Undulator Tracking Updates

- More careful derivation of quadrupole and sextupole terms

- Comparison with simulation

Magnetic Field (July 10)

- $B_x = B_0 [\cos(kz) + k^2/8 (r^2 \cos(kz) + 2x^2 \cos(kz)$
  $+ 2xy \sin(kz))]$

- $B_y = B_0 [\sin(kz) + k^2/8 (r^2 \sin(kz) + 2y^2 \sin(kz)$
  $+ 2xy \cos(kz))]$

- $B_z = B_0 [(k + k^3 r^2/8)*(y \cos(kz) - x \sin(kz))]$

0^{th} Order Solution (July 10)

- $x = -e B_0 / (\bar{v} \gamma m k^2) \sin(kz) + \Delta x'_0 z + \Delta x_0$
- $y = e B_0 / (\bar{v} \gamma m k^2) \cos(kz) + \Delta y'_0 z + \Delta y_0$
- ~~$z = \bar{v} z$~~
- $z = \bar{v} t$

Which Terms to Keep?

- Last month, only kept field terms linear in transverse offset – only appear in B_z field
- However, forces are generated from this B_z field by multiplying by transverse velocity – small relative to longitudinal velocity
- Terms in transverse fields quadratic in position are equally important
- Radius of e- orbit is closer in magnitude to the typical offsets of electron orbits than it is to the undulator period, so we are justified in keeping terms quadratic in phase-space coordinates at this order without fear that terms higher-ordered in rk will significantly alter our answers

Change in Transverse Angles

- $\Delta x' = \int_0^L -e/(\bar{v}\gamma m)\ (y' B_z - B_y)\ dz$

- $\Delta y' = \int_0^L -e/(\bar{v}\gamma m)\ (B_x - x' B_z)\ dz$

Linear and Quadratic Terms from Theory

$$\Delta x' = (eB_0/(\bar{v}\gamma m))^2 L \left[-1/2\,(\Delta x_0 + L/2\,\Delta x'_0) - 11/(8k)\,\Delta y'_0\right]$$
$$- eB_0L/(8m\bar{v}\gamma)\left[Lk(\Delta x'_0)^2 + 3kL(\Delta y'_0)^2 + 4\,\Delta x'_0\Delta y'_0\right.$$
$$\left. + 2k\,(\Delta x_0\Delta x'_0 + 3\,\Delta y_0\Delta y'_0)\right]$$

$$\Delta y' = (eB_0/(\bar{v}\gamma m))^2 L \left[-1/2\,(\Delta y_0 + L/2\,\Delta y'_0) + 5/(8k)\,\Delta x'_0\right]$$
$$+ eB_0L/(8m\bar{v}\gamma)\left[2(\Delta x'_0)^2 - 2(\Delta y'_0)^2 + 2Lk\,\Delta x'_0\Delta y'_0\right.$$
$$\left. + 2k\,(\Delta x_0\Delta y'_0 + \Delta x'_0\Delta y_0)\right]$$

