Advanced Topics in Accelerator Physics

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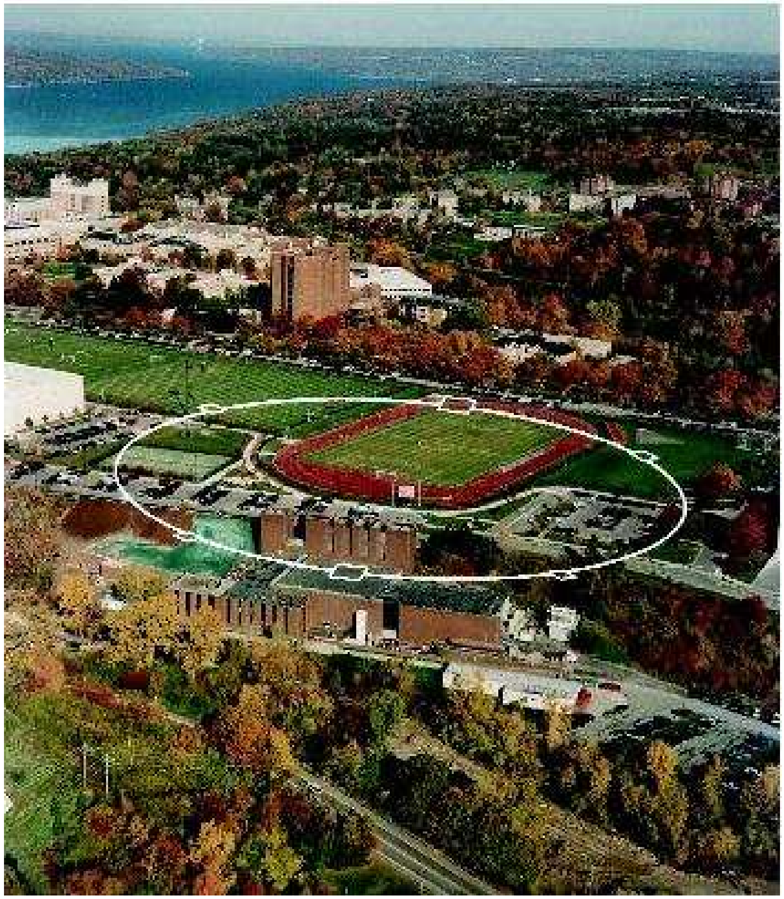

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Literature

Required:
The Physics of Particle Accelerators, Klaus Wille, Oxford University Press, 2000, ISBN: 19 850549 3

Optional:
Particle Accelerator Physics I, Helmut Wiedemann, Springer, 2nd edition, 1999, ISBN 3 540 64671 x

Particle Accelerator Physics II, Helmut Wiedemann, Springer, 2nd edition, 1999, ISBN 3 540 64504 7

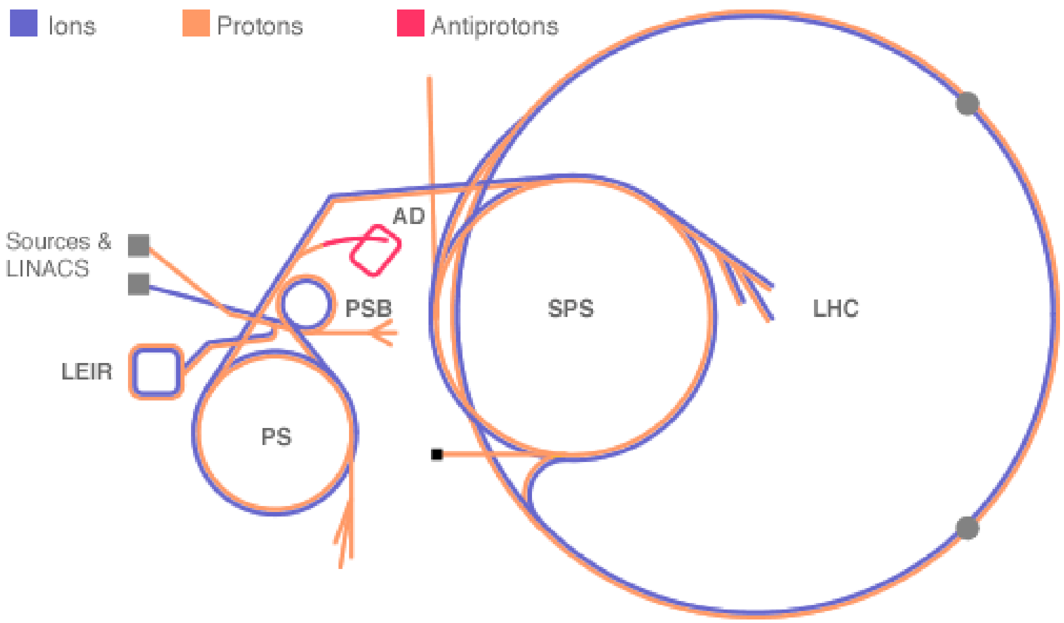
Related material:
Handbook of Accelerator Physics and Engineering, Alexander Wu Chao and Maury Tigner, 2nd edition, 2002, World Scientific, ISBN: 981 02 3858 4

What is accelerator physics

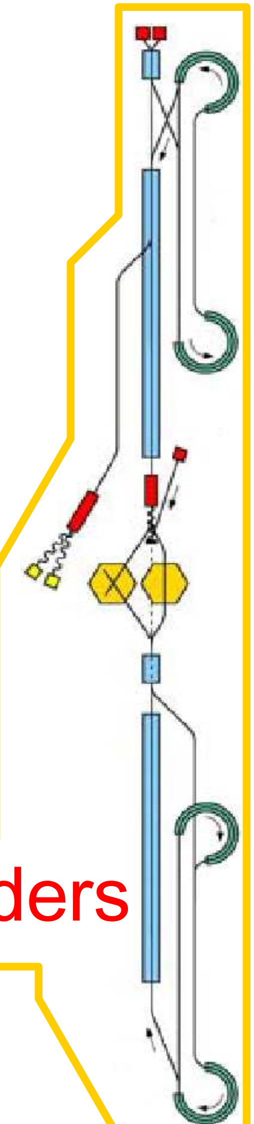
Accelerator Physics has applications in particle accelerators for high energy physics or for x-ray science, in spectrometers, in electron microscopes, and in lithographic devices. These instruments have become so complex that an empirical approach to properties of the particle beams is by no means sufficient and a detailed theoretical understanding is necessary. This course will cover theoretical aspects of charged particle beams and their practical relevance.

- Physics of beams (nonlinear dynamics, many particle systems)
- Physics of non-neutral plasmas (distribution dynamics)
- Physics involved in the technology:
  - Superconductivity in magnets and radiofrequency (RF) devices
  - Surface physics in particle sources, vacuum technology, RF devices
  - Material science in collimators, beam dumps, superconducting materials

Linear and circular accelerators

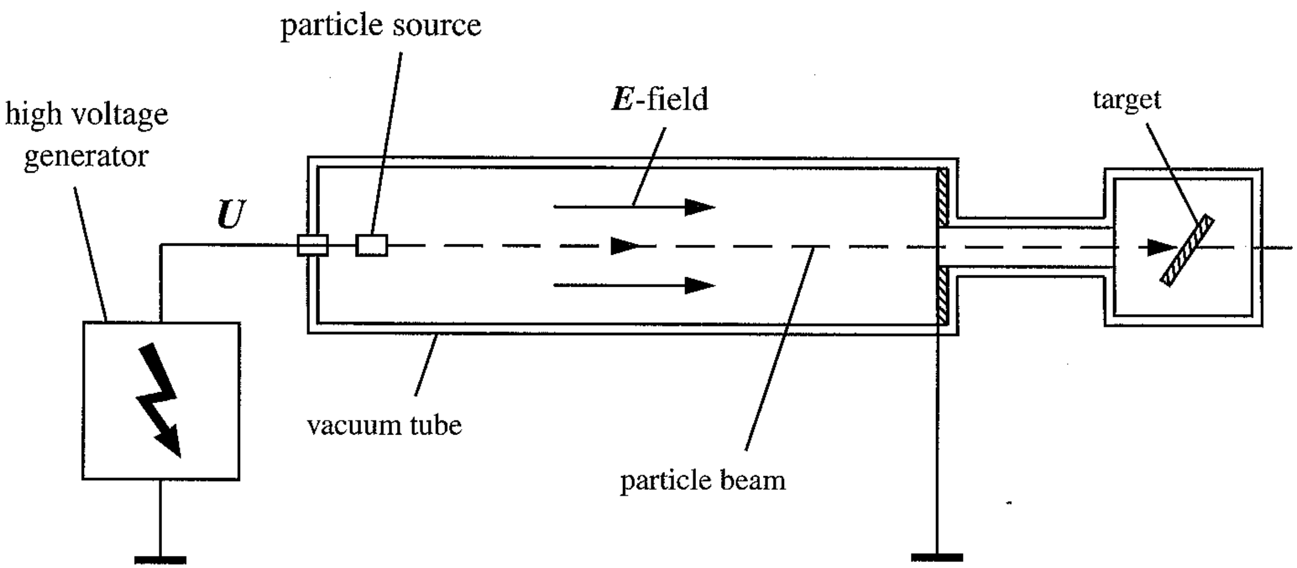


Linear Colliders



Three historic lines of accelerators

Direct Voltage Accelerators   Resonant Accelerators   Transformer Accelerator



Voltage 1MV
Charge Ze
Energy Z MeV

The energy limit is given by the maximum possible voltage. At the limiting voltage, electrons and ions are accelerated to such large energies that they hit the surface and produce new ions. An avalanche of charge carries causes a large current and therefore a breakdown of the voltage.

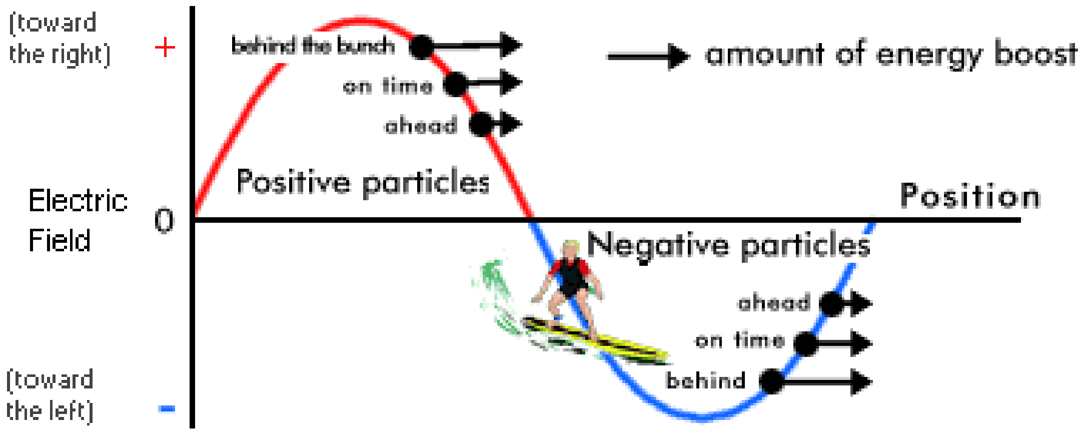
- 1930: van de Graaff builds the first 1.5MV high voltage generator
- 1932: Cockcroft and Walton: 700keV cascade generator (planned for 800keV) and use initially 400keV protons for $^7Li + p \mapsto {}^4He + {}^4He$ and $^7Li + p \mapsto {}^7Be + n$
- 1932: Marx Generator achieves 6MV at General Electrics

Three historic lines of accelerators

Direct Voltage Accelerators

Resonant Accelerators

Transformer Accelerator



- 1932: Lawrence and Livingston 1st cyclotron in Berkeley
- 1934: Cornell is the 2nd lab with Cyclotron (by Livingston in room Rf-B54)
- Microtrons
- 1928: Wideroe builds the first drift tube linear accelerator for Na^+ and K^+
- 1933: J.W. Beams uses resonant cavities for acceleration
- Alvarez builds the first Alvarez Linear Accelerator
- 1970: Kapchinskii and Teplyakov built the first RFQ

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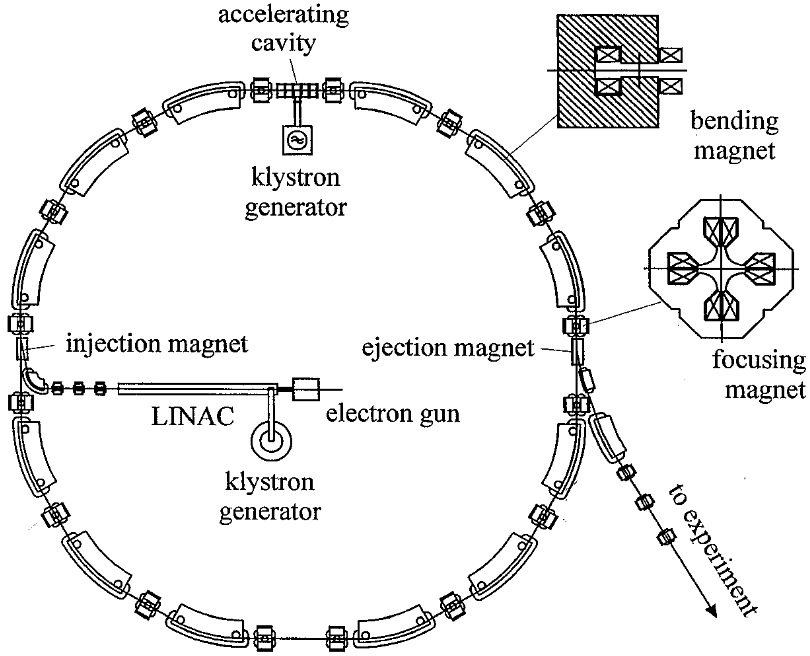
Three historic lines of accelerators

Transformer Accelerator

Direct Voltage Accelerators   Resonant Accelerators

- 1940: Kerst and Serber build a betatron for 2.3MeV electrons and understand betatron (transverse) focusing (in 1942: 20MeV)

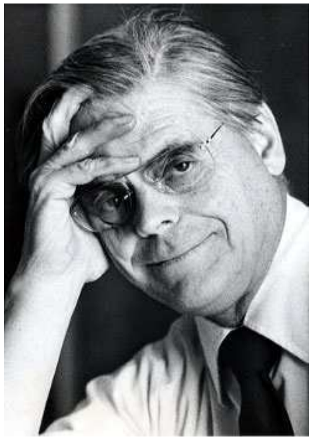


- 1946: Goward and Barnes build the first synchrotron (using a betatron magnet)
- 1949: Wilson et al. at Cornell are first to store beam in a synchrotron (later 300MeV, magnet of 80 Tons)
- 1954: Wilson et al. build first synchrotron with strong focusing for 1.1GeV electrons at Cornell, 4cm beam pipe height, only 16 Tons of magnets.

