}\cos\theta_0 + (-\sqrt{\beta_{p\beta}}\alpha_p + \frac{\alpha_p}{\sqrt{\beta_p}}\beta_p)\sin\theta_0\right) \\
\langle x_p x_k\rangle &= \frac{a^2}{2}\sqrt{\beta_k\beta_p}\,(\cos\theta_0)
\end{aligned}
$$

Next

$$
\begin{aligned}
\langle x_p x'_{k\beta}\rangle &= \langle -\frac{a^2}{\sqrt{\beta_k}}\Bigg(\alpha_k(\frac{x_p x_{p\beta}}{\sqrt{\beta_p}}\cos\theta_0 + (\sqrt{\beta_{p\beta}}x_p x'_{p\beta} + \\
&\quad \frac{\alpha_p}{\sqrt{\beta_p}}x_p x_{p\beta})\sin\theta_0) + \frac{x_p x_{p\beta}}{\sqrt{\beta_p}}\sin\theta_0 - (\sqrt{\beta_{p\beta}}x_p x'_{p\beta} + \frac{\alpha_p}{\sqrt{\beta_p}}x_p x_{p\beta})\cos\theta_0\Bigg)\rangle \\
\langle x_p x'_{k\beta}\rangle &= -\frac{1}{2}\frac{a^2}{\sqrt{\beta_k}}\Bigg(\alpha_k(\frac{\beta_p}{\sqrt{\beta_p}}\cos\theta_0 + (\sqrt{\beta_{p\beta}}(-\alpha_p) + \\
&\quad \frac{\alpha_p}{\sqrt{\beta_p}}\beta_p)\sin\theta_0) + \frac{\beta_p}{\sqrt{\beta_p}}\sin\theta_0 - (-\sqrt{\beta_{p\beta}}\alpha_p + \frac{\alpha_p}{\sqrt{\beta_p}}\beta_p)\cos\theta_0\Bigg) \\
\langle x_p x'_{k\beta}\rangle &= -\frac{a^2}{2}\frac{\sqrt{\beta_p}}{\sqrt{\beta_k}}(\alpha_k\cos\theta_0 + \sin\theta_0)
\end{aligned}
$$

Another term

$$
\begin{aligned}
\langle x'_p x_{k\beta}\rangle &= \langle x'_p \sqrt{\beta_k}\left(\frac{x_{p\beta}}{\sqrt{\beta_p}}\cos\theta_0 + (\sqrt{\beta_{p\beta}}x'_{p\beta} + \frac{\alpha_p}{\sqrt{\beta_p}}x_{p\beta})\sin\theta_0\right) \\
\langle x'_p x_{k\beta}\rangle &= \frac{a^2}{2}\sqrt{\beta_k}\left(\frac{-\alpha_p}{\sqrt{\beta_p}}\cos\theta_0 + (\sqrt{\beta_{p\beta}}\gamma_{p\beta} - \frac{\alpha_p^2}{\sqrt{\beta_p}})\sin\theta_0\right) \\
\langle x'_p x_{k\beta}\rangle &= \frac{a^2}{2}\sqrt{\frac{\beta_k}{\beta_p}}\left(\sin\theta_0 - \alpha_p\cos\theta_0\right)
\end{aligned}
$$

Finally

$$
\begin{aligned}
\langle x'_p x'_{k\beta}\rangle &= -x'_p\frac{1}{\sqrt{\beta_k}}\left(\alpha_k(\frac{x_{p\beta}}{\sqrt{\beta_p}}\cos\theta_0 + (\sqrt{\beta_{p\beta}}x'_{p\beta} + \frac{\alpha_p}{\sqrt{\beta_p}}x_{p\beta})\sin\theta_0) + \frac{x_{p\beta}}{\sqrt{\beta_p}}\sin\theta_0 - (\sqrt{\beta_{p\beta}}x'_{p\beta} + \frac{\alpha_p}{\sqrt{\beta_p}}x_{p\beta})\cos\theta_0\right) \\
&= -\frac{a^2}{2\sqrt{\beta_k}}\left(-\frac{\alpha_k\alpha_p}{\sqrt{\beta_p}}\cos\theta_0 + \alpha_k(\sqrt{\beta_p}\gamma_p - \frac{\alpha_p^2}{\sqrt{\beta_p}})\sin\theta_0 - \frac{\alpha_p}{\sqrt{\beta_p}}\sin\theta_0 - (\sqrt{\beta_p}\gamma_p - \frac{\alpha_p^2}{\sqrt{\beta_p}})\cos\theta_0\right) \\
&= -\frac{a^2}{2\sqrt{\beta_k}}\left(-\frac{\alpha_k\alpha_p}{\sqrt{\beta_p}}\cos\theta_0 + \frac{\alpha_k}{\sqrt{\beta_p}}\sin\theta_0 - \frac{\alpha_p}{\sqrt{\beta_p}}\sin\theta_0 - (\sqrt{\beta_p}\gamma_p - \frac{\alpha_p^2}{\sqrt{\beta_p}})\cos\theta_0\right) \\
&= -\frac{a^2}{2\sqrt{\beta_k}}\left(-\frac{\alpha_k\alpha_p}{\sqrt{\beta_p}}\cos\theta_0 - \frac{1}{\sqrt{\beta_p}}\cos\theta_0 + \frac{\alpha_k - \alpha_p}{\sqrt{\beta_p}}\sin\theta_0\right) \\
&= -\frac{a^2}{2\sqrt{\beta_k\beta_p}}\left((-1-\alpha_k\alpha_p)\cos\theta_0 + (\alpha_k - \alpha_p)\sin\theta_0\right)
\end{aligned}
$$