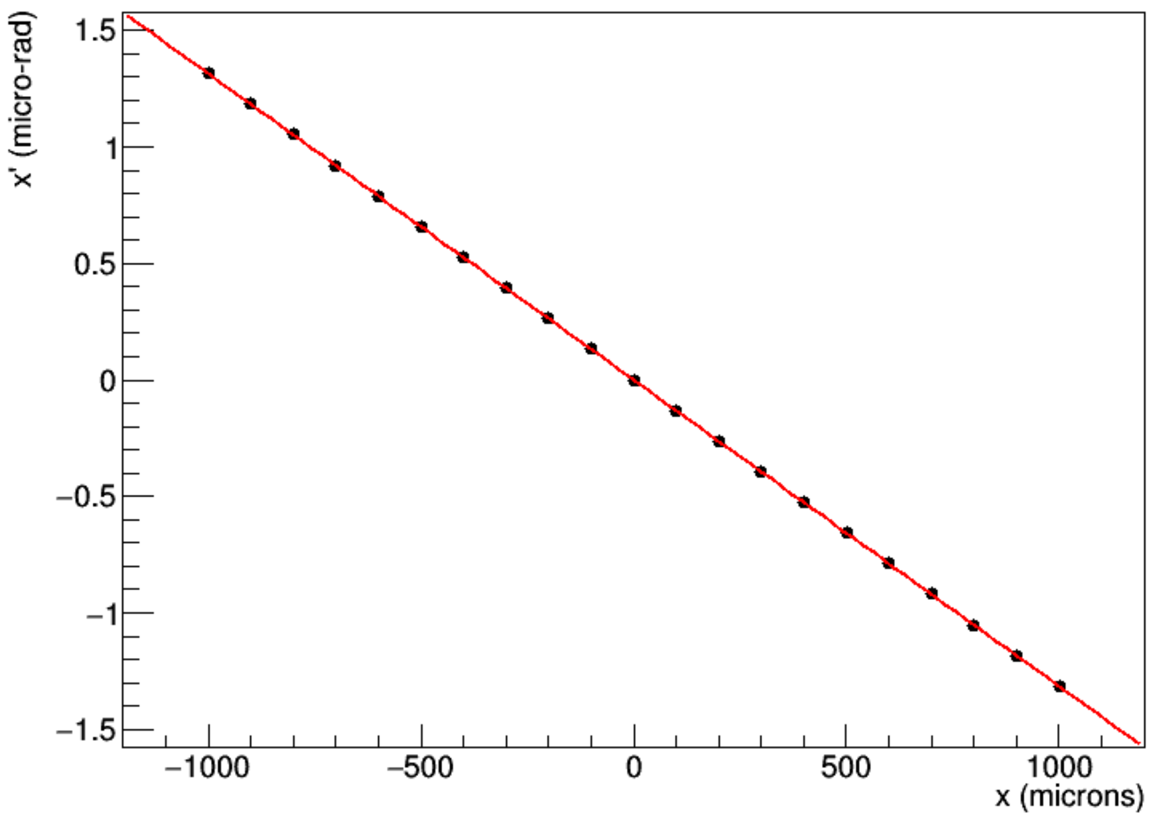
Points to Note

- x,y not symmetric – undulator has choice of which way the field points at the start
- Quadrupolar focusing dependent on position beam would have at undualtor center
- Initial y' causes shift in x', and vice versa
- Quadratic coupling terms important relative to linear ones when $L \Delta x'_0$ is comparable to r_0 – ie, ~100 μrad
- I do not consider S.H. Kim's higher-order fields (B_3, etc)

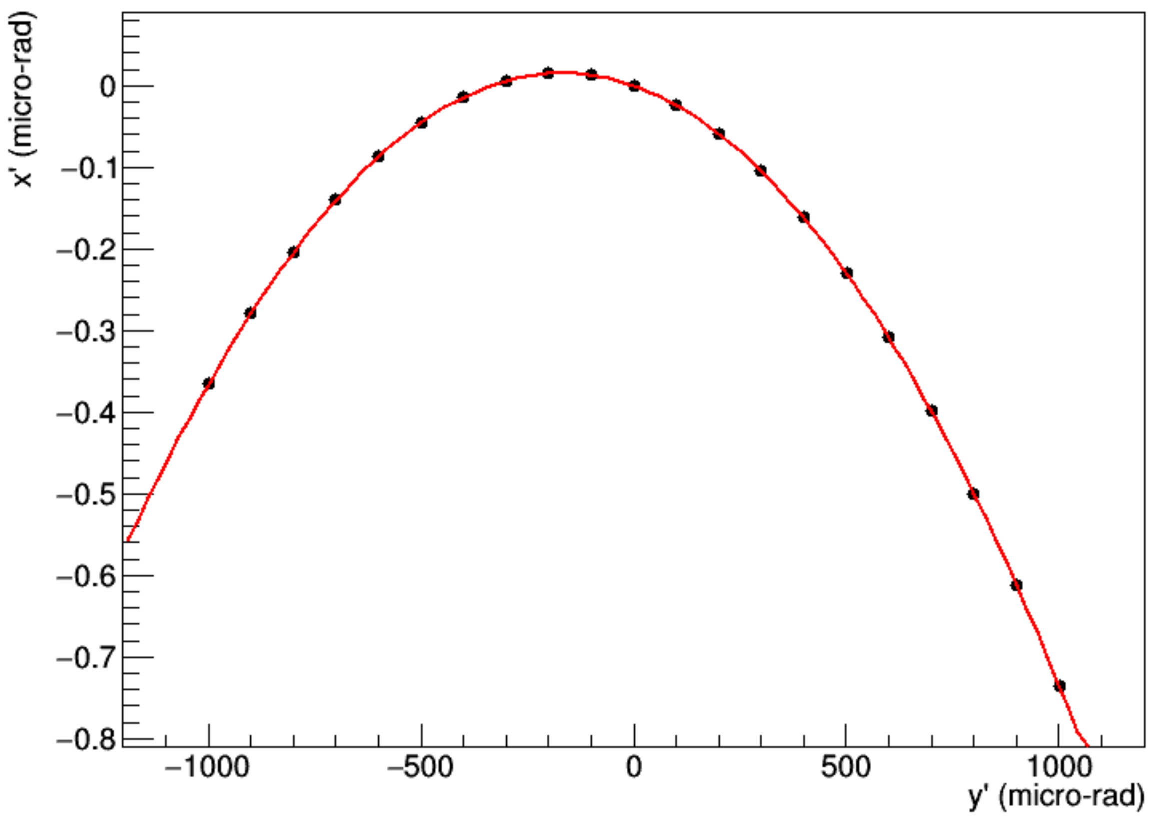
Simulations

- Run Vardan's code with different initial x, y, px, and py – look at shifts in final px, py – perform quadratic fits
- To avoid issues with interpolation, calculate field directly when needed by tracker
- Can keep S. H. Kim's higher fields, or just use B_1
- Notation – use px = x', py=y'

px(x) (B_1)



px(py) (B_1)


x' (micro-rad)
0
−0.1
−0.2
−0.3
−0.4
−0.5
−0.6
−0.7
−0.8
−1000
−500
0
500
1000
y' (micro-rad)