Robert R Wilson, Architecture

Science Ed Center, FNAL (1990)

Wilson Hall, FNAL



Robert R Wilson
USA 1914-2000

Robert R Wilson, Sculpture

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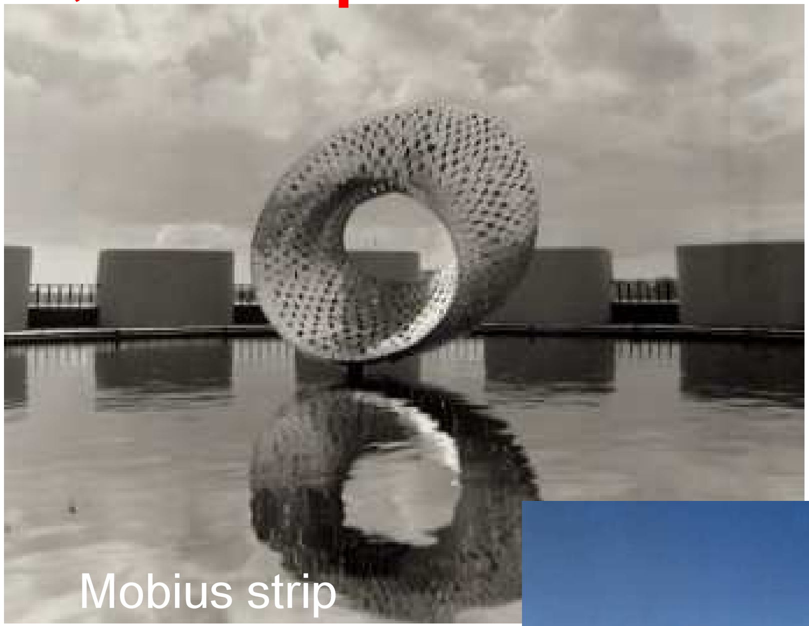
Broken symmetry


Mobius strip


Tractricious


π lines


Hyperbolic obelisk

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Limits of Synchrotrons

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$$\rho = \frac{p}{qB} \quad \Rightarrow$$ The rings become too long

Protons with p = 20 TeV/c , B = 6.8 T would require a 87 km SSC tunnel
Protons with p = 7 TeV/c , B = 8.4 T require CERN's 27 km LHC tunnel

$$P_{\text{radiation}} = \frac{c}{6\pi\varepsilon_0} N \frac{q^2}{\rho^2} \gamma^4 \quad \Downarrow$$

Energy needed to compensate
Radiation becomes too large

Electron beam with p = 0.1 TeV/c in CERN's 27 km LEP tunnel radiated 20 MW
Each electron lost about 4GeV per turn, requiring many RF accelerating sections.

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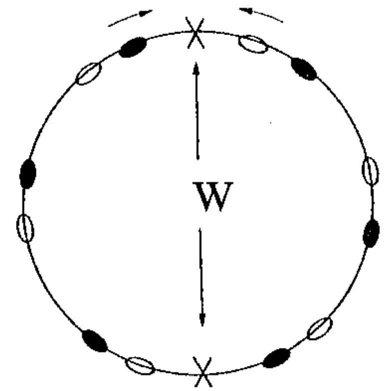
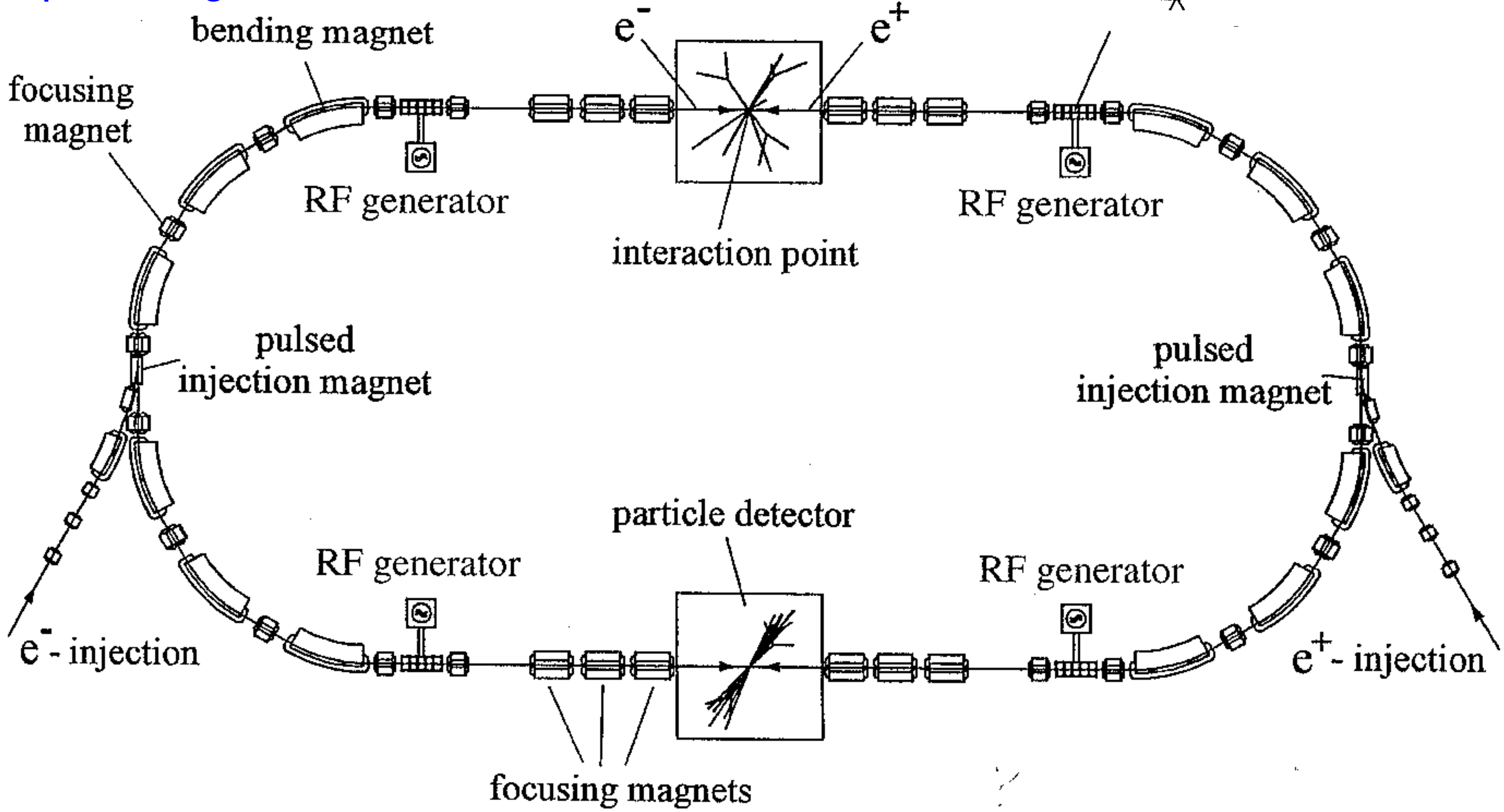
Elements of a Collider



Chalenges:

- Saving one beam while injection another
- Avoiding collisions outside the detectors.
- Compensating the forces between e^+ and e^- beams



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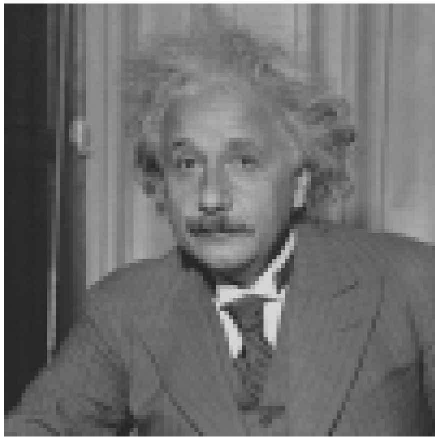
Special Relativity

$$E = mc^2$$



Albert Einstein, 1879-1955
Nobel Prize, 1921
Time Magazine Man of the Century

Four-Vectors:
Quantities that transform according to the Lorentz transformation when viewed from a different inertial frame.

Examples:

$$X^\mu \in \{ct, x, y, z\}$$

$$P^\mu \in \{\tfrac{1}{c}E, p_x, p_y, p_z\}$$

$$\Phi^\mu \in \{\tfrac{1}{c}\phi, A_x, A_y, A_z\}$$

$$J^\mu \in \{c\rho, j_x, j_y, j_z\}$$

$$K^\mu \in \{\tfrac{1}{c}\omega, k_x, k_y, k_z\}$$

$$X^\mu \in \{ct, x, y, z\} \quad \Rightarrow \quad X^\mu X_\mu = (ct)^2 - \vec{x}^{\,2} = \text{const.}$$

$$P^\mu \in \{\tfrac{1}{c}E, p_x, p_y, p_z\} \Rightarrow \quad P^\mu P_\mu = \left(\frac{E}{c}\right)^2 - \vec{p}^{\,2} = (m_0 c)^2 = \text{const.}$$

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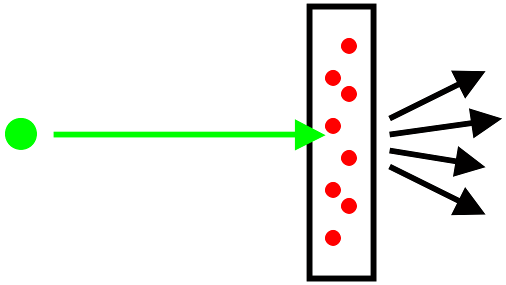
Available Energy

$$\frac{1}{c^2}E_{\rm cm}^2 = (P_1^\mu + P_2^\mu)_{\rm cm}(P_{1\mu} + P_{2\mu})_{\rm cm}$$
$$= (P_1^\mu + P_2^\mu)(P_{1\mu} + P_{2\mu})$$
$$= \frac{1}{c^2}(E_1 + E_2)^2 - (p_{z1} - p_{z2})^2$$
$$= 2(\frac{E_1E_2}{c^2} + p_{z1}p_{z2}) + (m_{01}c)^2 + (m_{02}c)^2$$

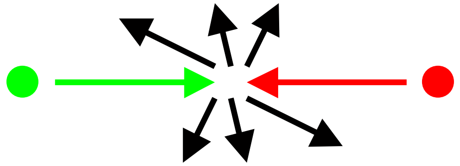
Operation of synchrotrons: fixed target experiments where some energy is in the motion of the center off mass of the scattering products



$$E_1 >> m_{01}c^2, m_{02}c^2; p_{z2} = 0; E_2 = m_{02}c^2 \quad \Rightarrow \quad E_{\rm cm} = \sqrt{2E_1m_{02}c^2}$$

Operation of colliders:

the detector is in the center of mass system



$$E_1 >> m_{01}c^2; E_2 >> m_{02}c^2 \quad \Rightarrow \quad E_{\rm cm} = 2\sqrt{E_1E_2}$$

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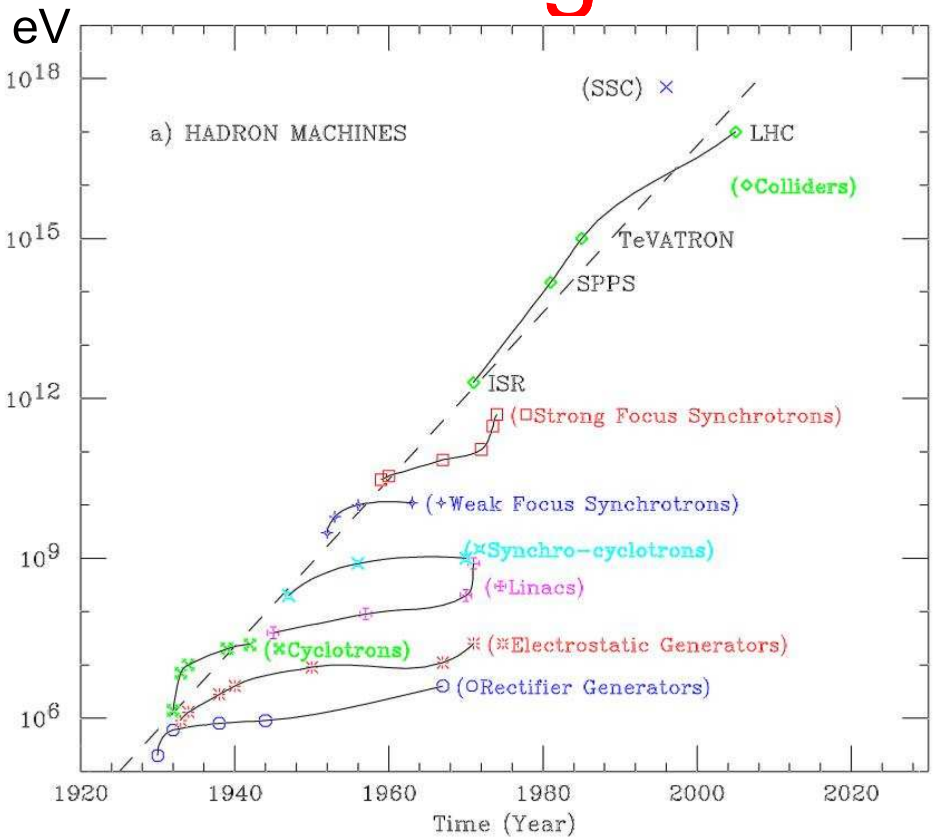
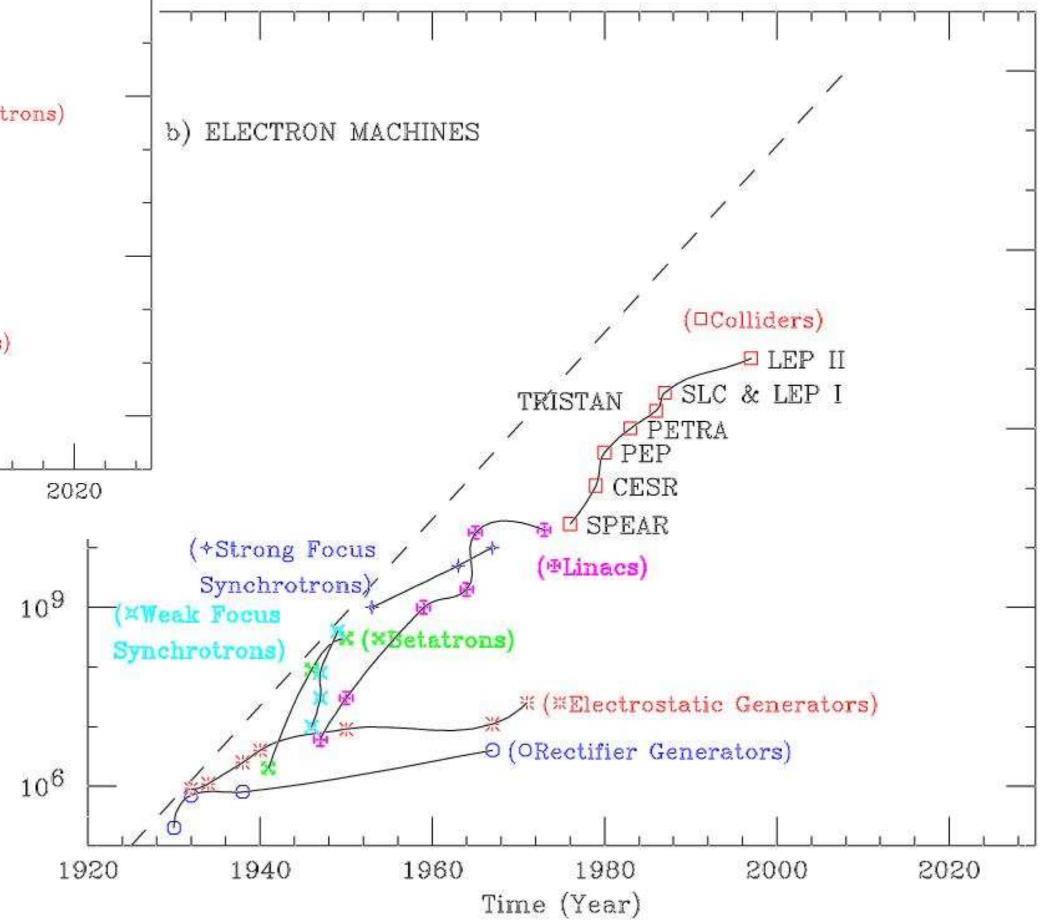
The Livingston Chart