Now we can write Equation 53 Step 1

$$
\begin{aligned}
2a\Delta a &= -2\xi k(M_{51}x_{p\beta} + M_{52}x'_{p\beta})(\beta x'_{k\beta}\eta'_k + \gamma x_{k\beta}\eta_k + \alpha(x_{k\beta}\eta'_k + x'_{k\beta}\eta_k)) \\
&= -2\xi k(M_{51}\left(x_p\beta x'_k\eta'_k + \gamma_k\eta_k x_p x_k + \alpha_k(\eta'_k x_p x_k + \eta_k x_p x'_k)\right) + \\
&\quad M_{52}\left(\beta_k\eta'_k x'_p x'_k + \gamma_k\eta_k x'_p x_k + \alpha(\eta'_k x'_p x_k + \eta_k x'_p x'_k)\right)
\end{aligned}
$$

Step 2

$$
\begin{aligned}
&= -2\xi k(M_{51}(\eta'_k\left(\beta_k x_p x'_k + \alpha_k x_p x_k\right) + \eta_k\left(\gamma_k x_p x_k + \alpha_k x_p x'_k\right)) + \\
&\quad M_{52}(\eta'_k\left(\beta_k x'_p x'_k + \alpha x'_p x_k\right) + \eta_k\left(\gamma_k x'_p x_k + \alpha x'_p x'_k\right))
\end{aligned}
$$

Step 3

$$
\begin{aligned}
&= -2\xi k\frac{a^2}{2}(M_{51}\left(-\sqrt{\beta_p\beta_k}.(\alpha_k\cos\theta_0 + \sin\theta_0)\eta'_k + \gamma_k\eta_k\sqrt{\beta_k\beta_p}\cos\theta_0\right. \\
&\quad \left.+\alpha_k(\sqrt{\beta_k\beta_p}\eta'_k\cos\theta_0 - \sqrt{\frac{\beta_p}{\beta_k}}\alpha_k(\alpha_k\cos\theta_0 + \sin\theta_0)\eta_k\right) \\
&\quad +M_{52}\left((1+\alpha_k\alpha_p)\cos\theta_0 + (\alpha_p - \alpha_k)\sin\theta_0)\sqrt{\frac{\beta_k}{\beta_p}}\eta'_k + \sqrt{\frac{\beta_k}{\beta_p}}(\sin\theta_0 - \alpha_p\cos\theta_0)\gamma_k\eta_k + \right. \\
&\quad \left.\alpha_k(\sqrt{\frac{\beta_k}{\beta_p}}(\sin\theta_0 - \alpha_p\cos\theta_0)\eta'_k + \alpha_k\eta_k\frac{1}{\sqrt{\beta_k\beta_p}}((1+\alpha_k\alpha_p)\cos\theta_0 + (\alpha_p - \alpha_k)\sin\theta_0)\right)
\end{aligned}
$$

Step 4

$$= \quad -2\xi k\frac{a^2}{2}(M_{51}\left(-\sqrt{\beta_p\beta_k}\sin\theta_0\eta_k' + \sqrt{\frac{\beta_p}{\beta_k}}\eta_k\left(\cos\theta_0 - \alpha_k\sin\theta_0\right)\right)\right)$$

$$+M_{52}\left(\sqrt{\frac{\beta_k}{\beta_p}}\eta_k'(\cos\theta_0 + \alpha_p\sin\theta_0) + \frac{1}{\sqrt{\beta_k\beta_p}}\eta_k\left(\sin\theta_0(1+\alpha_k\alpha_p) + (\alpha_k - \alpha_p)\cos\theta_0\right)\right)$$

Step 5

$$= \quad -2\xi k\frac{a^2}{2}(M_{51}\left(-\sqrt{\beta_p\beta_k}\sin\theta_0\eta_k' + \sqrt{\frac{\beta_p}{\beta_k}}\eta_k(\cos\theta_0 - \alpha_k\sin\theta_0)\right)$$

$$+M_{52}\left(\sqrt{\frac{\beta_k}{\beta_p}}\eta_k'(\cos\theta_0 + \alpha_p\sin\theta_0) + \sqrt{\frac{1}{\beta_k\beta_p}}\eta_k(\sin\theta_0(1+\alpha_k\alpha_p) + \cos\theta_0(\alpha_k - \alpha_p))\right)$$

If we have symmetry

$$= \quad -2\xi k\frac{a^2}{2}(M_{51}\left(-\beta_p\sin\theta_0\eta_k' + \eta_k(\cos\theta_0 - \alpha_k\sin\theta_0) + M_{52}\left(\eta_k'(\cos\theta_0 + \alpha_p\sin\theta_0) + \frac{\eta_k}{\beta}(\sin\theta_0(1-\alpha^2) + 2\alpha_k\cos\theta_0\right)\right.$$

and if $\theta_0 = \pi$

$$= \quad 2\xi k\frac{a^2}{2}(M_{51}\eta_k + M_{52}\eta_k' + 2\frac{\eta_k}{\beta}\alpha_k)$$

And if $\theta_0 = \pi/2$

$$= \quad -2\xi k\frac{a^2}{2}(M_{51}\left(-\beta_p\eta_k' + \eta_k(-\alpha_k) + M_{52}\left(\eta_k'(\alpha_p) + \frac{\eta_k}{\beta}((1-\alpha^2)\right)\right.$$