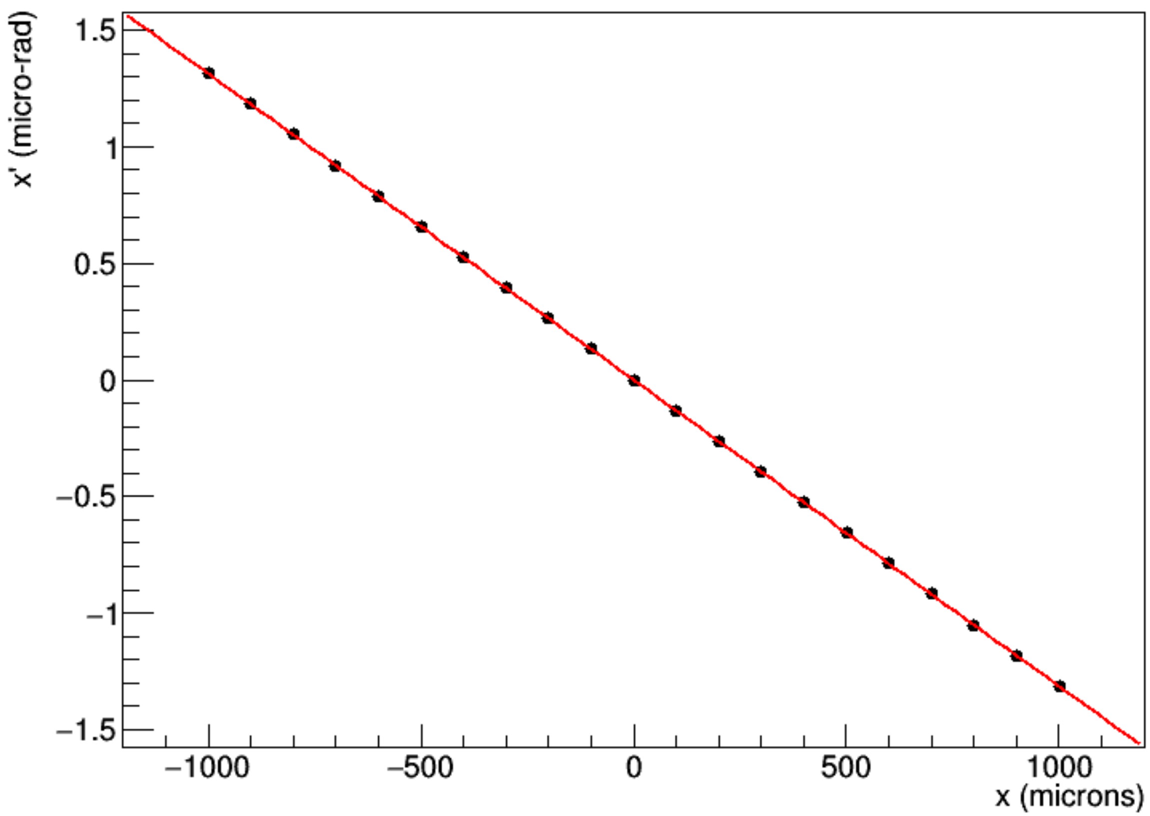
Ignore Higher-Order Fields

Units in μm, μrad	Theory	Simulation
px(x)	-0.00131444	-0.00131462
px(y)	0	1.6903e-06
px(px)	0.999146	0.999145
px(py)	-0.000186972	-0.000184747
px(x^2)	0	9.2884e-11
px(x*y)	0	2.21322e-10
px(x*px)	-2.82549e-07	-2.82157e-07
px(x*py)	0	1.21839e-10
px(y^2)	0	1.89002e-10
px(y*px)	0	9.1179e-11
px(y*py)	-8.47646e-07	-8.46815e-07
px(px^2)	-1.83657e-07	-1.83434e-07
px(px*py)	-2.92298e-08	-2.90827e-08
px(py^2)	-5.5097e-07	-5.50713e-07