Comparison:

highest energy cosmic rays have a few 10^{20}eV



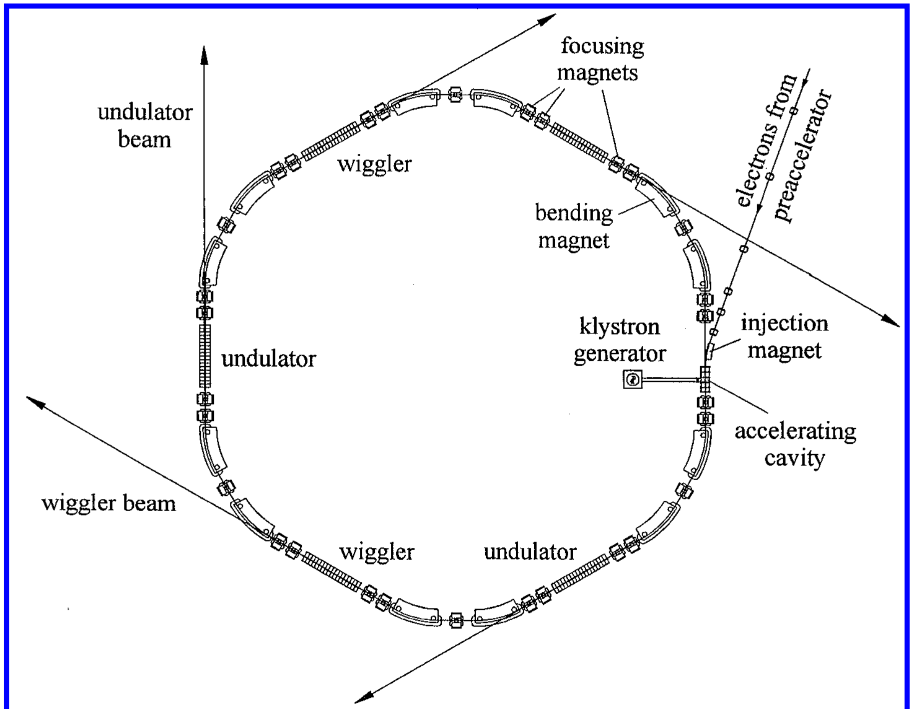
Energy that would be needed in a fixed target experiment versus the year of achievement

$$E_1 = \frac{E_{cm}^2}{2m_{02}c^2}$$

4 Generations of Light Sources

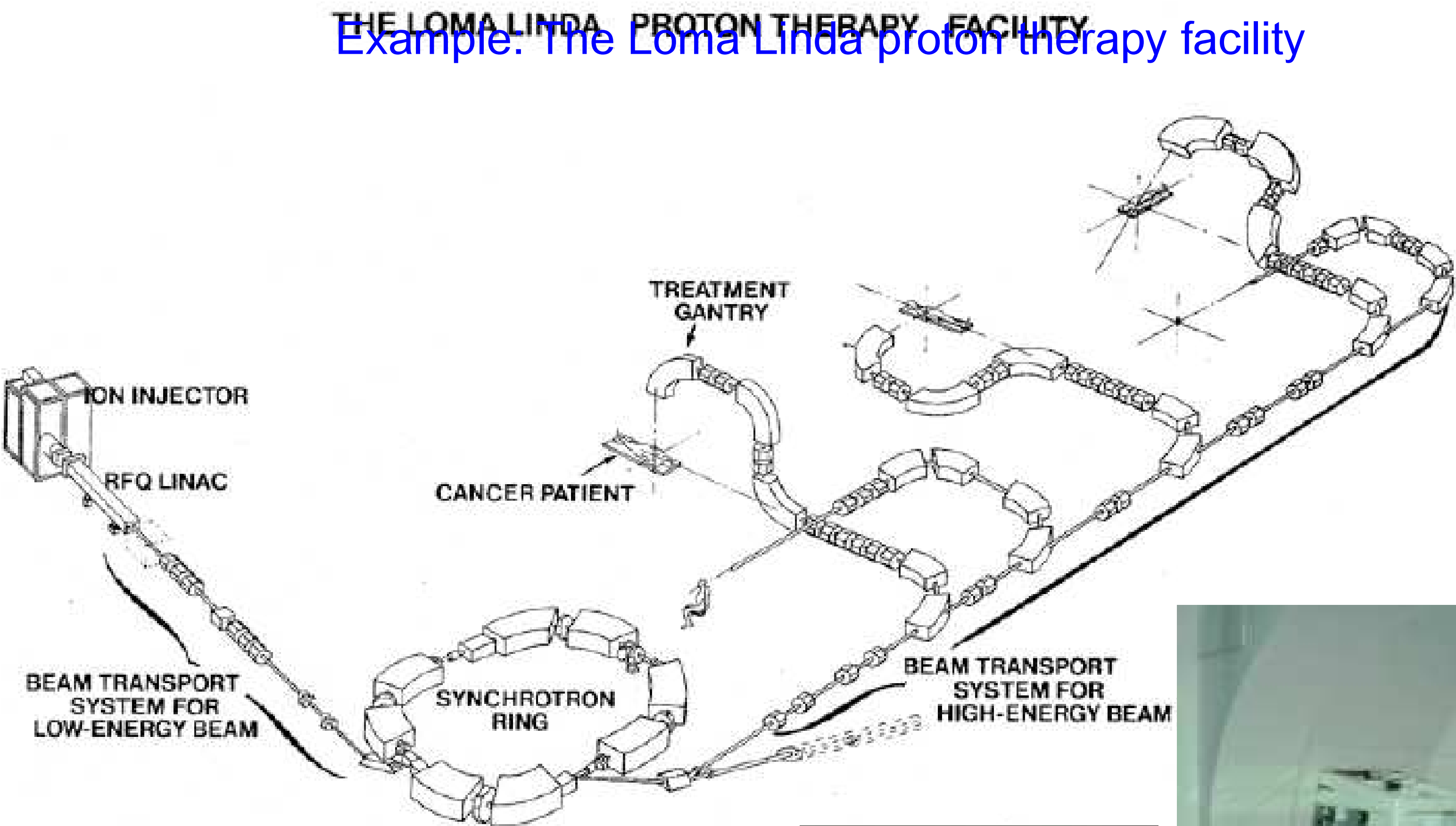
- 1952: First accurate measurement of synchrotron radiation power by Dale Corson with the Cornell 300MeV synchrotron. Later also first measurements of spectrum and polarization with worlds 1st SR beam line.

- 1st Generation (1970s): Many HEP rings are parasitically used for X-ray production
- 2nd Generation (1980s): Many dedicated X-ray sources (light sources)
- 3rd Generation (1990s): Several rings with dedicated radiation devices (wigglers and undulators)
- 3.5th Generation: Construction of Energy Recovery Linac (ERL) with Undulators
- 4th Generation: Construction of Free Electron Lasers (FELs) driven by LINACs



Beamlines

Example: The Loma Linda proton therapy facility



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Accelerators of the World

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Sorted by Location

Europe

AGOR	Accelerateur Groningen-ORsay, KVI Groningen, Netherlands
ANKA	Ångströmquelle Karlsruhe, Karlsruhe, Germany (Forschungsgruppe Synchrotronstrahlung (FGS))
ASTRID	Aarhus Storage Ring in Denmark, ISA, Aarhus, Denmark
BESSY	Berliner Elektronenspeicherring-Gesellschaft für Synchrotronstrahlung, Germany (BESSY I status, BESSY II status)
BINP	Budker Institute for Nuclear Physics, Novosibirsk, Russian Federation (VEPP-2M collider, VEPP-4M collider (status))
CERN	Centre Europeen de Recherche Nucleaire, Geneva, Suisse (LEP & SPS Status, LHC, CLIC, PS-Division, SL-Division)
COSY	Cooler Synchrotron, IKP, FZ Jülich, Germany (COSY Status)
CYCLONE	Cyclotron of Louvain la Neuve, Louvain-la-Neuve, Belgium
DELTA	Dortmund Electron Test Accelerator, U of Dortmund, Germany (DELTA Status)
DESY	Deutsches Elektronen Synchrotron, Hamburg, Germany (HERA, PETRA and DORIS status, TESLA)
ELBE	ELectron source with high Brilliance and low Emittance, FZ Rossendorf, Germany
ELETTRA	Trieste, Italy (ELETTRA status)
ELSA	Electron Stretcher Accelerator, Bonn University, Germany (ELSA status)
ESRF	European Synchrotron Radiation Facility, Grenoble, France (ESRF status)
GANIL	Grand Accélérateur National d'Ions Lourds, Caen, France
GSI	Gesellschaft für Schwerionenforschung, Darmstadt, Germany
IHEP	Institute for High Energy Physics, Protvino, Moscow region, Russian Federation
INFN	Istituto Nazionale di Fisica Nucleare, Italy, LNF - Laboratori Nazionali di Frascati (DAFNE, other accelerators), LNL - Laboratori Nazionali di Legnaro (Tandem, CN Van de Graaff, AN 2000 Van de Graaff), LNS - Laboratori Nazionali del Sud, Catania, (Superconducting Collider & Van de Graaff Tandem)
ISIS	Rutherford Appleton Laboratory, Oxford, U.K. (ISIS Status)
ISL	IonenStrahlLabor am HMI, Berlin, Germany
JINR	Joint Institute for Nuclear Research, Dubna, Russian Federation (U-200, U-400, U-400M, Storage Ring, LHE Synchrophasotron / Nuclotron)
JYFL	Jyväskylän Yliopiston Fysiikan Laitos, Jyväskylä, Finland
KTH	Kungl Tekniska Högskola (Royal Institute of Technology), Stockholm, Sweden (Alfén Lab electron accelerators)

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Accelerators of the World

LMU/TUM	Accelerator of LMU and TU Muenchen, Munich, Germany
LURE	Laboratoire pour l'Utilisation du Rayonnement Electromagnétique, Orsay, France (DCI, Super-ACO status, CLIO)
MAMI	Mainzer Microtron, Mainz U, Germany
MAX-Lab	Lund University, Sweden
MSL	Manne Siegbahn Laboratory, Stockholm, Sweden (CRYRING)
NIKHEF	Nationaal Instituut voor Kernfysica en Hoge-Energie Fysica, Amsterdam, Netherlands (AmPS *closed!*)
PSI	Paul Scherrer Institut, Villigen, Switzerland (PSI status, SLS *under construction*)
S-DALINAC	Darmstadt University of Technology, Germany (S-DALINAC status)
SRS	Synchrotron Radiation Source, Daresbury Laboratory, Daresbury, U.K. (SRS Status)
TSL	The Svedberg Laboratory, Uppsala University, Sweden (CELSIUS)
TSR	Heavy-Ion Test Storage Ring, Heidelberg, Germany

North America

88" Cycl.	88-Inch Cyclotron, Lawrence Berkeley Laboratory (LBL), Berkeley, CA
ALS	Advanced Light Source, Lawrence Berkeley Laboratory (LBL), Berkeley, CA (ALS Status)
ANL	Argonne National Laboratory, Chicago, IL (Advanced Photon Source APS [status], Intense Pulsed Neutron Source IPNS [status], Argonne Tandem Linac Accelerator System ATLAS)
BNL	Brookhaven National Laboratory, Upton, NY (AGS, ATF, NSLS, RHIC)
CAMD	Center for Advanced Microstructures and Devices
CHESS	Cornell High Energy Synchrotron Source, Cornell University, Ithaca, NY
CLS	Canadian Light Source, U of Saskatchewan, Saskatoon, Canada
CESR	Cornell Electron-positron Storage Ring, Cornell University, Ithaca, NY (CESR Status)
FNAL	Fermi National Accelerator Laboratory , Batavia, IL (Tevatron)
IAC	Idaho accelerator center, Pocatello, Idaho
IUCF	Indiana University Cyclotron Facility, Bloomington, Indiana
JLab	aka TJNAF, Thomas Jefferson National Accelerator Facility (formerly known as CEBAF), Newport News, VA
LAC	Louisiana Accelerator Center, U of Louisiana at Lafayette, Louisiana
LANL	Los Alamos National Laboratory
MIT-Bates	Bates Linear Accelerator Center, Massachusetts Institute of Technology (MIT)
NSCL	National Superconducting Cyclotron Laboratory, Michigan State University
ORNL	Oak Ridge National Laboratory (EN Tandem Accelerator), Oak Ridge, Tennessee
SBSL	Stony Brook Superconducting Linac, State University of New York (SUNY)
SLAC	Stanford Linear Accelerator Center (Linac, NLC - Next Linear Collider, PEP - Positron Electron Project (finished), PEP-II - asymmetric B Factory (in commissioning), SLC - SLAC Linear electron positron Collider, SPEAR - Stanford Positron Electron Asymmetric Ring (actually SPEAR-II, see SSRL), SSRL - Stanford Synchrotron Radiation Laboratory)
SNS	Spallation Neutron Source, Oak Ridge, Tennessee
SRC	Synchrotron Radiation Center, U of Wisconsin - Madison (Aladdin Status)
SURF II	Synchrotron Ultraviolet Radiation Facility, National Institute of Standards and Technology (NIST), Gaithersburg, Maryland
TASCC	Tandem Accelerator Superconducting Cyclotron (Canada) *(closed!)*
TRIUMF	TRI-University Meson Facility / National Meson Research Facility, Vancouver, BC (Canada)

South America

LNLS	Laboratorio Nacional de Luz Sincrotron, Campinas SP, Brazil
TANDAR	Tandem Accelerator, Buenos Aires, Argentina

