EUROPEAN ORGANIZATION FOR NUCLEAR RESEARCH

CERN - SL DIVISION

SL-Note 95-125 (RF)

On the longitudinal echo in a continuous beam

E. Shaposhnikova

Abstract

An alternative method to calculate a longitudinal echo in a continuous beam is suggested. It is based on the analysis of the exact solution for the evolution of the longitudinal beam distribution and can be used for analytic estimations of the effect.

Geneva, Switzerland
December 1995

1 Introduction

The echo effect in plasma has been known for a long time [1]. This phenomenon should exist in accelerator beams as well, [2], both in the transverse and longitudinal planes of motion. The experimental observation of echo signals associated with two RF excitations of the continuous beam were made in Fermilab, [3], with the hope of measuring in the future diffusion rates or intensity effects in the machine. It was suggested that these experiments be started at the CERN SPS as well, [4],[5]. The detailed analysis done in [4], both by an analytical method and by numerical simulations, allows one to define the dependence of the longitudinal echo signal on different parameters of the system.

In this note another method to calculate the longitudinal echo effect is suggested which is based on the analysis of an exact solution describing the evolution of an arbitrary initial distribution after two successive RF excitations. The work was initiated by discussions of the proposal [5] with its authors and is an alternative approach to that given in [4].

2 Evolution of the distribution function

The longitudinal echo can be observed in the continuous beam at some well defined time and frequency after two consequent RF kicks occurred with some time delay one after another at different frequencies.

According to Liouville's theorem phase space density doesn't change along the particle trajectories. Then if the initial coordinates of the particles can be expressed as functions of the latest coordinates and time, we can describe the evolution of an arbitrary initial distribution function. Below we shall show that, with some assumptions, in the case of a longitudinal echo the evolution of the distribution function can be found in this way.

Let us consider the motion of the particles during the periods of time which are shown schematically in Fig.1. There are two different types of motion which can be described by two following systems of equations.

We assume that the time when the kick is applied is short enough that particles don't change significantly their azimuthal position. Then during the kick the equations of motion are

$$\dot{\theta} = 0, \tag{1}$$

$$\dot{p} = \frac{\omega_0 e V_n}{2\pi E_s} \sin(h_n \theta). \tag{2}$$

Here θ is the azimuthal position of the particle in the coordinate system of the beam, $p = (E - E_s)/E_s$ is the relative deviation in energy, ω_0 is the revolution frequency, V_n is amplitude and h_n is the harmonic number of the RF voltage during kick number n.

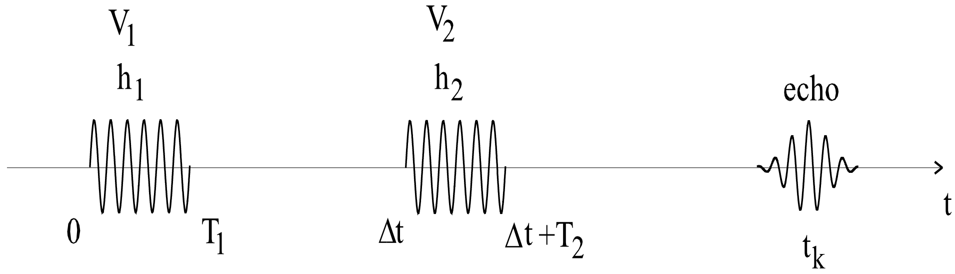

Figure 1: Time scale in the longitudinal echo effect.

To describe the motion after the kick we have

$$\dot{\theta} = k_0 p, \tag{3}$$
$$\dot{p} = 0, \tag{4}$$

where

$$k_0 = \frac{\omega_0 \eta}{\beta^2}, \qquad \eta = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}.$$

Now let us introduce parameter

$$u_n = \frac{eV_n}{E_s}\frac{\omega_0 T_n}{2\pi} = \frac{eV_n}{E_s} l_n \tag{5}$$

where T_n is the length of the n-th RF kick in time and l_n is the length in revolution periods.