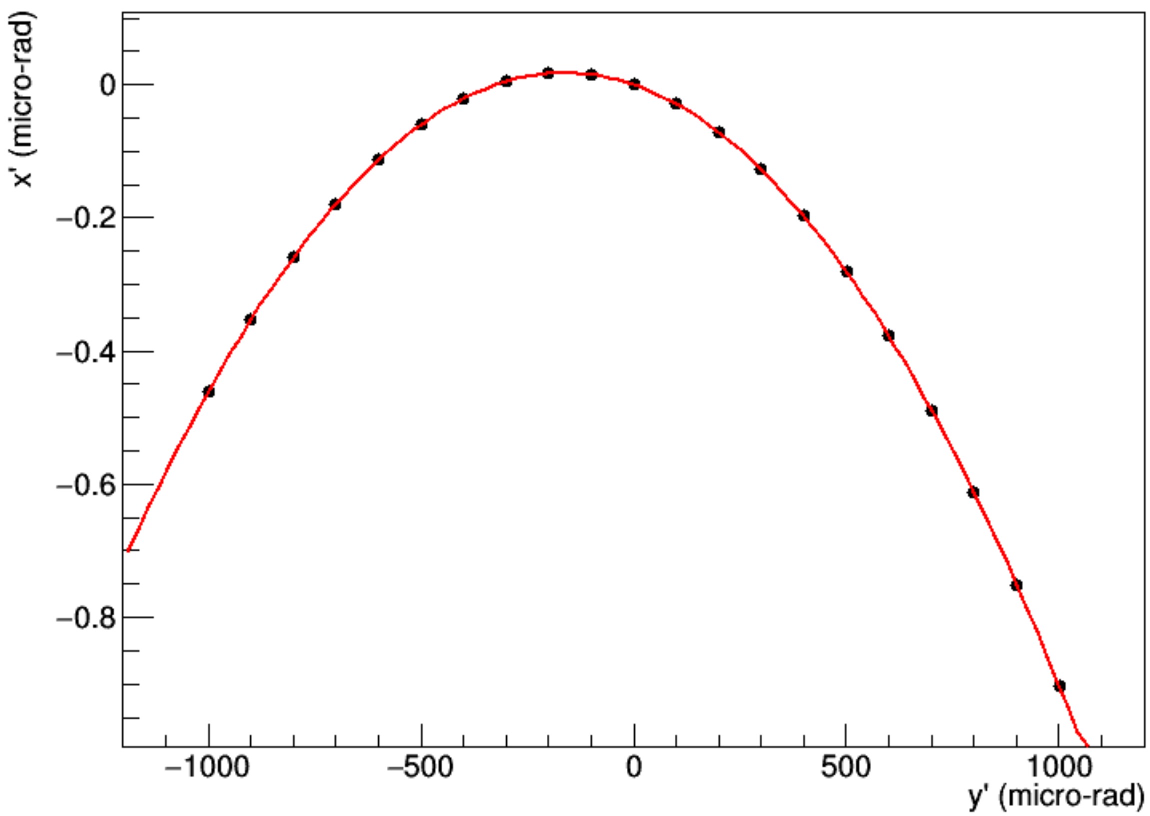
Ignore Higher-Order Fields

Units in μm, μrad	Theory	Simulation
py(x)	0	-2.32346e-06
py(y)	-0.00131444	-0.00131386
py(px)	8.49871e-05	8.31456e-05
py(py)	0.999146	0.999146
py(x^2)	0	-1.9466e-10
py(x*y)	0	-1.86284e-10
py(x*px)	0	-4.03241e-10
py(x*py)	2.82549e-07	2.82258e-07
py(y^2)	0	-4.37897e-11
py(y*px)	2.82549e-07	2.82272e-07
py(y*py)	0	-2.42267e-10
py(px^2)	1.46149e-08	1.4358e-08
py(px*py)	3.67313e-07	3.67156e-07
py(py^2)	-1.46149e-08	-1.46834e-08

px(x) (7th Order)



px(py) (7^{th} Order)



Use Higher-Order Fields

Units in µm, µrad	Theory (1st Order)	Simulation
px(x)	-0.00131444	-0.00131429
px(y)	0	1.92226e-06
px(px)	0.999146	0.999146
px(py)	-0.000186972	-0.000220748
px(x^2)	0	2.47091e-11
px(x*y)	0	6.03192e-10
px(x*px)	-2.82549e-07	-8.0522e-08
px(x*py)	0	3.45142e-10
px(y^2)	0	2.3453e-10
px(y*px)	0	4.05954e-10
px(y*py)	-8.47646e-07	-1.04865e-06
px(px^2)	-1.83657e-07	-5.2271e-08
px(px*py)	-2.92298e-08	-2.16221e-08
px(py^2)	-5.5097e-07	-6.81879e-07

