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Longitudinal echo in a continuous beam. (Extension to more general case).

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First measurements of echo signals associated with two RF excitations of the continuous beam have been made in the SPS, [1]. While this new data has confirmed understanding of the phenomenon in general, [2]-[4], some unexpected results still await an explanation. This short note is an extension of [4] and presents calculations of the echo effect in a more general way, without using perturbation theory. Only the assumption about the length of the RF kicks (they must be short) is still necessary. Below we use results and definitions from [4].

The longitudinal echo can be observed in the continuous beam at some well defined time and frequency after two consecutive RF kicks, with time delay Δt between them, have occurred at different harmonics (h_1 and h_2) of the revolution frequency.

At the moment t (the first RF kick is at $t = 0$) the beam current in the machine is

$$I(\theta, t) = e\omega_0 \int_{-\infty}^{\infty} F(p, \theta, t)\, dp \tag{1}$$

which can be expanded in the Fourier series

$$I(\theta, t) = \sum_{n=-\infty}^{\infty} I_n(t) \mathrm{e}^{in\theta}, \tag{2}$$

where

$$I_n(t) = \frac{e\omega_0}{2\pi} \int_{-\infty}^{\infty} \int_{-\pi}^{\pi} F(p, \theta, t) \mathrm{e}^{-in\theta}\, d\theta\, dp \tag{3}$$

and $F(p, \theta, t)$ is the longitudinal distribution function in the beam at the moment t.

According to Liouville's theorem phase space density doesn't change along the particle trajectories

$$F(p, \theta, t) = F_0[p_0(p, \theta, t), \theta_0(p, \theta, t)], \tag{4}$$

where $F_0(p_0, \theta_0)$ is the initial distribution function. Below we assume that the initial distribution function of the continuous beam is a function of p_0 only. Then if later coordinates of the particles (p, θ) can be expressed as functions of the initial coordinates (p_0, θ_0) and time, $\theta = \theta(\theta_0, p_0, t)$, we can write for the n-th harmonic of the beam current at the moment t the following expression

$$I_n(t) = \frac{e\omega_0}{2\pi} \int_{-\infty}^{\infty} \int_{-\pi}^{\pi} F_0(p_0) \mathrm{e}^{-in\theta(\theta_0, p_0, t)}\, d\theta_0\, dp_0. \tag{5}$$

The solution for the particle motion found in [4] can be presented for $t > \Delta t$ in the form

$$\theta = \theta_0 + k_0 p_0 t + z_1 \sin\psi_1 + z_2 \sin(\psi_2 + y\sin\psi_1), \tag{6}$$

where we introduced the definitions

$$\begin{aligned}
\psi_1 &= h_1\theta_0, & (7)\\
\psi_2 &= h_2(\theta_0 + k_0 p_0 \Delta t), & (8)\\
z_1 &= k_0 u_1 t, & (9)\\
z_2 &= k_0 u_2 (t - \Delta t), & (10)\\
y &= h_2 k_0 u_1 \Delta t & (11)
\end{aligned}$$

and used the notations

$$k_0 = \frac{\omega_0 \eta}{\beta^2}, \qquad u_n = \frac{eV_n}{E_s}\frac{\omega_0 T_n}{2\pi}.$$

Here V_n and T_n are the amplitude and the length of the n-th RF kick. This solution is valid only under the assumption that there is negligible change in the position of the particles during the kick, which leads to the requirements discussed in [2]-[4]:

$$T_n \ll \frac{2\pi}{\omega_{s0}}, \qquad T_n \ll \frac{\pi}{h_n k_0 p_{max}},$$

where p_{max} is the maximum value of p in the beam.

Now using solution (6) we can rewrite formula (5) for the n-th harmonic of the beam current at the moment t as

$$I_n(t) = \frac{e\omega_0}{2\pi}\int_{-\infty}^{\infty}\int_{-\pi}^{\pi} F_0(p_0) \mathrm{e}^{-in[\theta_0 + k_0 p_0 t + z_1\sin\psi_1 + z_2\sin(\psi_2 + y\sin\psi_1)]}\, d\theta_0\, dp_0. \tag{12}$$

To proceed further let us consider separately the following expressions:

$$A = \mathrm{e}^{-inz_1\sin\psi_1} \quad \text{and} \quad B = \mathrm{e}^{-inz_2\sin(\psi_2 + y\sin\psi_1)}. \tag{13}$$

After some transformation these expressions can be presented as

$$\begin{aligned}
A &= \sum_{k=-\infty}^{\infty} (-i)^k J_k(nz_1)\cos[k(\psi_1 - \frac{\pi}{2})], & (14)\\
B &= \sum_{m=-\infty}^{\infty} (-i)^m J_m(nz_2) \sum_{l=-\infty}^{\infty} (-1)^l J_l(my)\cos[m(\psi_2 - \frac{\pi}{2}) - l\psi_1], & (15)
\end{aligned}$$

where $J_k(x)$ is the Bessel function of order k.