Now for the particle with initial coordinates (θ_0, p_0) we can write solutions for each of four regions shown in Fig.1 in the following form:

$0 < t < T_1$

$$\theta_1 = \theta_0, \tag{6}$$
$$p_1 = p_0 + u_1 \sin{(h_1 \theta_0)}, \tag{7}$$

$T_1 < t < \Delta t$

$$\theta_2 = \theta_1 + k_0 p_1 t, \tag{8}$$
$$p_2 = p_1, \tag{9}$$

$\Delta t < t < \Delta t + T_2$

$$\theta_3 = \theta_2, \tag{10}$$
$$p_3 = p_2 + u_2 \sin{(h_2 \theta_2)}, \tag{11}$$

$t > \Delta t + T_2$

$$\theta_4 = \theta_3 + k_0 p_3 (t - \Delta t), \quad (12)$$

$$p_4 = p_3. \quad (13)$$

Now initial coordinates (p_0, θ_0) can be expressed as functions of coordinates at any moment t. Just before the second kick at $t = \Delta t$ they are

$$\theta_0 = \theta_2 - k_0 p_2 \Delta t, \quad (14)$$

$$p_0 = p_2 - u_1 \sin [h_1(\theta_2 - k_0 p_2 \Delta t)]. \quad (15)$$

In the same way for the coordinates of the particles after the second kick we have

$$\theta_2 = \theta_4 - k_0 p_4 (t - \Delta t), \quad (16)$$

$$p_2 = p_4 - u_2 \sin \{h_2[\theta_4 - k_0 p_4 (t - \Delta t)]\} \quad (17)$$

Combining these last two solutions we can finally write for $t > \Delta t$

$$\theta_0 = \theta - k_0 pt + u_2 k_0 \Delta t \sin \phi_2, \quad (18)$$

$$p_0 = p - u_1 \sin (\phi_1 + h_1 u_2 k_0 \Delta t \sin \phi_2) - u_2 \sin \phi_2, \quad (19)$$

where we redefined the coordinates (p_4, θ_4) as (p, θ) and introduced phases

$$\phi_1 = h_1(\theta - k_0 pt), \quad (20)$$

$$\phi_2 = h_2[\theta - k_0 p(t - \Delta t)]. \quad (21)$$

As a result the distribution function of the beam after the second kick can be found from the initial distribution function $F_0(p, \theta)$:

$$F(p, \theta, t) = F_0[p_0(p, \theta, t), \theta_0(p, \theta, t)]. \quad (22)$$

Together with expressions (18) and (19) this is an exact solution describing the evolution of an arbitrary initial distribution after two short RF excitations. In principle all information about the echo effect can be obtained from an analysis of this solution. In the next section we will consider examples for two given initial distribution functions.

Now let us recall the fact that this solution is valid only under the assumption, made at the beginning, that there is negligible change in the position of the particles during the kick. Below we will analyse this requirement in more detail.

Voltage applied to the beam during a kick creates potential wells. The depth of the potential well in energy is

$$p_b = \frac{2\omega_{s0}}{h_n k_0},$$

where ω_{s0} is the frequency of linear synchrotron oscillations inside this bucket

$$\omega_{sn}^2 = \frac{h_n k_0 u_n}{T_n} = \frac{h_n \omega_0^2 \eta e V_n}{2\pi E_s \beta^2}.$$

If the maximum energy deviation in the beam p_{max} is larger than the depth of the potential well p_b, then the character of particle motion depends on the value of p. Particles with $p < p_b$ are captured inside the potential well and others, with $p > p_b$, move outside the bucket. The first group of particles doesn't change significantly their position during the kick if the time length of the kick T_n is much less than one quarter of the period of linear synchrotron oscillations:

$$T_n << \frac{2\pi}{\omega_{s0}}.$$

For particles outside the bucket the requirement is [4]

$$T_n << \frac{\pi}{h_n k_0 p_{max}}.$$

3 Echo effect

3.1 Time and frequency of the echo signal

An echo has the nature of a resonance phenomenon. At some particular moment of time after two consecutive RF excitations, the dependence on p in the distribution function for some longitudinal harmonics disappears and a coherent signal can be observed. To see this effect let us analyse first the solution (19) for p_0

$$p_0 = p - \delta p = p - \delta p_1 - \delta p_2, \tag{23}$$

where we defined

$$\begin{aligned} \delta p_1 &= u_1 \sin(\phi_1 + x \sin\phi_2), \\ \delta p_2 &= u_2 \sin\phi_2, \\ x &= h_1 u_2 k_0 \Delta t. \end{aligned}$$

The first term in this solution can be rewritten in the form

$$\begin{aligned} \delta p_1 &= u_1[\sin\phi_1 \cos(x \sin\phi_2) + \cos\phi_1 \sin(x\sin\phi_2)] \qquad (24) \\ &= u_1\{\sin\phi_1[J_0(x) + 2\sum_{k=1}^{\infty} J_{2k}(x)\cos 2k\phi_2] + 2\cos\phi_1 \sum_{k=0}^{\infty} J_{2k+1}(x)\sin(2k+1)\phi_2\}, \end{aligned}$$

where $J_k(x)$ is the Bessel function of order k.