Asia

BEPC	Beijing Electron-Positron Collider, Beijing, China
KEK	National Laboratory for High Energy Physics ("Koh-Ene-Ken"), Tsukuba, Japan (KEK-B, PF, JLC)
NSC	Nuclear Science Centre, New Delhi, India (15 UD Pelletron Accelerator)
PLS	Pohang Light Source, Pohang, Korea
RIKEN	Institute of Physical and Chemical Research ("Rikagaku Kenkyusho"), Hirosawa, Wako, Japan
SESAME	Synchrotron-light for Experimental Science and Applications in the Middle East, Jordan (under construction)
SPring-8	Super Photon ring - 8 GeV, Japan
SRRC	Synchrotron Radiation Research Center, Hsinchu, Taiwan (SRRC Status)
UVSOR	Ultraviolet Synchrotron Orbital Radiation Facility, Japan
VECC	Variable Energy Cyclotron, Calcutta, India

Africa

NAC	National Accelerator Centre, Cape Town, South Africa

Sorted by Accelerator Type

Electrons

Stretcher Ring/Continuous Beam facilities

ELSA (Bonn U), JLab, MAMI (Mainz U), MAX-Lab, MIT-Bates, PSR (SAL), S-DALINAC (TH Darmstadt), SLAC

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Accelerators of the World

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Synchrotron Light Sources

ANKA (FZK), ALS (LBL), APS (ANL), ASTRID (ISA), BESSY, CAMD (LSU), CHESS (Cornell Wilson Lab), CLS (U of Saskatchewan), DELTA (U of Dortmund), ELBE (FZ Rossendorf), Elettra, ELSA (Bonn U), ESRF, HASYLAB (DESY), LURE, MAX-Lab, LNLS, NSLS (BNL), PF (KEK), UVSOR (IMS), PLS, S-DALINAC (TH Darmstadt), SESAME, SLS (PSI), SPEAR (SSRL, SLAC), SPring-8, SRC (U of Wisconsin), SRRC, SRS (Daresbury), SURF II (NIST)

Other

Alfén Lab (KTH), IAC

Protons

88" Cyclotron (LBL), CELSIUS (TSL), COSY (FZ Jülich), IPNS (ANL), ISL (HMI), ISIS, IUCF, LHC (CERN), NAC, PS (CERN), PSI, SPS (CERN)

Light and Heavy Ions

88" Cyclotron (LBL), AGOR, ASTRID (ISA), ATLAS (ANL), CELSIUS (TSL), CRYRING (MSL), CYCLONE, EN Tandem (ORNL), GANIL, GSI, ISL (HMI), IUCF, JYFL, LAC, LHC (CERN), LHE Synchrophasotron / Nuclotron (JINR), LMU/TUM, LNL (INFN), LNS (INFN), NAC, NSC, PSI, RHIC (BNL), SBSL, SNS, SPS (CERN), TANDAR, TSR, U-200 / U-400 / U-400M / Storage Ring (JINR), VECC

Collider

BEPC, CESR, DAFNE (LNF), HERA (DESY), LEP (CERN), LHC (CERN), PEP / PEP-II (SLAC), SLC (SLAC), KEK-B (KEK), TESLA (DESY), Tevatron (FNAL), VEPP-2M, VEPP-4M (BINP)

The Future

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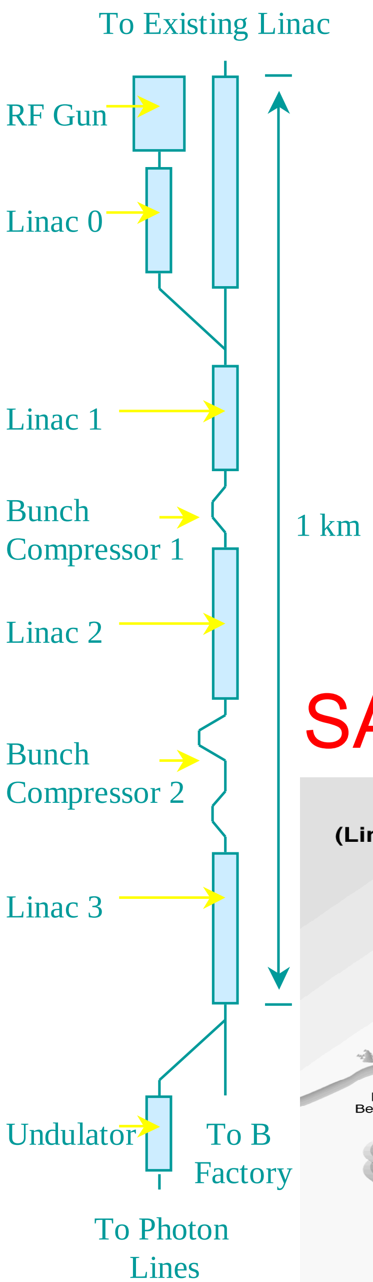
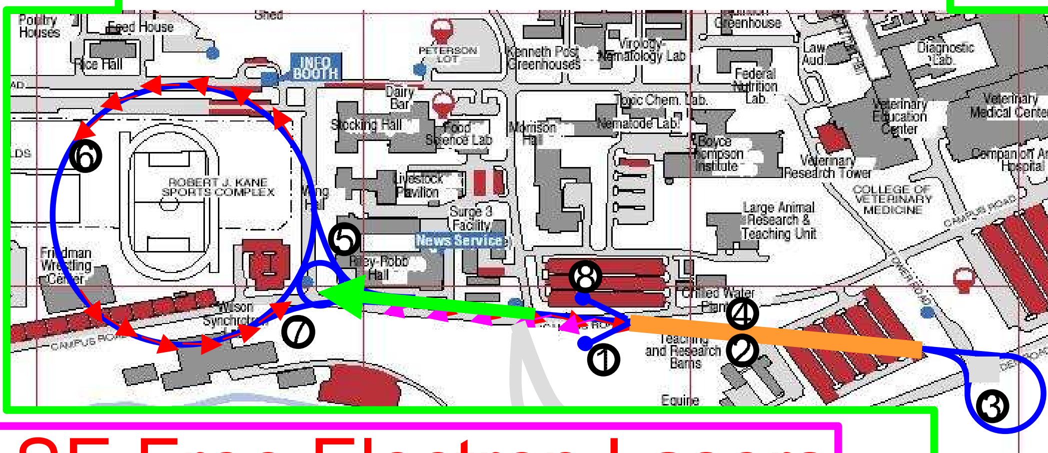
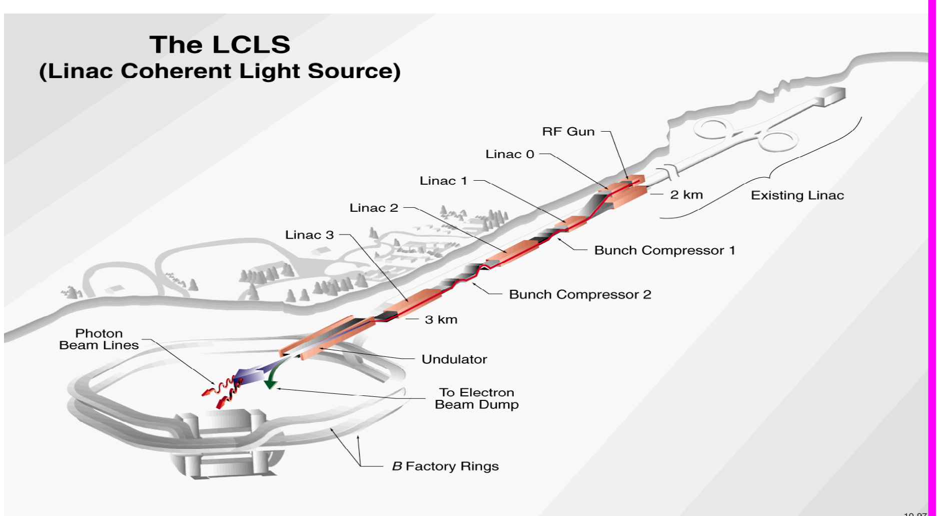


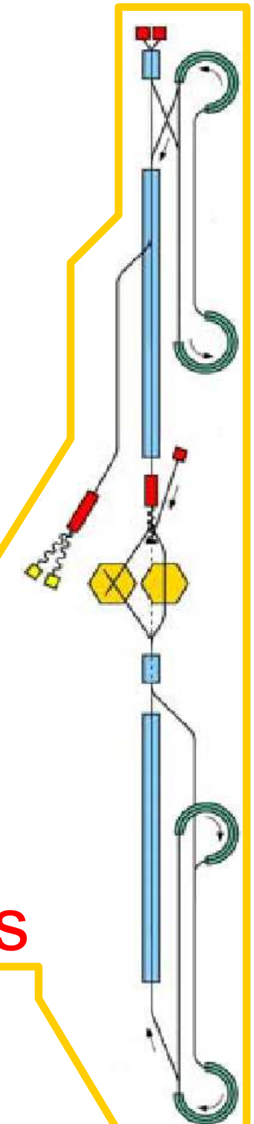
Energy Recovery Linacs



SASE Free Electron Lasers



Linear Colliders



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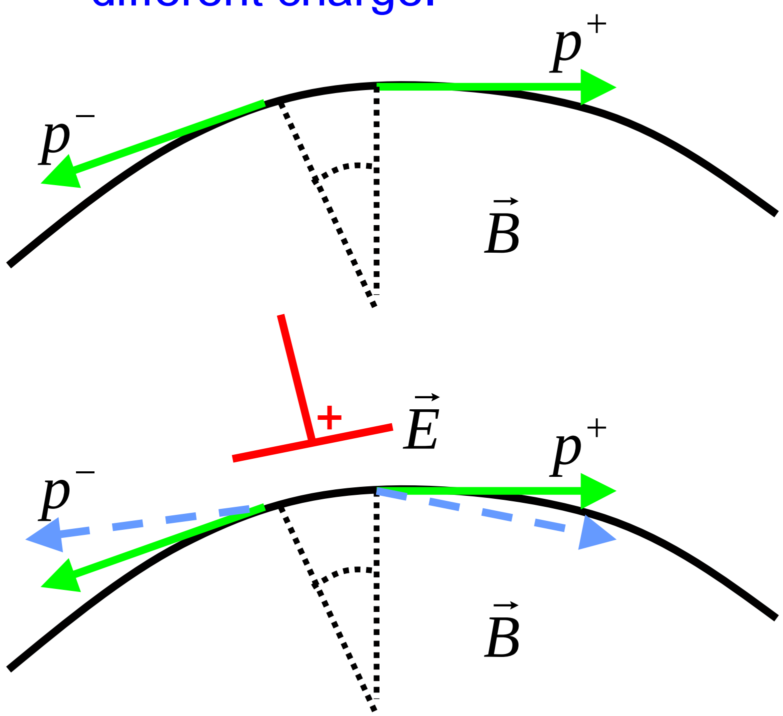
Macroscopic Fields in Accelerators

$$\frac{d}{dt}\vec{p} = q(\vec{E} + \vec{v} \times \vec{B})$$

E has a similar effect as v B.
For relativistic particles B = 1T has a similar effect as
E = cB = 3 10^8 V/m , such an
Electric field is beyond technical limits.

- Electric fields are only used for very low energies or
- For separating two counter rotating beams with different charge.



Electrostatic separators at CESR

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Magnetic Fields in Accelerators

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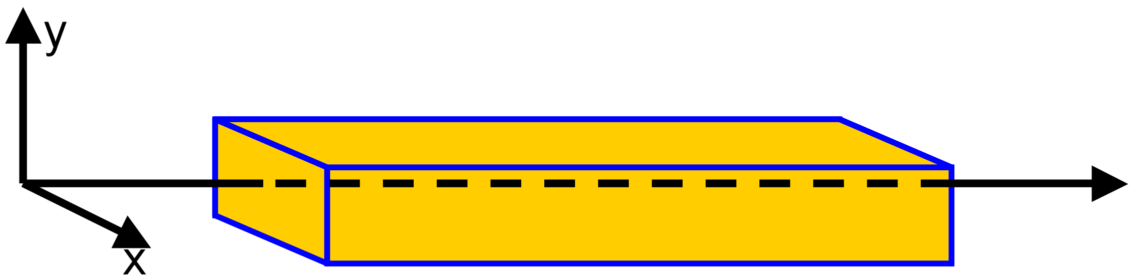
Static magnetic fileds: $\partial_t \vec{B} = 0\,; \quad \vec{E} = 0$ Charge free space: $\vec{j} = 0$

$$\vec{\nabla}\times\vec{B} = \mu_0(\vec{j} + \varepsilon_0 \partial_t \vec{E}) = 0 \quad \Rightarrow \quad \vec{B} = -\vec{\nabla}\psi(\vec{r})$$

$$\vec{\nabla}\cdot\vec{B} = 0 \quad \Rightarrow \quad \vec{\nabla}^2\psi(\vec{r}) = 0$$



(x=0,y=0) is the beam's design curve

For finite fields on the design curve,
Ψ can be power expanded in x and y: $\psi(x,y,z) = \sum_{n,m=0}^{\infty} b_{nm}(z) x^n y^m$

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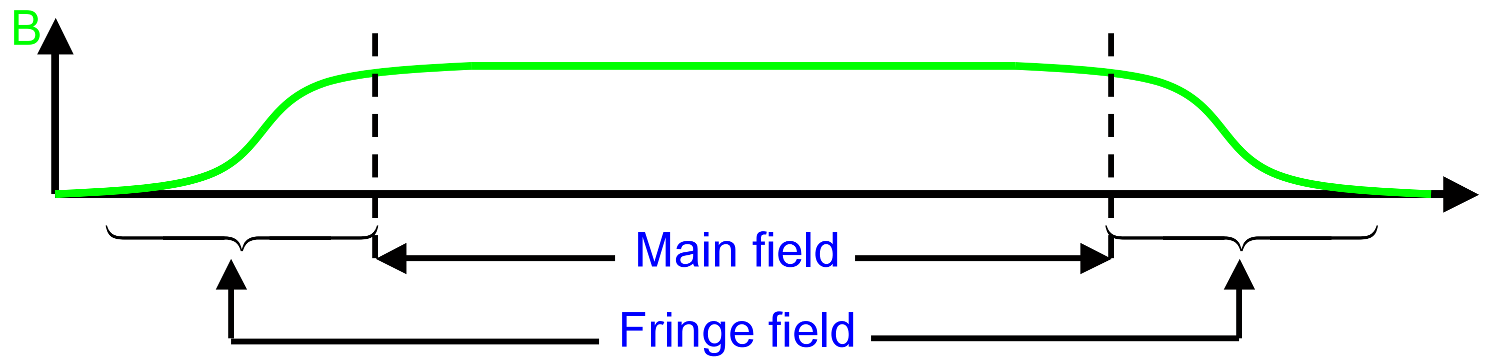
Complex Potentials

$$w = x + iy \quad , \quad \overline{w} = x - iy$$

$$\partial_x = \partial_w + \partial_{\overline{w}} \; , \quad \partial_y = i\partial_w - i\partial_{\overline{w}} = i(\partial_w - \partial_{\overline{w}})$$

$$\vec{\nabla}^2 = \partial_x^2 + \partial_y^2 + \partial_z^2 = (\partial_w + \partial_{\overline{w}}) - (\partial_w - \partial_{\overline{w}}) + \partial_z^2 = 4\partial_w \partial_{\overline{w}} + \partial_z^2$$