Use Higher-Order Fields

Units in μm, μrad	Theory (1st Order)	Simulation
py(x)	0	-2.08519e-06
py(y)	-0.00131444	-0.00131426
py(px)	8.49871e-05	4.71624e-05
py(py)	0.999146	0.999146
py(x^2)	0	-8.07269e-11
py(x*y)	0	-5.28086e-11
py(x*px)	0	-1.37789e-10
py(x*py)	2.82549e-07	8.05578e-08
py(y^2)	0	-1.87595e-10
py(y*px)	2.82549e-07	8.06309e-08
py(y*py)	0	-3.18941e-10
py(px^2)	1.46149e-08	1.80055e-08
py(px*py)	3.67313e-07	1.04854e-07
py(py^2)	-1.46149e-08	-1.83833e-08

Conclusions

- If simulate using just the B_1 field, our theory does a very good job matching the simulations

- If we include higher orders, we still capture the transverse focusing, but the cross-terms are significantly altered – initial attempts to include these terms were unsuccessful

Backup Slides

- Useful integrals
- Lots of algebra

Mathematical Identities
(L is Integer Number of Periods)

$\int_0^L z \, dz = L^2/2$

$\int_0^L z \cos(kz) \, dz = 0$

$\int_0^L z \sin(kz) \, dz = -L/k$

$\int_0^L z \sin^2(kz) \, dz = L^2/4$

$\int_0^L z \cos^2(kz) \, dz = L^2/4$

$\int_0^L z \sin(kz) \cos(kz) \, dz = -L/(4k)$

$\int_0^L z^2 \cos(kz) \, dz = 2L/k^2$

$\int_0^L z^2 \sin(kz) \, dz = -L^2/k$

Integral of y' B_z

$\int_0^L [-eB_0/(\gamma\bar{v}mk) \sin(kz) + \Delta y'_0]$ *

$[eB_0^2/(\bar{v}\gamma mk) - B_0 k(\Delta x'_0 z + \Delta x_0)\sin(kz)$

$+ B_0 k(\Delta y'_0 z + \Delta y_0)\cos(kz)]\, dz$

$= eB_0^2/(\gamma m\bar{v})\, [L^2/4\, \Delta x'_0 + L/2\, \Delta x_0 + L/(4k)\, \Delta y'_0]$

$+ eB_0^2 L/(\bar{v}\gamma mk)\, \Delta y'_0 + B_0 L\, \Delta x'_0 \Delta y'_0$

$= eB_0^2 L/(\bar{v}\gamma mk)\, [kL/4\, \Delta x'_0 + k/2\, \Delta x_0 + 5/4\, \Delta y'_0]$

$+ B_0 L\, \Delta x'_0 \Delta y'_0$

Integral of B_y

$\int_0^L B_0 \sin(kz) + B_0 k^2/8\,[$

$(-eB_0/(\bar{v}\gamma m k^2)\sin(kz) + \Delta x'_0 z + \Delta x_0)^2 \sin(kz)$

$+ 3(eB_0/(\bar{v}\gamma m k^2)\cos(kz) + \Delta y'_0 z + \Delta y_0)^2 \sin(kz)$

$+ 2(-eB_0/(\bar{v}\gamma m k^2)\sin(kz) + \Delta x'_0 z + \Delta x_0)$

$(eB_0/(\bar{v}\,\gamma\, m\, k^2)\cos(kz) + \Delta y'_0 z + \Delta y_0)\cos(kz))]\,dz$

Define $r_0 = eB_0/(\bar{v}\gamma m k^2)$

Integral of B_y (cont.)

$= B_0 k^2/8\ [-L^2/k\ (\Delta x'_0)^2 - r_0 L^2/2\ \Delta x'_0 - 2L/k\ \Delta x_0 \Delta x'_0$

$- r_0 L\ \Delta x_0 - 3L^2/k\ (\Delta y'_0)^2 - 3 r_0 L/(2k)\ \Delta y'_0$

$- 6L/k\ \Delta y_0 \Delta y'_0 + r_0 L/(2k)\ \Delta y'_0 + r_0 L^2/2\ \Delta x'_0$

$+ 4L/k^2\ \Delta x'_0 \Delta y'_0 + r_0 L\ \Delta x_0]$

$= B_0 k^2/8\ [- r_0 L/k\ \Delta y'_0 - L^2/k\ (\Delta x'_0)^2 - 2L/k\ \Delta x_0 \Delta x'_0$

$- 3L^2/k\ (\Delta y'_0)^2 - 6L/k\ \Delta y_0 \Delta y'_0 + 4L/k^2\ \Delta x'_0 \Delta y'_0]$

$\Delta x'$

$\Delta x' = e/(\bar{v}\gamma m)\ \{B_0 k^2/8\ [-\ r_0 L/k\ \Delta y'_0 - L^2/k\ (\Delta x'_0)^2$

$-\ 2L/k\ \Delta x_0 \Delta x'_0 - 3L^2/k\ (\Delta y'_0)^2 - 6L/k\ \Delta y_0 \Delta y'_0 + 4L/k^2\ \Delta x'_0 \Delta y'_0]$