In the expression (24) the dependence on p in the phases

$$\phi_1 - k\phi_2 = (h_1 - kh_2)\theta + k_0 p[kh_2(t - \Delta t) - h_1 t] \tag{25}$$

will disappear respectively at the moments of time

$$t_k^* = \frac{kh_2 \Delta t}{kh_2 - h_1} \tag{26}$$

where $k = 1, 2...$ and the effect exists only for $kh_2 > h_1$. This means that at these particular moments of time, after integration over p, the coherent signal at the harmonic $(kh_2 - h_1)$ can be restored. The shape of the echo signal in time is defined by the form of the initial distribution function. Depending on the value of x, harmonics with different k will be more or less exposed. Below, as an example, we will consider the case $k = 1$ with harmonic $(h_2 - h_1) = h_{21}$. The contribution to this harmonic from (24) will be

$$\delta p_1 = u_1 J_1(x) \sin(\phi_2 - \phi_1), \tag{27}$$

Note that usually the initial distribution function is a function of $p_0^2 = (p - \delta p_1 - \delta p_2)^2$. Then as follows from (23) the contribution to the echo effect at harmonic h_{21} will come also from the term $\delta p_1 \delta p_2$. Indeed using again the expansion (24) we get

$$2\delta p_1 \delta p_2 = u_1 u_2 [J_0(x) - J_2(x)] \cos(\phi_2 - \phi_1) + ... \tag{28}$$

As can be seen from the following expression

$$(\delta p_1)^2 = u_1^2 \sin^2(\phi_1 + x \sin\phi_2) = \frac{u_1^2}{2}\{1 - \cos[2(\phi_1 + x\sin\phi_2)]\} \tag{29}$$

the Fourier spectrum of this term doesn't contain the harmonic h_{21} in which we are interested. Note that at the same time resonance exists at harmonic $2h_{21}$.

3.2 Shape of the echo signal

The amplitude of the echo signal at the current harmonic $(h_{21}\theta)$ can be calculated accurately for an arbitrary initial distribution function by integrating solution (22) over p and using its expansion in the Fourier series. For a continuous beam the initial distribution function can often be assumed to be only a function of energy deviation p_0, which simplifies calculations.

In the case when the initial distribution function $F_0(p_0)$ can be expanded with respect to $\delta p = \delta p_1 + \delta p_2$ in the convergent Taylor series

$$F(p, \theta, t) = F_0(p) - \delta p \frac{dF_0}{dp} + \frac{(\delta p)^2}{2}\frac{d^2 F_0}{dp^2} + ... \tag{30}$$

one should get the same results as obtained in [4]. Inserting expressions (27) and (28) in this series, for the part of the distribution function responsible for the echo effect at harmonics h_{21} we have

$$\delta f(p, \theta, t) = -u_1 J_1(x) \frac{dF_0}{dp} \cdot \sin(\phi_2 - \phi_1) + \frac{u_1 u_2}{2}[J_0(x) - J_2(x)] \frac{d^2 F_0}{dp^2} \cdot \cos(\phi_2 - \phi_1) + ... \tag{31}$$

Convergence of the Taylor series depends on the form of the initial distribution function. For a Gaussian distribution

$$F_0(p_0) = \mathcal{F}_0 \exp(-\frac{p_0^2}{2\sigma_p^2}) \tag{32}$$

it converges if $(u_1 + u_2) << \sigma_p$.

Let us estimate now the shape of the echo signal. According to (25) the phase difference $(\phi_2 - \phi_1)$ can be presented as

$$\phi_2 - \phi_1 = (h_2 - h_1)\theta - p\tau, \tag{33}$$

where we introduced the parameter τ, proportional to the time interval from the time t^*:

$$\tau = k_0(h_2 - h_1)(t - t^*).$$

If we assume that an initial distribution function is symmetric in energy

$$F_0(p) = F_0(-p),$$

then after integration over p in (31) the contribution to the echo signal from terms

$$\frac{dF_0}{dp} \cdot \cos{(p\tau)} \quad \text{and} \quad \frac{d^2 F_0}{dp^2} \cdot \sin{(p\tau)}$$

is zero in the first approximation. Integrating (31) by parts we obtain for the amplitude of the current harmonic h_{21} the following expression

$$\mathcal{J}(t) \simeq e\omega_0 u_1 \left\{ J_1(x) + \frac{\tau u_2}{2}[J_0(x) - J_2(x)] \right\} \tau \int_{-p_m}^{p_m} F_0(p)\cos(p\tau)dp, \tag{34}$$

where p_m is maximum value of p in the initial distribution. Here we assumed that $(\delta p)_{max} = (u_1 + u_2) << p_m$, $F_0(p_m) \sim 0$ and also neglected small oscillating terms.