In the main field:

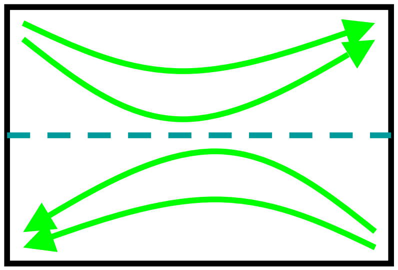
$$\psi(r,\varphi) = \sum_{\nu=1}^{\infty} \mathrm{Im}\{\Psi_\nu \overline{w}^\nu\} + |\Psi_0| - \tfrac{1}{4}|\Psi_0''| w\overline{w} + \dots$$

Main Field Potential

The isolated multipole: $\psi = -r^{\nu}|\Psi_{\nu}|\sin(\nu[\varphi - \vartheta_{\nu}])$

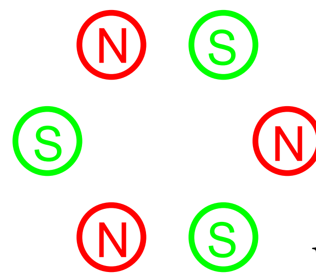
Where the rotation ϑ_{ν} of the coordinate system is set to 0 for midplane symmetry

$$B_x(x,-y,z) = -B_x(x,y,z)$$
$$B_y(x,-y,z) = \quad B_y(x,y,z)$$
$$B_z(x,-y,z) = -B_z(x,y,z)$$



$$\psi(x,y,z) = -\psi(x,-y,z)$$

The index ν describes C_{ν} Symmetry around the z-axis $\vec{e}_z$ due to a sign change after $\Delta\varphi = \frac{\pi}{\nu}$



$\nu = 3$

The potentials of different multipole components Ψ_{ν} have

a) Different rotation symmetry C_{ν}

b) Different radial dependence r^{ν}

Multipoles in Accelerators

Dipole:

$$\psi = \Psi_1 \,\mathrm{Im}\{x - iy\} = \Psi_1 \cdot (-y) \quad \Rightarrow \quad \vec{B} = -\vec{\nabla}\psi = \Psi_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Quadrupole:

$$\psi = \Psi_2 \,\mathrm{Im}\{(x - iy)^2\} = \Psi_3 \cdot (-2xy) \quad \Rightarrow \quad \vec{B} = -\vec{\nabla}\psi = \Psi_3 \, 2 \begin{pmatrix} y \\ x \end{pmatrix}$$

Sextupole:

$$\psi = \Psi_3 \,\mathrm{Im}\{(x - iy)^3\} = \Psi_3 \cdot (y^3 - 3x^2 y) \quad \Rightarrow \quad \vec{B} = -\vec{\nabla}\psi = \Psi_3 \, 3 \begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix}$$

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The CESR Tunnel

Real Quadrupoles

The coils show that this is an upright quadrupole not a rotated or skew quadrupole.

Real Sextupoles

Higher order Multipoles

$$\psi = \Psi_n \,\mathrm{Im}\{(x - iy)^n\} = \Psi_n \cdot (\ldots - i\, n\, x^{n-1} y) \quad \Rightarrow \quad \vec{B}(y=0) = \Psi_n\, n \begin{pmatrix} 0 \\ x^{n-1} \end{pmatrix}$$

Multipole strength: $k_n = \frac{q}{p} \partial_x^n B_y \Big|_{x,y=0} = \frac{q}{p} \Psi_{n+1}\,(n+1)!$ units: $\frac{1}{\mathrm{m}^{n+1}}$

p/q is also called Bρ and used to describe the energy of multiply charge ions

Names: dipole, quadrupole, sextupole, octupole, decapole, duodecapole, …

Higher order multipoles come from

- Field errors in magnets
- Magnetized materials
- From multipole magnets that compensate such erroneous fields
- To compensate nonlinear effects of other magnets
- To stabilize the motion of many particle systems
- To stabilize the nonlinear motion of individual particles

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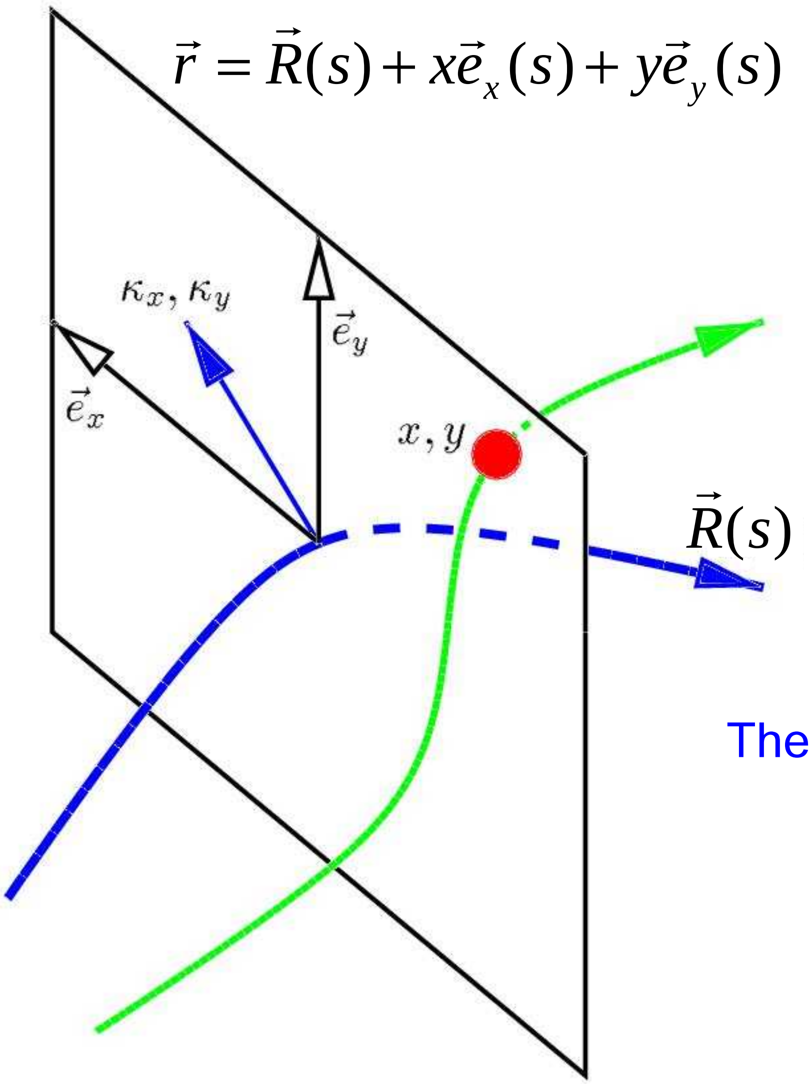
The comoving Coordinate System

$$\vec{r} = \vec{R}(s) + x\vec{e}_x(s) + y\vec{e}_y(s)$$



$$|d\vec{R}| = ds$$

$$\vec{e}_s \equiv \tfrac{d}{ds}\vec{R}(s)$$

The time dependence of a particle's motion is often not as interesting as the trajectory along the accelerator length "s".

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The 4D Equation of Motion

$$\frac{d^2}{dt^2}\vec{r} = \vec{f}_r(\vec{r}, \tfrac{d}{dt}\vec{r}, t)$$

3 dimensional ODE of 2nd order can be changed to a

6 dimensional ODE of 1st order:

$$\left.\begin{aligned} \frac{d}{dt}\vec{r} &= \frac{1}{m\gamma}\vec{p} = \frac{c}{\sqrt{p^2-(mc)^2}}\vec{p} \\ \frac{d}{dt}\vec{p} &= \vec{F}(\vec{r}, \vec{p}, t) \end{aligned}\right\} \quad \frac{d}{dt}\vec{Z} = \vec{f}_Z(\vec{Z}, t), \quad \vec{Z} = (\vec{r}, \vec{p})$$

If the force does not depend on time, as in a typical beam line magnet, the energy is conserved so that one can reduce the dimension to 5. The equation of motion is then autonomous.

Furthermore, the time dependence is often not as interesting as the trajectory along the accelerator length “s”. Using “s” as the independent variable reduces the dimensions to 4. The equation of motion is then no longer autonomous.

$$\frac{d}{ds}\vec{z} = \vec{f}_z(\vec{z}, s), \quad \vec{z} = (x, y, p_x, p_y)$$

The 6D Equation of Motion

Usually one prefers to compute the trajectory as a function of "s" along the accelerator even when the energy is not conserved, as when accelerating cavities are in the accelerator.

Then the energy "E" and the time "t" at which a particle arrives at the cavities are important. And the equations become 6 dimensional again:

$$\tfrac{d}{ds}\vec{z} = \vec{f}_z(\vec{z}, s)\,, \quad \vec{z} = (x, y, p_x, p_y, -t, E)$$

But: $\vec{z} = (\vec{r}, \vec{p})$ is an especially suitable variable, since it is a phase space vector so that its equation of motion comes from a Hamiltonian, or by variation principle from a Lagrangian.

$$\delta\int\left[p_x\dot{x} + p_y\dot{y} + p_s\dot{s} - H(\vec{r}, \vec{p}, t)\right]dt = 0 \quad \Rightarrow$$ Hamiltonian motion

$$\delta\int\left[p_x x' + p_y y' - H\,t' + p_s(x, y, p_x, p_y, t, H)\right]ds = 0 \quad \Rightarrow$$ Hamiltonian motion

The new canonical coordinates are: $\vec{z} = (x, y, p_x, p_y, -t, E)$ with $\mathrm{E} = \mathrm{H}$

The new Hamiltonian is: $K = -p_s(\vec{z}, s)$

6 Dimensional Phase Space

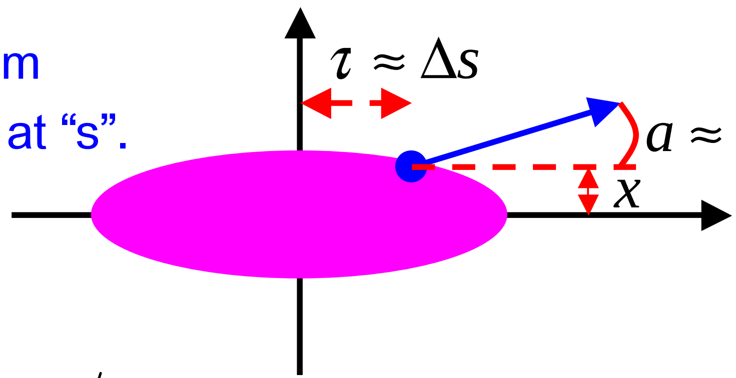
Using a reference momentum p_0 and a reference time t_0:

$$\vec{z} = (x, a, y, b, \tau, \delta)$$

$$a = \frac{p_x}{p_0}, \quad b = \frac{p_y}{p_0}, \quad \delta = \frac{E - E_0}{E_0}, \quad \tau = (t_0 - t)\frac{c^2}{v_0} = (t_0 - t)\frac{E_0}{p_0}$$

Usually p_0 is the design momentum of the beam

And t_0 is the time at which the bunch center is at “s”.



$$\left.\begin{aligned} x' &= \partial_{p_x} K \\ p_x' &= -\partial_x K \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} x' &= \partial_a K/p_0\,, \quad a' = -\partial_x K/p_0 \\ y' &= \partial_b K/p_0\,, \quad b' = -\partial_y K/p_0 \end{aligned}\right.$$

$$-t' = \partial_E K \quad \Rightarrow \quad \tau' = \tfrac{c^2}{v_0}\partial_\delta K/E_0 = \partial_\delta K/p_0$$

$$E' = -\partial_{-t} K \quad \Rightarrow \quad \delta' = -\tfrac{1}{E_0}\partial_\tau K\,\tfrac{c^2}{v_0} = -\partial_\tau K/p_0$$

New Hamiltonian:

$$\tilde{H} = K/p_0$$

Simplified Equation of Motion

Only bend in the horizontal plane: $\kappa_y = 0\,, \quad \kappa_x = \kappa = 1/\rho$

Only magnetic fields: $\vec{E} = 0$

Mid-plane symmetry: $B_x(x,y,s) = -B_x(x,-y,s)\,, \quad B_y(x,y,s) = B_y(x,-y,s)$

$$a' = -x\kappa^2 - \tfrac{q}{p_0}\partial_x B_y x + \delta\beta_0^{-2}\kappa \quad \Rightarrow \quad x'' = -x(\kappa^2 + k) + \delta\beta_0^{-2}\kappa$$

$$b' = \tfrac{q}{p_0}\partial_y B_x y \quad \Rightarrow \quad y'' = k\,y$$

$$\tau' = -x\beta_0^{-2}\kappa + \tfrac{1}{\gamma_0^2}\beta_0^{-4}\delta$$

$$\delta' = 0$$

Hamiltonian:

$$H = \tfrac{1}{2}a^2 + \tfrac{1}{2}b^2 + \tfrac{1}{2}k(x^2 - y^2) + \tfrac{1}{2}\kappa^2 x^2 - \beta_0^{-2}\kappa x\delta + \tfrac{1}{2}\tfrac{1}{\gamma_0^2}\beta_0^{-4}\delta^2$$