For simplicity let us consider first the most interesting case when

$$n = h_2 - h_1.$$

Then in (15) we can keep only the terms with $m = \pm 1$ in the series and rewrite it in the form

$$B = -2iJ_1(nz_2) \sum_{l=-\infty}^{\infty} (-1)^l J_l(y) \sin(\psi_2 - l\psi_1). \tag{16}$$

Multiplication of expressions A and B will give

$$A \cdot B = -2iJ_1(nz_2) \sum_{k=-\infty}^{\infty} (-i)^k J_k(nz_1) \sum_{l=-\infty}^{\infty} (-1)^l J_l(y) \sin[\psi_2 - (l-k)\psi_1 - k\frac{\pi}{2}]. \tag{17}$$

Summation over l can be eliminated by taking into account the fact that we are interested only in harmonics with $l - k = 1$. Then we obtain

$$A \cdot B = J_1(nz_2) \sum_{k=-\infty}^{\infty} \left[e^{i\Delta\psi} - e^{-i(\Delta\psi + \pi k)} \right] J_k(nz_1) J_{k+1}(y), \tag{18}$$

where $\Delta\psi = \psi_2 - \psi_1 = (h_2 - h_1)\theta + h_2 p k_0 \Delta t$.

Using Neumann's theorem for the summation of Bessel functions:

$$\sum_{k=-\infty}^{\infty} J_k(nz_1) J_{k+1}(y) = J_1(y - nz_1)$$

we can sum over k in (18) and present (12) as

$$I_n(t) = \frac{e\omega_0}{2\pi} J_1(nz_2) \int_{-\infty}^{\infty} \int_{-\pi}^{\pi} F_0(p) e^{-in(\theta + k_0 p t)} \left[e^{i\Delta\psi} J_1(y - nz_1) - e^{-i\Delta\psi} J_1(y + nz_1) \right] d\theta\, dp. \tag{19}$$

Here we dropped the index "0" for initial coordinates.

After integration over θ in (19) we get

$$I_n(t) = e\omega_0 J_1(nz_2) J_1(y - nz_1) \int_{-\infty}^{\infty} F_0(p) e^{-ipk_0(nt - h_2\Delta t)}\, dp. \tag{20}$$

Finally after substituting expressions for z_1, z_2 and y in (20) we can write for the amplitude of the echo signal at harmonic $(h_2 - h_1)$

$$\mathcal{J}(t) = 2e\omega_0 |J_1(u_1\tau)\, J_1(x + u_2\tau)| \cdot |f(\tau)|, \tag{21}$$

where we used the notations

$$\tau = k_0(h_2 - h_1)(t - t^*), \tag{22}$$

$$x = h_1 u_2 k_0 \Delta t. \tag{23}$$

The echo signal can be observed at times close to t^*, where

$$t^* = \frac{h_2 \Delta t}{h_2 - h_1}.$$

However as can be seen from the expression (21) at exactly $t = t^*$ the amplitude of the signal is always zero.

The function $f(\tau)$ can be defined from the Fourier transform of the initial distribution

$$f(\tau) = \int_{-\infty}^{\infty} F_0(p) \mathrm{e}^{-ip\tau}\, dp. \tag{24}$$

For an initial distribution which is symmetric in energy, $F_0(p) = F_0(-p)$, function $f(\tau)$ becomes

$$f(\tau) = \int_{-\infty}^{\infty} F_0(p) \cos(p\tau)\, dp. \tag{25}$$

For small values of $u_n\tau$ we can use the Bessel function expansion

$$J_1(u_1\tau) = \frac{u_1\tau}{2} + ..., \tag{26}$$

$$J_1(x + u_2\tau) = J_1(x) + u_2\tau J_1'(x) + ... \tag{27}$$

with $2J_1'(x) = J_0(x) - J_2(x)$. In this case formula (21) gives the same answer as found in [4].

In the more general (but nondegenerate) case when

$$n = \mu h_2 - \nu h_1,$$

the amplitude of the n-th beam current harmonic becomes

$$\mathcal{J}(t) = 2e\omega_0 |J_\nu(u_1\tau)\, J_\mu(\nu x + u_2\tau)| \cdot |f(\tau)|, \tag{28}$$

with

$$\tau = k_0 n(t - t^*), \tag{29}$$

$$t^* = \frac{\mu}{n} h_2 \Delta t. \tag{30}$$

References

[1] O.Brüning, T.Linnecar, F.Ruggiero, W.Scandale and E.Shaposhnikova, MD Note 1996, to be published.

[2] O.Brüning, On the possibility of measuring longitudinal echos in the SPS, CERN SL/95-83 (AP), 1995.

[3] O.Brüning, F.Ruggiero and W.Scandale, A proposal to measure longitudinal beam echos in the SPS, CERN SL-Note 95-115 (AP), 1995.

[4] E.Shaposhnikova, On the longitudinal echo in a continuous beam, CERN SL-Note 95-125 (RF).