For a Gaussian distribution we can write finally the approximate formula which describes the shape of the echo signal

$$\mathcal{J}(t) = \mathcal{J}_0\, u_1 \left\{ J_1(x) + \frac{\tau u_2}{2}[J_0(x) - J_2(x)] \right\} \tau \cdot \exp[-\frac{\tau^2\sigma_p^2}{2}], \tag{35}$$

where $\mathcal{J}_0 = (Ne\omega_0)/(2\pi)$ is average current.

In the general case the echo signal has an asymmetric double peak shape in time. The two peaks become identical when $J_1(x)$ has a maxima. This was observed in [4]. This fact can be explained by taking into account the relation

$$J_0(x) - J_2(x) = 2J_1'(x).$$

As we can see the second term in (35), which produces the asymmetry, equal zero when the first term has a maxima.

The distance between identical peaks is defined by the relation $\tau = 2/\sigma_p$, which corresponds to the time interval

$$t_{12} = \frac{2}{\sigma_p k_0 (h_2 - h_1)}. \tag{36}$$

The maximum amplitude of the echo signal at harmonic h_{21} is

$$\mathcal{J}_{max} = \mathcal{J}_0 \frac{u_1}{\sigma_p} J_1(x_j) \mathrm{e}^{-1/2}, \tag{37}$$

where x_j, $(j = 1, 2...)$, is one of the values of the argument x at which the Bessel function $J_1(x)$ has a maxima. For this case the echo signal described by function

$$\mathcal{J}(s) = \mathcal{J}_{max}\, s \cdot \exp\,(-\frac{s^2}{2}), \tag{38}$$

where $s = \sigma_p \tau = \sigma_p k_0 h_{21}(t - t^*)$, is shown in Fig.2.

It is interesting to compare the echo shape for different initial distributions. Let us consider a distribution function which belongs to the binomial family:

$$F_0(p_0) = \mathcal{F}_0 \cdot \left(1 - \frac{p_0^2}{p_m^2}\right)^{\mu}, \quad p_0 < p_m. \tag{39}$$

Analytic estimations can easily be done for the case $\mu = 1$. For the analysis of the echo effect we can use the exact solution

$$F(p, \theta, t) = \mathcal{F}_0 \cdot \left[1 - \frac{(p - \delta p)^2}{p_m^2}\right], \tag{40}$$

with $\delta p(p, \theta, t)$ defined by (23). Using expressions (27)-(28) and integrating (40) over p, we obtain for the amplitude of the echo signal at harmonic h_{21}

$$\mathcal{J}(t) = \mathcal{J}_0 \frac{3u_1}{p_m} \left\{ J_1(x) \cdot \frac{\sin y - y \cos y}{y^2} + \frac{u_2}{2p_m} \cdot [J_0(x) - J_2(x)] \cdot \frac{\sin y - y \cos y}{y} \right\}. \tag{41}$$

where $y = p_m \tau = p_m k_0 (h_2 - h_1)(t - t^*)$. To get this formula we assumed that $(u_1 + u_2) << p_m$. For the distribution function (39) the echo signal is also symmetric only at its maximum value, when the Bessel function $J_1(x)$ has a maxima. For this situation the shape of the echo signal $\mathcal{J}(y)$ calculated using expression (41) is shown in Fig.3. For the maximum value of the echo signal we can write

$$\mathcal{J}_{max} \simeq \mathcal{J}_0 \frac{3u_1}{p_m} J_1(x_j) \cdot 0.44. \tag{42}$$

When $J_1(x) = 0$, the amplitude of the echo signal is close to a minima.

Aknowledgements

The author would like to thank O.Brüning, T.Linnecar and F.Ruggiero for useful discussions of this work.

References

[1] E.M.Lifshitz and L.P.Pitaivskii, Physical Kinetics, Pergamon Press, 1981.