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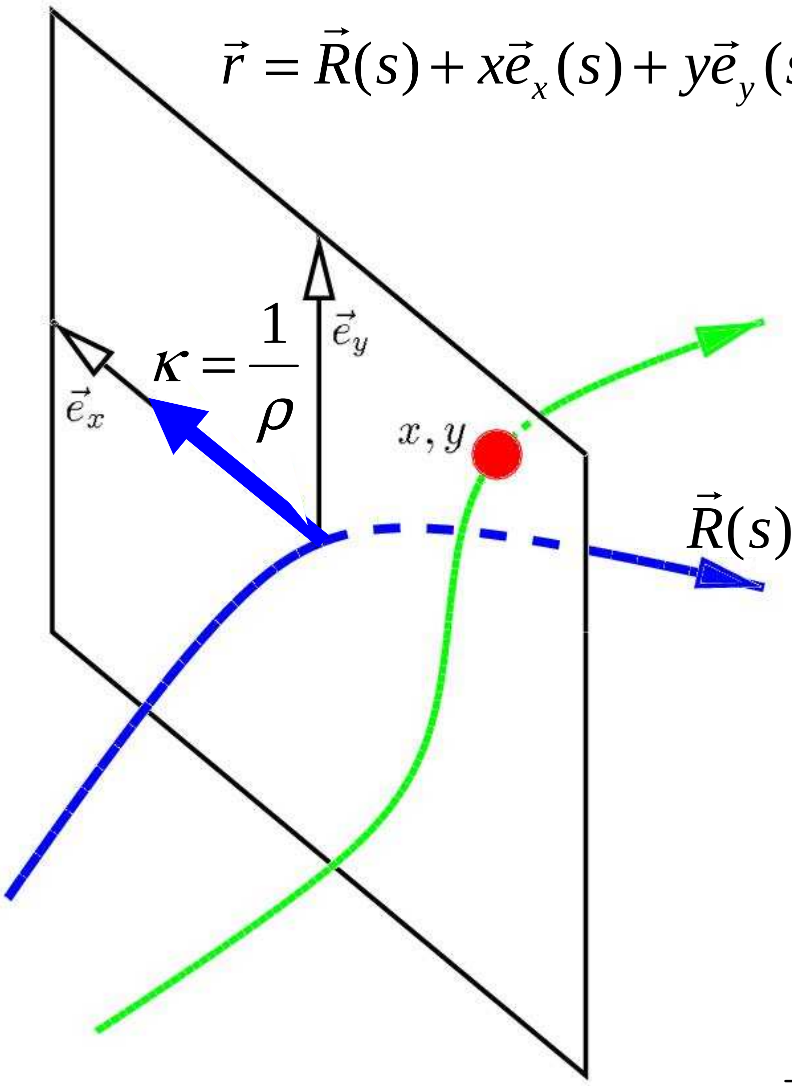
The Curvilinerar System for κ_y=0



$$\vec r = \vec R(s) + x\vec e_x(s) + y\vec e_y(s)$$

$$\frac{d}{ds}\vec e_s = -\kappa_x \vec e_x$$

$$\frac{d}{ds}\vec e_x = \kappa_x \vec e_s$$

$$\frac{d}{ds}\vec e_y = 0$$

$$\frac{d}{ds}\vec r = x'\vec e_x + y'\vec e_y + \underbrace{(1 + x\kappa_x)}_{h}\vec e_s$$

$$\left(\frac{d\vec r}{dt}\right)_s = h\frac{ds}{dt}$$

$$\vec p = p_s\vec e_s(s) + p_x\vec e_x(s) + p_y\vec e_y(s)$$

$$\frac{d}{ds}\vec p = (p_x' - p_s\kappa)\vec e_x + p_y'\vec e_y + (p_s' + p_x\kappa)\vec e_s$$

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Motivation for v=c

$$x'' = -x(\kappa^2 + k) + \delta\kappa$$
$$y'' = k\,y$$
$$\tau' = -\kappa x$$
$$\delta' = 0$$

$$\tfrac{d}{dt}\vec{p} = q\tfrac{d}{dt}\vec{r}\times\vec{B}\,,\quad \vec{B} = \tfrac{p}{q}\kappa\begin{pmatrix}0\\1\\0\end{pmatrix} + \tfrac{p}{q}k\begin{pmatrix}y\\x\\0\end{pmatrix}$$

$$\tfrac{d}{dt}p_y = \tfrac{p_s}{m\gamma}pk\,y \quad \& \quad \left\{\begin{array}{l} ds = c\,dt + O \\ p = m\gamma c \\ p_s = m\gamma c + O^2 \end{array}\right\}, \quad \Rightarrow \quad y'' = k\,y + O^2$$

$$\left(\tfrac{d}{dt}\vec{p}\right)_x = \tfrac{ds}{dt}\left(\tfrac{d}{ds}p_x - p_s\kappa\right) = -\left(\tfrac{d\vec{r}}{dt}\right)_s p(\kappa + kx) = -h\tfrac{ds}{dt}\,p(\kappa + kx)$$

$$\Rightarrow \quad \tfrac{d}{ds}p_x = -p[(h-1)\kappa + kx] + O^2 \quad \Rightarrow \quad x'' = -(\kappa^2 + k)x + O^2$$

$$\tau = \int_0^s [1 - \sqrt{h^2 + x'^2 + y'^2}]ds = -\int_0^s \kappa x\,ds$$

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Significance of Hamiltonian

The equations of motion can be determined by one function:

$$\tfrac{d}{ds}x = \partial_{p_x} H(\vec{z},s)\,, \quad \tfrac{d}{ds}p_x = -\partial_x H(\vec{z},s)\,, \quad \ldots$$

$$\tfrac{d}{ds}\vec{z} = \underline{J}\,\vec{\partial} H(\vec{z},s) = \vec{F}(\vec{z},s) \quad \text{with} \quad \underline{J} = \mathrm{diag}(\underline{J}_2)\,, \quad \underline{J}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The force has a Hamiltonian Jacobi Matrix:

A linear force: $\vec{F}(\vec{z},s) = \underline{F}(s)\cdot\vec{z}_0$

The Jacobi Matrix of a linear force: $\underline{F}(s)$

The general Jacobi Matrix : $F_{ij} = \partial_{z_j} F_i$ or $\underline{F} = \left(\vec{\partial}\,\vec{F}^T\right)^T$

Hamiltonian Matrices: $\underline{F}\,\underline{J} + \underline{J}\,\underline{F}^T = 0$

Proof : $F_{ij} = \partial_{z_j} F_i = \partial_{z_j} J_{ik}\partial_{z_k} H = J_{ik}\,\partial_k \partial_j H \quad \Rightarrow \quad \underline{F} = \underline{J}\,\underline{DH}$

$$\underline{F}\,\underline{J} + \underline{J}\,\underline{F}^T = \underline{J}\,\underline{DJH} + \underline{J}\,\underline{D}^T\,\underline{J}^T H = 0$$

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Hamiltonian F ↦ Hamiltonian H

Sofar shown: Hamiltonian ↦ Hamiltonian F

$$\vec{z}' = \vec{F}(\vec{z}, s) \quad \text{with} \quad F_{ij} = \partial_{z_j} F_i \quad \text{and} \quad \underline{F}\,\underline{J} + \underline{J}\,\underline{F}^T = 0$$

The vector $\vec{h} = \underline{J}\vec{F}$ therefore has a symmetric Jacobian $h_{ij} = \partial_{z_j} J_{ik} F_k = J_{ik} F_{kj}$

$$\underline{h} = \underline{J}\,\underline{F} = -\underline{F}^T \underline{J} = \underline{h}^T$$

The potential theorem:

Any vector valued function $\vec{h}(\vec{z})$ with $\partial_{z_j} h_i(\vec{z}) = \partial_{z_i} h_j(\vec{z})$

can be written as the gradient of a potential: $\vec{h}(\vec{z}) = \vec{\partial} H(\vec{z})$

Where H is the path independent integral $H(\vec{z}) = \int_0^{\vec{z}} \vec{h}(\vec{z}_0) d\vec{z}_0$

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H ↦ Symplectic Flows

The flow of a Hamiltonian equation of motion has a symplectic Jacobi Matrix

The flow or transport map: $\vec{z}(s) = \vec{M}(s, \vec{z}_0)$

A linear flow: $\vec{z}(s) = \underline{M}(s) \cdot \vec{z}_0$

The Jacobi Matrix of a linear flow: $\underline{M}(s)$

The general Jacobi Matrix : $M_{ij} = \partial_{z_{0j}} M_i$ or $\underline{M} = (\vec{\partial}_0 \vec{M}^T)^T$

The Symplectic Group SP(2N) : $\underline{M}\,\underline{J}\,\underline{M}^T = \underline{J}$

$$\tfrac{d}{ds}\vec{z} = \tfrac{d}{ds}\vec{M}(s, \vec{z}_0) = \underline{J}\,\vec{\nabla} H = \vec{F}$$

$$\tfrac{d}{ds} M_{ij} = \partial_{z_{0j}} F_i(\vec{z}, s) = \partial_{z_{0j}} M_k \partial_{z_k} F_i(\vec{z}, s)$$

$$\tfrac{d}{ds}\underline{M}(s, \vec{z}_0) = \underline{F}(\vec{z}, s)\underline{M}(s, \vec{z}_0)$$

$$K = \underline{M}\,\underline{J}\,\underline{M}^T$$

$$\tfrac{d}{ds}\underline{K} = \tfrac{d}{ds}\underline{M}\,\underline{J}\,\underline{M}^T + \underline{M}\,\underline{J}\,\tfrac{d}{ds}\underline{M}^T = \underline{F}\,\underline{M}\,\underline{J}\,\underline{M}^T + \underline{M}\,\underline{J}\,\underline{M}^T\underline{F}^T = \underline{F}\,\underline{K} + \underline{K}\,\underline{F}^T$$

$\underline{K} = \underline{J}$ is a solution. Since this is a linear ODE , $\underline{K} = \underline{J}$ is the unique solution.

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Symplectic Flows ↦ H

For every symplectic transport map there is a Hamilton function

The flow or transport map: $\vec{z}(s) = \vec{M}(s, \vec{z}_0)$

Vector to compute force: $\vec{h}(\vec{z}, s) = -\underline{J}\left[\frac{d}{ds}\vec{M}(s, \vec{z}_0)\right]_{\vec{z}_0 = \vec{M}^{-1}(\vec{z}, s)}$

Since then: $\frac{d}{ds}\vec{z} = \underline{J}\vec{h}(\vec{z}, s)$

There is a Hamilton function H with: $\vec{h} = \vec{\partial} H$

If and only if: $\partial_{z_j} h_i = \partial_{z_i} h_j \quad \Rightarrow \quad \underline{h} = \underline{h}^T$

$$\underline{M}\,\underline{J}\,\underline{M}^T = \underline{J} \quad \Rightarrow \quad \begin{cases} \frac{d}{ds}\underline{M}\,\underline{J}\,\underline{M}^T = -\underline{M}\,\underline{J}\frac{d}{ds}\underline{M}^T \\ \underline{M}^{-1} = -\underline{J}\,\underline{M}^T\underline{J} \end{cases}$$

$$\vec{h} \circ \vec{M} = -\underline{J}\frac{d}{ds}\vec{M}$$

$$\underline{h}(\vec{M})\underline{M} = -\underline{J}\frac{d}{ds}\underline{M}$$

$$\underline{h}(\vec{M}) = -\underline{J}\frac{d}{ds}\underline{M}\,\underline{M}^{-1} = \underline{J}\frac{d}{ds}\underline{M}\,\underline{J}\,\underline{M}^T\underline{J} = -\underline{J}\,\underline{M}\,\underline{J}\frac{d}{ds}\underline{M}^T\underline{J} = \underline{M}^{-T}\frac{d}{ds}\underline{M}^T\underline{J} = \underline{h}^T$$

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Hamiltonian and Symplecticity

Note that the exponent of a Hamiltonian matrix is symplectic.

$$\vec{z}\,' = \underline{F}\,\vec{z} \quad \Rightarrow \quad \vec{z}(s) = \sum_{n=0}^{\infty} \frac{1}{n!} s^n \vec{z}^{[n]}(0) = \sum_{n=0}^{\infty} \frac{1}{n!} s^n \underline{F}^n \vec{z}_0 = e^{s\underline{F}} \vec{z}_0$$

$$\underline{M}(s) = e^{s\underline{F}}$$

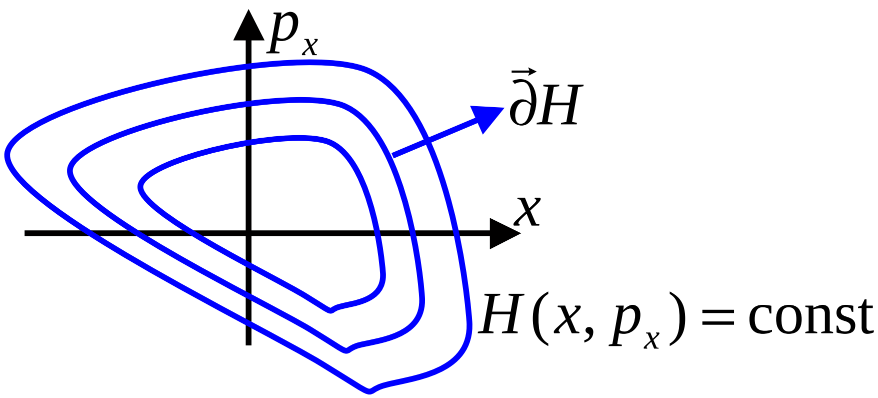
In fact, any symplectic matrix (that has a logarithm) can be written as the exponent of a Hamiltonian matrix.

$$e^{s\underline{F}} \underline{J} e^{s\underline{F}^T} = \underline{J} \quad \Rightarrow \quad \underline{F} e^{s\underline{F}} \underline{J} e^{s\underline{F}^T} + e^{s\underline{F}} \underline{J}\,\underline{F}^T e^{s\underline{F}^T} = 0 \quad \Rightarrow \quad \underline{F}\,\underline{J} + \underline{J}\,\underline{F}^T = 0$$

And every symplectic matrix has a logarithm.

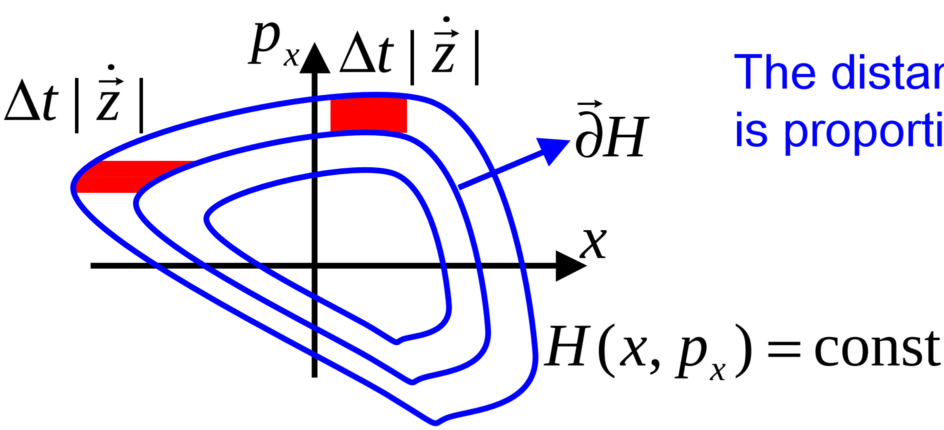
Phase space density in 2D

- Phase space trajectories move on surfaces of constant energy



$$\tfrac{d}{ds}\vec{z} = \underline{J}\,\vec{\partial}H \quad \Rightarrow \quad \tfrac{d}{ds}\vec{z} \perp \vec{\partial}H$$

- A phase space volume does not change when it is transported by Hamiltonian motion.