$-\ r_0 B_0 kL\ [kL/4\ \Delta x'_0 + k/2\ \Delta x_0 + 5/4\ \Delta y'_0] - B_0 L\ \Delta x'_0 \Delta y'_0\}$

$= e/(\bar{v}\gamma m)\ \{-r_0 B_0 k^2 L/2\ (\Delta x_0 + L/2\ \Delta x'_0) - 11/8\ r_0 B_0 kL\ \Delta y'_0$

$-\ B_0 k^2/8\ [L^2/k\ (\Delta x'_0)^2 + 2L/k\ \Delta x_0 \Delta x'_0 + 3L^2/k\ (\Delta y'_0)^2$

$+\ 6L/k\ \Delta y_0 \Delta y'_0 + 4L/k^2\ \Delta x'_0 \Delta y'_0]\}$

Integral of $x' B_z$

$$\int_0^L [-eB_0/(\gamma\bar{v}mk) \cos(kz) + \Delta x'_0] *$$

$$[eB_0^2/(\bar{v}\gamma mk) - B_0 k(\Delta x'_0 z + \Delta x_0)\sin(kz)$$

$$+ B_0 k(\Delta y'_0 z + \Delta y_0)\cos(kz)]\, dz$$

$$= eB_0^2/(\gamma m\bar{v})\ [-L/(4k)\ \Delta x'_0 - L^2/4\ \Delta y'_0 - L/2\ \Delta y_0]$$

$$+ eB_0^2 L/(\bar{v}\gamma mk)\ \Delta x'_0 + B_0 L\ (\Delta x'_0)^2$$

$$= eB_0^2 L/(\bar{v}\gamma mk)\ [3/4\ \Delta x'_0 - kL/4\ \Delta y'_0 - k/2\ \Delta y_0]$$

$$+ B_0 L\ (\Delta x'_0)^2$$

Integral of B_x

$$\int_0^L B_0 \cos(kz) + B_0 k^2/8 \, [$$

$$3(-r_0 \sin(kz) + \Delta x'_0 z + \Delta x_0)^2 \cos(kz)$$

$$+ (r_0 \cos(kz) + \Delta y'_0 z + \Delta y_0)^2 \cos(kz)$$

$$+ 2(-r_0 \sin(kz) + \Delta x'_0 z + \Delta x_0)$$

$$(r_0 \cos(kz) + \Delta y'_0 z + \Delta y_0) \sin(kz))] \, dz$$

Integral of B_x (cont.)

$= B_0 k^2/8\ [6L/k^2(\Delta x'_0)^2 + 3Lr_0/(2k)\ \Delta x'_0$

$+ 2L/k^2(\Delta y'_0)^2 + L^2 r_0/2\ \Delta y'_0 + Lr_0\ \Delta y_0$

$- L^2 r_0/2\ \Delta y'_0 - Lr_0\ \Delta y_0 - Lr_0/(2k)\ \Delta x'_0$

$- 2L^2/k\ \Delta x'_0 \Delta y'_0 - 2L/k\ (\Delta x_0 \Delta y'_0 + \Delta x'_0 \Delta y_0)]$

$= B_0 k^2/8\ [Lr_0/k\ \Delta x'_0 + 2L/k^2(\Delta y'_0)^2 + 6L/k^2(\Delta x'_0)^2$

$- 2L^2/k\ \Delta x'_0 \Delta y'_0 - 2L/k\ (\Delta x_0 \Delta y'_0 + \Delta x'_0 \Delta y_0)]$

$\Delta y'$

$$\Delta y' = e/(\bar{v}\gamma m) \{r_0 B_0 kL [3/4\ \Delta x'_0 - kL/4\ \Delta y'_0 - k/2\ \Delta y_0]$$

$$+ B_0 L (\Delta x'_0)^2 - B_0 k^2/8 [L r_0/k\ \Delta x'_0 + 2L/k^2 (\Delta y'_0)^2$$