[2] G.V.Stupakov, Echo effect in Hadron Colliders, SSC Report SSCL-579, July 1992.

[3] P.L.Colestock, L.K.Spenzouris and F.Ostiguy, Nonlinear wave phenomena in coasting beams, PAC, Dallas, 1995.

[4] O.Brüning, On the possibility of measuring longitudinal echos in the SPS, CERN SL/95-83 (AP), 1995.

[5] O.Brüning, F.Ruggiero and W.Scandale, A proposal to measure longitudinal beam echos in the SPS, CERN SL-Note 95-115 (AP), 1995.

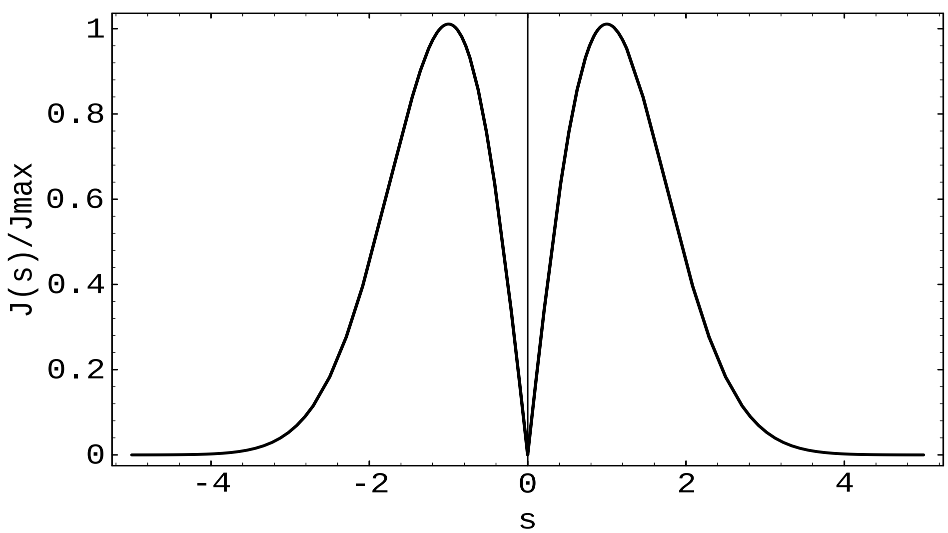

Figure 2: Shape of the echo signal at harmonic h_{21} as a function of $s = \sigma_p k_0 h_{21}(t - t^*)$ for the case of maximum amplitude (Gaussian distribution).

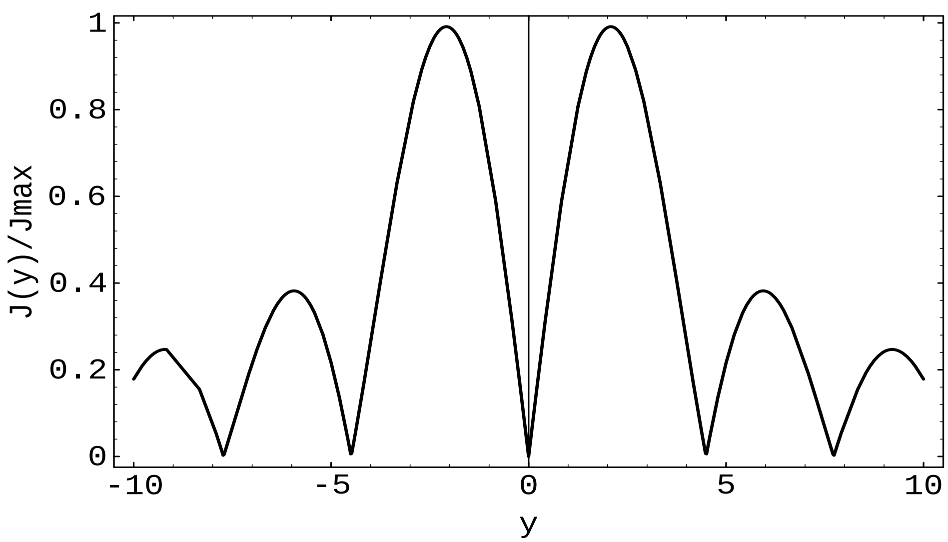

Figure 3: Shape of the echo signal at harmonic h_{21} as a function of $y = p_m k_0(h_2 - h_1)(t - t^*)$ calculated from (41) for the case of maximum amplitude (binomial distribution with $\mu = 1$).