The distance d of lines with equal energy is proportional to $1/|\vec{\partial}H| \propto |\dot{\vec{z}}\,|^{-1}$

$$d * \Delta t\,|\dot{\vec{z}}\,| = \text{const}$$

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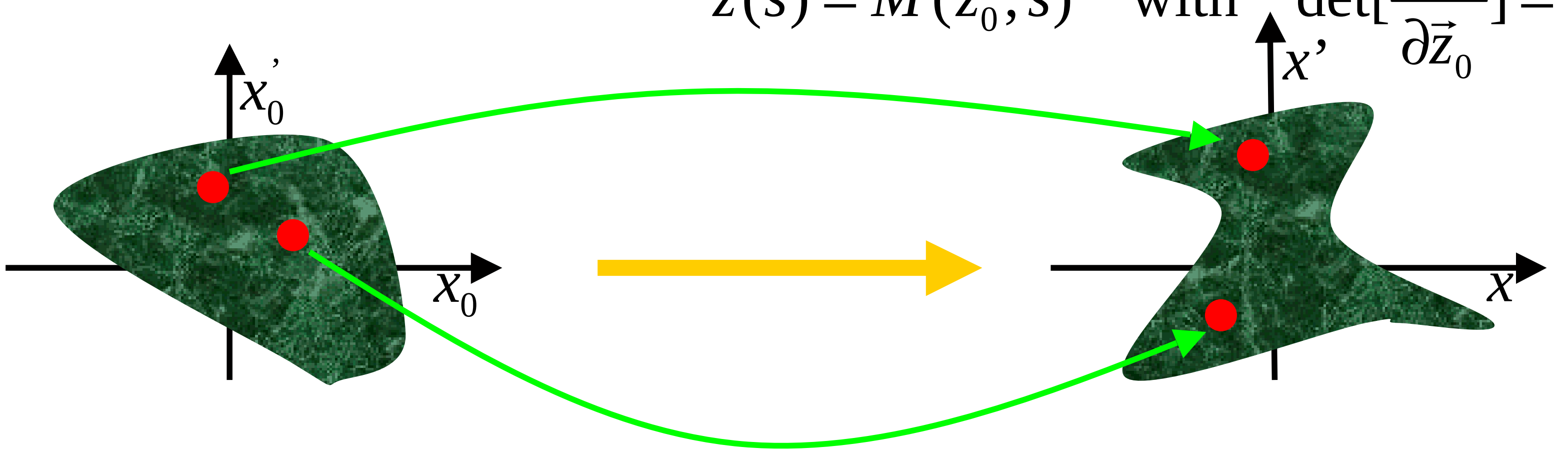
Lioville's Theorem

- A phase space volume does not change when it is transported by Hamiltonian motion.

$$\vec{z}(s) = \vec{M}(\vec{z}_0, s) \quad \text{with} \quad \det[\frac{\partial \vec{M}}{\partial \vec{z}_0}] = +1$$



Volume = $V = \iint_V d^n \vec{z} = \iint_{V_0} \left| \frac{\partial \vec{z}}{\partial \vec{z}_0} \right| d^n \vec{z}_0 = \iint_{V_0} |\underline{M}| d^n \vec{z}_0 = \iint_{V_0} d^n \vec{z}_0 = V_0$

Hamiltonian Motion → $V = V_0$

But Hamiltonian requires symplecticity, which is much more than just $\det[\underline{M}(s)] = +1$

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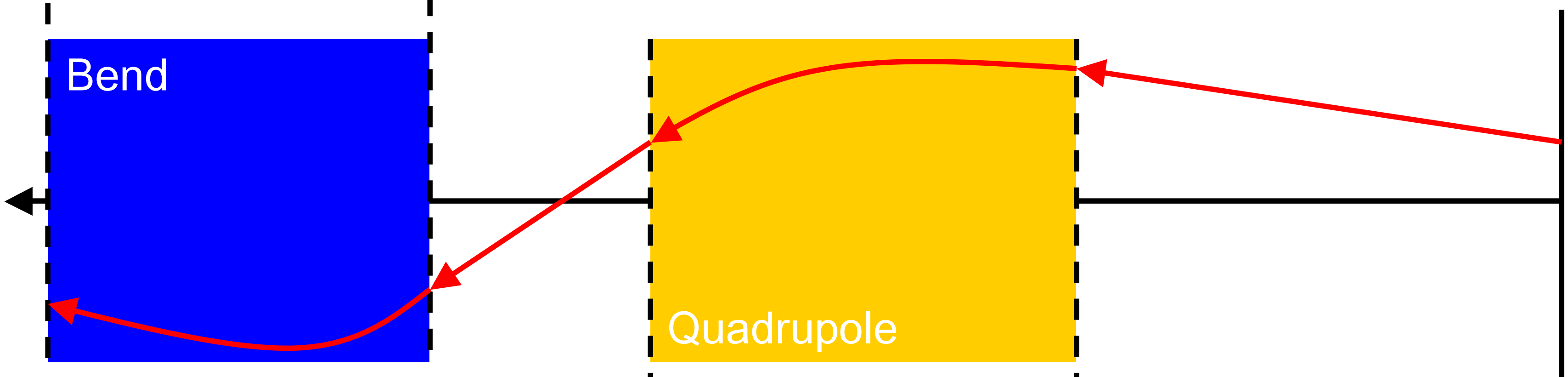
Matrix Solutions

Linear equation of motion: $\vec{z}' = \underline{F}(s)\vec{z}$

Matrix solution of the starting condition $\vec{z}(0) = \vec{z}_0$

$$\vec{z} = \underline{M}_{\text{bend}}(L_4)\underline{M}_{\text{drift}}(L_3)\underline{M}_{\text{quad}}(L_2)\underline{M}_{\text{drift}}(L_1)\vec{z}_0$$

$$\vec{z} = \underline{M}_{\text{drift}}(L_3)\underline{M}_{\text{quad}}(L_2)\underline{M}_{\text{drift}}(L_1)\vec{z}_0$$



$$\vec{z} = \underline{M}_{\text{drift}}(L_1)\vec{z}_0 \qquad \vec{z}_0$$

$$\vec{z} = \underline{M}_{\text{quad}}(L_2)\underline{M}_{\text{drift}}(L_1)\vec{z}_0$$

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The Drift

$$\begin{pmatrix} x' \\ a' \\ y' \\ b' \\ \tau' \\ \delta' \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ b \\ 0 \\ \frac{1}{\gamma_0^2}\beta_0^{-4}\delta \\ 0 \end{pmatrix}$$

Note that in nonlinear expansion $x' \neq a$ so that the drift does not have a linear transport map even though $x(s) = x_0 + x_0' s$ is completely linear.

$$\frac{1}{\gamma^2} << 1 \quad \Rightarrow \quad \begin{pmatrix} x \\ a \\ y \\ b \\ \tau \\ \delta \end{pmatrix} = \begin{pmatrix} x_0 + s a_0 \\ a \\ y_0 + s b_0 \\ b_0 \\ \tau_0 \\ \delta_0 \end{pmatrix} = \begin{pmatrix} 1 & s & \underline{0} & & \underline{0} & \\ 0 & 1 & & & & \\ \underline{0} & & 1 & s & \underline{0} & \\ & & 0 & 1 & & \\ \underline{0} & & \underline{0} & & 1 & 0 \\ & & & & 0 & 1 \end{pmatrix} \vec{z}_0$$

The Dipole Equation of Motion

$$\frac{1}{\gamma^2} << 1 \quad \Rightarrow \quad \begin{aligned} x'' &= -x\,\kappa^2 + \delta\kappa \\ y'' &= 0 \\ \tau' &= -x\,\kappa \end{aligned}$$

Homogeneous solution:

$$x_H'' = -x_H\,\kappa^2 \quad \Rightarrow \quad x_H = A\cos(\kappa s) + B\sin(\kappa s)$$ (natural ring focusing)

Variation of constants:

$$x = A(s)\cos(\kappa\, s) + B(s)\sin(\kappa\, s)$$

$$x' = -A\kappa\sin(\kappa s) + B\kappa\cos(\kappa s) + \underbrace{A'\cos(\kappa s) + B'\sin(\kappa s)}_{\equiv 0}$$

$$x'' = -\kappa^2 x \underbrace{- A'\kappa\sin(\kappa s) + B'\kappa\cos(\kappa s)}_{=\delta\kappa} = -\kappa^2 x + \delta\kappa$$

$$\begin{pmatrix} \cos(\kappa s) & \sin(\kappa s) \\ -\sin(\kappa s) & \cos(\kappa s) \end{pmatrix} \begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} 0 \\ \delta\beta_0^{-2} \end{pmatrix}$$

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The Dipole

$$\begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} \cos(\kappa s) & -\sin(\kappa s) \\ \sin(\kappa s) & \cos(\kappa s) \end{pmatrix} \begin{pmatrix} 0 \\ \delta \end{pmatrix}$$

$$\begin{pmatrix} A \\ B \end{pmatrix} = \delta\, \kappa^{-1} \begin{pmatrix} \cos(\kappa s) \\ \sin(\kappa s) \end{pmatrix} + \begin{pmatrix} A_H \\ B_H \end{pmatrix} \quad \text{with} \quad x = A\cos(\kappa s) + B\sin(\kappa s)$$

$$\tau' = -x\kappa$$

$$\underline{M} = \begin{pmatrix} \begin{matrix} \cos(\kappa s) & \frac{1}{\kappa}\sin(\kappa s) \\ -\kappa\sin(\kappa s) & \cos(\kappa s) \end{matrix} & \underline{0} & \begin{matrix} 0 & \kappa^{-1}[1-\cos(\kappa s)] \\ 0 & \sin(\kappa s) \end{matrix} \\ \underline{0} & \begin{matrix} 1 & s \\ 0 & 1 \end{matrix} & \underline{0} \\ \begin{matrix} -\sin(\kappa s) & \kappa^{-1}[\cos(\kappa s)-1] \\ 0 & 0 \end{matrix} & \underline{0} & \begin{matrix} 1 & \kappa^{-1}[\sin(\kappa s)-s\kappa] \\ 0 & 1 \end{matrix} \end{pmatrix}$$

Time of Flight from Symplecticity

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$$\underline{M} = \begin{pmatrix} \underline{M}_4 & \vec{0} & \vec{D} \\ \vec{T}^T & 1 & M_{56} \\ \vec{0}^T & 0 & 1 \end{pmatrix}$$

is in SU(6) and therefore $\underline{M}\,\underline{J}\,\underline{M}^T = \underline{J}$

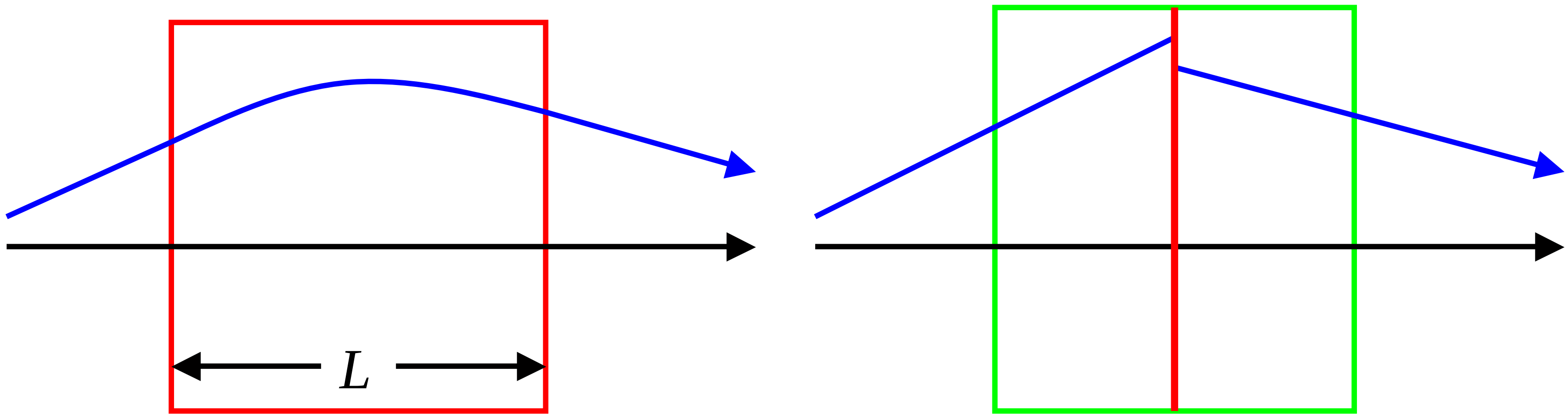
$$\begin{pmatrix} \underline{M}_4 \underline{J}_4 & -\vec{D} & \vec{0} \\ \vec{T}^T \underline{J}_4 & -M_{56} & 1 \\ \vec{0}^T & -1 & 0 \end{pmatrix} \begin{pmatrix} \underline{M}_4^T & \vec{T} & \vec{0} \\ \vec{0}^T & 1 & 0 \\ \vec{D}^T & M_{56} & 1 \end{pmatrix} = \begin{pmatrix} \underline{J}_4 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \underline{M}_4 \underline{J}_4 \underline{M}_4^T & \underline{M}_4 \underline{J}_4 \vec{T} - \vec{D} & \vec{0} \\ \vec{T}^T \underline{J}_4 \underline{M}_4^T + \vec{D}^T & 0 & 1 \\ \vec{0}^T & -1 & 0 \end{pmatrix} = \begin{pmatrix} \underline{J}_4 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\vec{T} = -\underline{J}_4 \underline{M}_4^{-1} \vec{D}$$

It is sufficient to compute the 4D map $\underline{M}_4$, the Dispersion $\vec{D}$ and the time of flight term M_{56}

The Thin Lens Approximation



$$x'' = [-(k+\kappa^2)x + \kappa\,\delta]L\,\delta_D(s)$$

$$y'' = k\,y\,L\,\delta_D(s)$$

$$\tau' = -x\,\kappa\,L\,\delta_D(s)$$

$$\underline{M}^{\text{thin}}_{0^- \to 0^+} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -(k+\kappa^2)L & 1 & 0 & 0 & 0 & \kappa L \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & k L & 1 & 0 & 0 \\ -\kappa L & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\underline{M} \approx \underline{D}(\tfrac{L}{2})\underline{M}^{\text{thin}}_{0^- \to 0^+}\underline{D}(\tfrac{L}{2})$$

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Thin Combined Function Bend

$$\underline{M}_6 = \begin{pmatrix} \underline{M}_x & \underline{0} & \vec{0}\,\vec{D} \\ \underline{0} & \underline{M}_y & \underline{0} \\ \underline{0} & \underline{0} & \underline{1} \end{pmatrix}$$

Weak magnet limit: $\kappa s << 1$

$$\underline{M}_x = \begin{pmatrix} \cos(\sqrt{K}\,s) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}\,s) \\ -\sqrt{K}\sin(\sqrt{K}\,s) & \cos(\sqrt{K}\,s) \end{pmatrix} \qquad \underline{M}_x^{\text{thin}} = \begin{pmatrix} 1 & 0 \\ -K\,s & 1 \end{pmatrix}$$

$$\underline{M}_y = \begin{pmatrix} \cosh(\sqrt{k}\,s) & \frac{1}{\sqrt{k}}\sinh(\sqrt{k}\,s) \\ \sqrt{k}\sinh(\sqrt{k}\,s) & \cosh(\sqrt{k}\,s) \end{pmatrix} \longrightarrow \underline{M}_y^{\text{thin}} = \begin{pmatrix} 1 & 0 \\ k\,s & 1 \end{pmatrix}$$

$$\vec{D} = \begin{pmatrix} \frac{\kappa}{K}[1-\cos(\sqrt{K}s)] \\ \frac{\kappa}{\sqrt{K}}\sin(\sqrt{K}s) \end{pmatrix} \qquad \vec{D} = \begin{pmatrix} 0 \\ \kappa\,s \end{pmatrix}$$