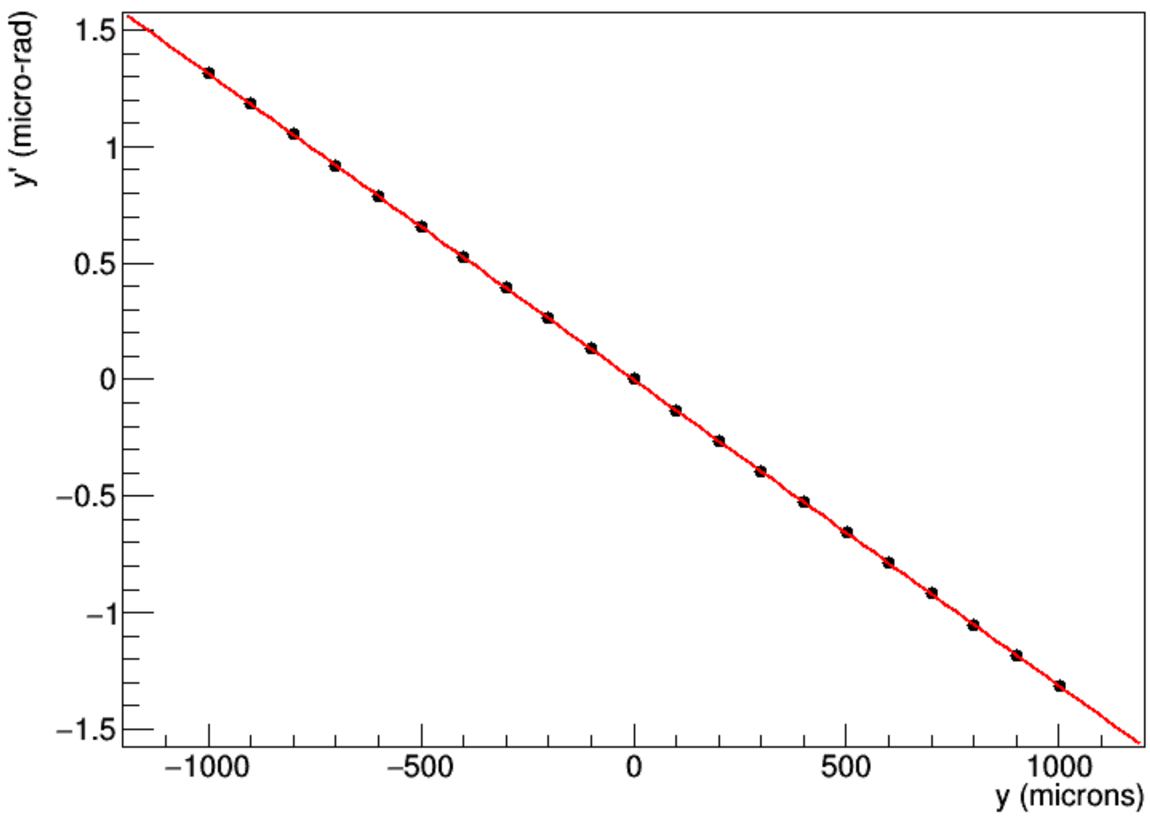
$$+ 6L/k^2 (\Delta x'_0)^2 - 2L^2/k\ \Delta x'_0 \Delta y'_0 - L/(2k) (\Delta x_0 \Delta y'_0 + \Delta x'_0 \Delta y_0)]\}$$

$$= e/(\bar{v}\gamma m) \{-r_0 B_0 k^2 L/2 (\Delta y_0 + L/2\ \Delta y'_0) + 5/8\ r_0 B_0 kL\ \Delta x'_0$$

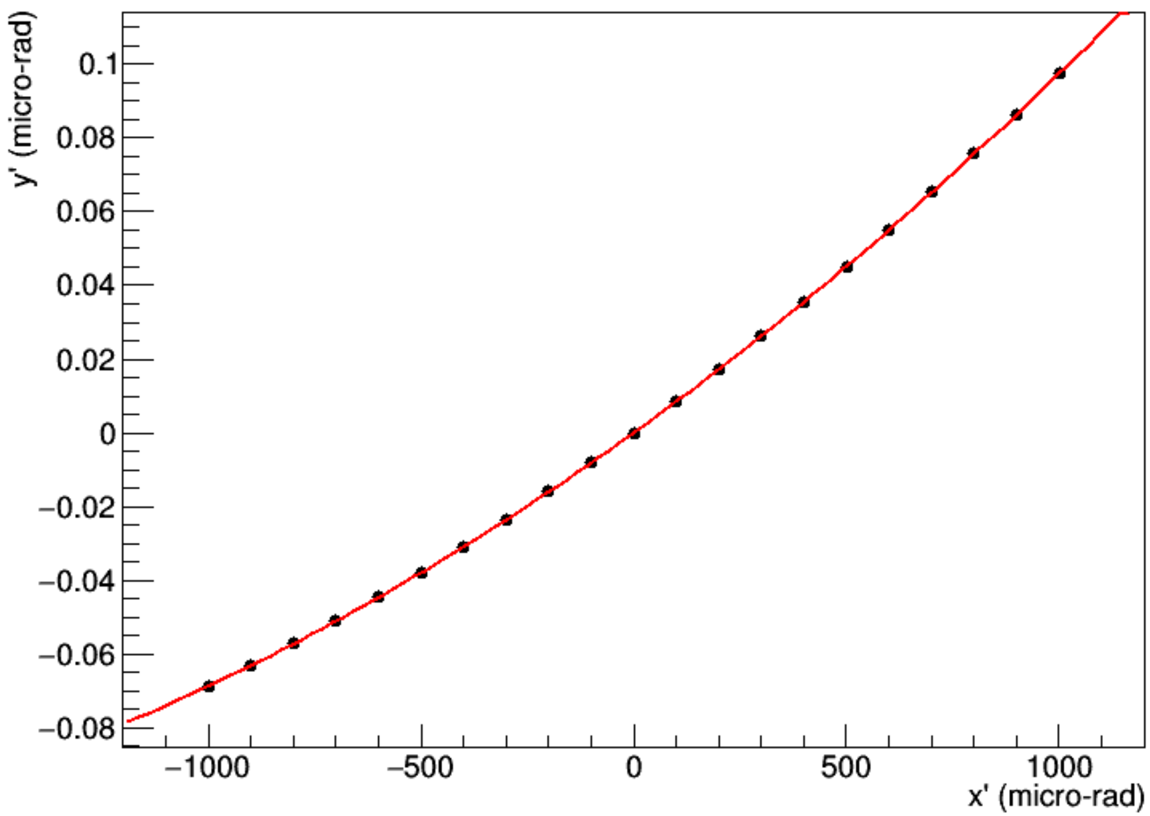
$$+ B_0 k^2/8 [2L/k^2 (\Delta x'_0)^2 - 2L/k^2 (\Delta y'_0)^2 + 2L^2/k\ \Delta x'_0 \Delta y'_0$$

$$+ 2L/k (\Delta x_0 \Delta y'_0 + \Delta x'_0 \Delta y_0)] \}$$

py(y) (B_1)

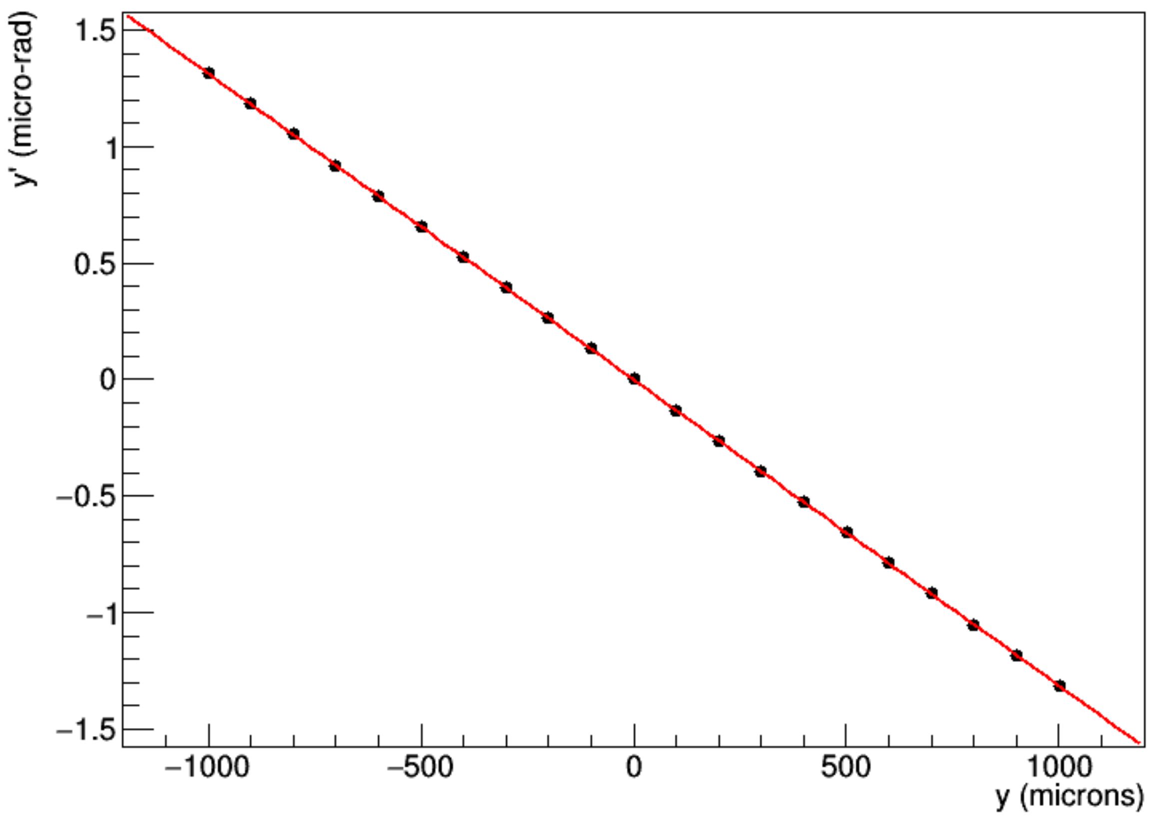


py(px) (B_1)



y' (micro-rad)
0.1
0.08
0.06
0.04
0.02
0
−0.02
−0.04
−0.06
−0.08
−1000
−500
0
500
1000
x' (micro-rad)

py(y) (7th Order)



py(px) (7^{th} Order)