Beta Function and Betatron Phase

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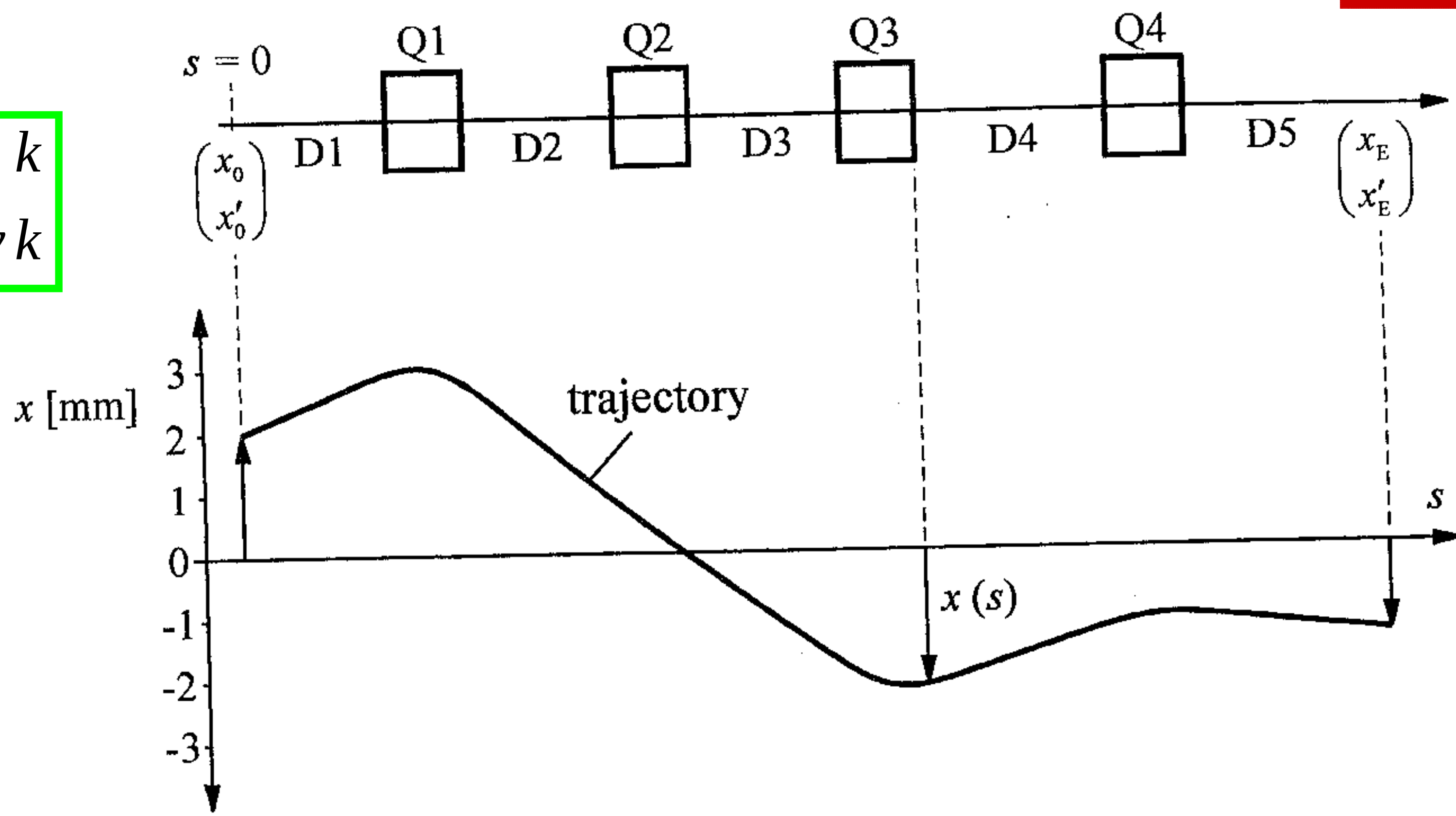
$$x'' = -x\,k$$
$$y'' = \;\; y\,k$$



$$x(s) = M_{11}(s)x_0 + M_{12}(s)x_0'$$
$$x(s) = \sqrt{2J\beta(s)}\,\sin(\psi(s) + \phi_0)$$

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Twiss Parameters

$$\left.\begin{aligned} &x'' = -k\,x \\ &x(s) = \sqrt{2J\beta(s)}\,\sin(\psi(s)+\phi_0) \\ &\psi' = \tfrac{1}{\beta} \\ &\beta' = -2\alpha \end{aligned}\right\} \quad \alpha' + \gamma = k\beta$$

$\alpha, \beta, \gamma, \psi$ are called Twiss parameters.

What are the initial conditions?

In linear accelerators:

The beam distribution is used to define the initial twiss parameters.

$$\beta(0) = \beta_0$$
$$\alpha(0) = \alpha_0$$

In ring accelerators:

In equilibrium the beam distribution Is periodic and therefore periodic Boundary conditions are used.

$$\beta(s+L) = \beta(s)$$

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Phase Space Ellipse

Particles with a common J and different ϕ all lie on an ellipse in phase space:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \sqrt{2J}\begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix}\begin{pmatrix} \sin(\psi(s)+\phi_0) \\ \cos(\psi(s)+\phi_0) \end{pmatrix}$$

(Linear transform of a circle)

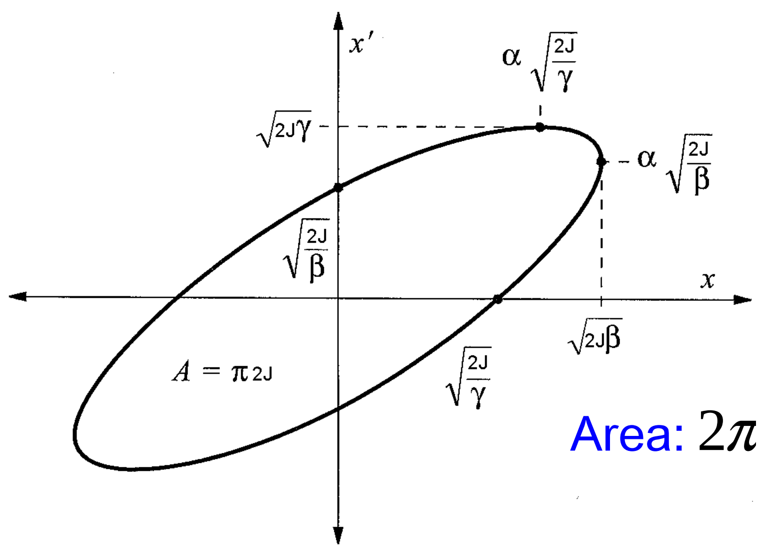
$$x_{\max} = \sqrt{2J\beta} \text{ at } x' = -\alpha\sqrt{\tfrac{2J}{\beta}}$$

$$(x,x')\begin{pmatrix} \frac{1}{\sqrt{\beta}} & \frac{\alpha}{\sqrt{\beta}} \\ 0 & \sqrt{\beta} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix}\begin{pmatrix} x \\ x' \end{pmatrix} = (x,x')\begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix}\begin{pmatrix} x \\ x' \end{pmatrix} = 2J$$

(Quadratic form)

$$\beta\gamma - \alpha^2 = 1$$

Area: $2\pi J$



What β is for x, γ is for x'

$$x'_{\max} = \sqrt{2J\gamma} \text{ at } x = -\alpha\sqrt{\tfrac{2J}{\gamma}}$$

Area: $2\pi J$ → $\int_0^{2\pi}\int_0^{J} dJ\,d\phi = 2\pi J = \iint dx\,dx'$

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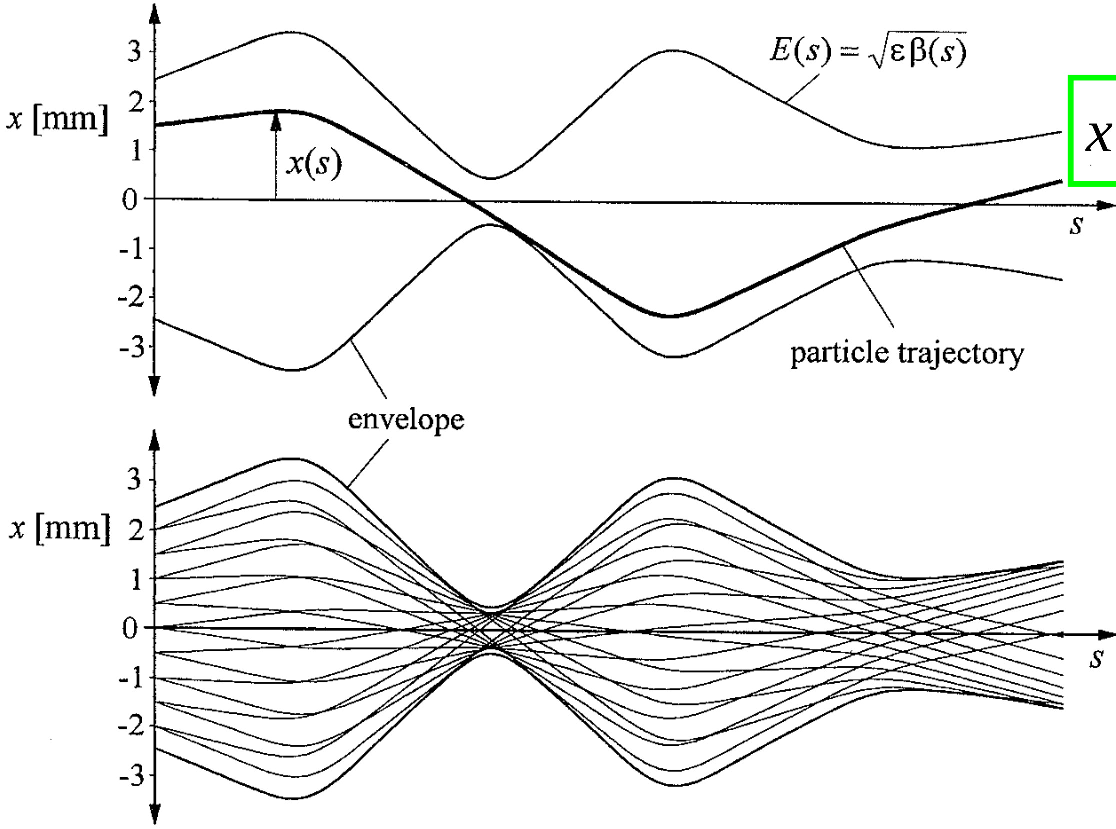
The Beam Envelope



$$x(s)=\sqrt{2J\beta(s)}\sin(\psi(s)+\phi_0)$$

In any beam there is a distribution of initial parameters. If the particles with the largest J are distributed in ϕ over all angles, then the envelope of the beam is described by $\sqrt{2J_{max}\beta(s)}$

The initial conditions of β and α are chosen so that this is approximately the case.

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Invariant of Motion

$$x(s) = \sqrt{2J\beta(s)}\sin(\psi(s) + \phi_0)$$

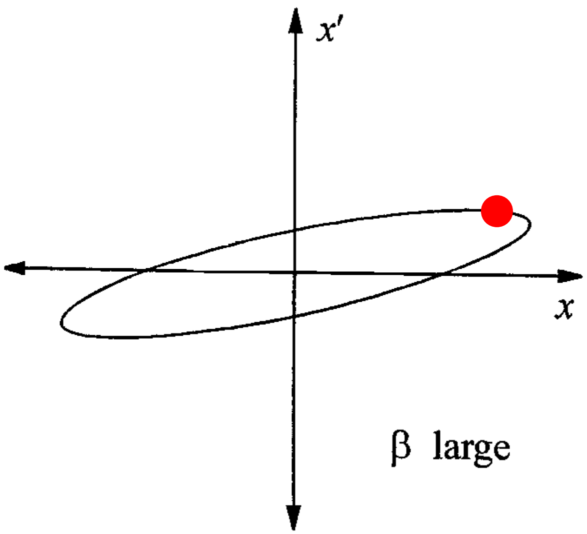
Where J and ϕ are given by the starting conditions x_0 and x'_0.

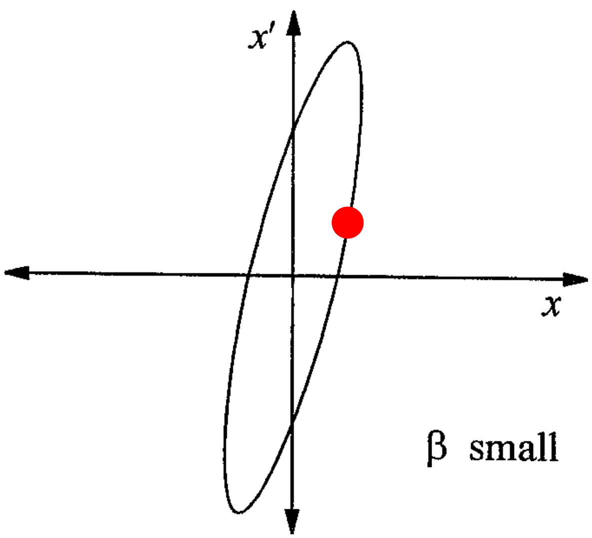
$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = 2J$$

Leads to the invariant of motion:

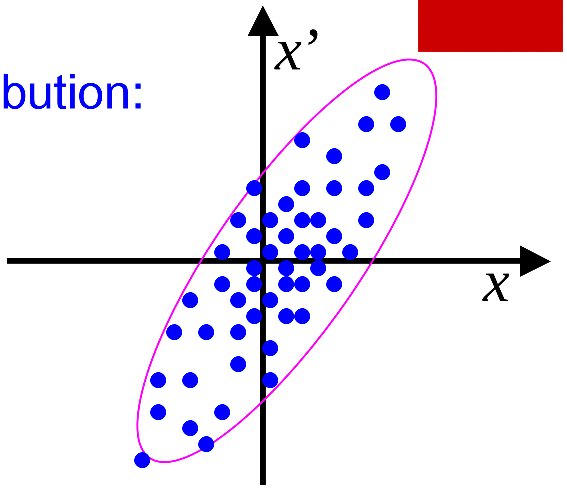
$$f(x, x', s) = \gamma(s)x^2 + 2\alpha(s)xx' + \beta(s)x'^2 \quad \Rightarrow \quad \tfrac{d}{ds} f = 0$$

It is called the Courant-Snyder invariant.





Phase Space Distribution



Often one can fit a Gauss distribution to the particle distribution:

$$\rho(x,x') = \frac{1}{2\pi\varepsilon} e^{-\frac{\gamma x^2 + 2\alpha x x' + \beta x'^2}{2\varepsilon}}$$

The equi-density lines are then ellipses. And one chooses the starting conditions for β and α according to these ellipses!

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \sin\phi_0 \\ \cos\phi_0 \end{pmatrix} \qquad \rho(J,\phi_0) = \frac{1}{2\pi\varepsilon} e^{-\frac{J}{\varepsilon}}$$

$$\langle 1 \rangle = \frac{1}{2\pi\varepsilon} \int_0^{2\pi}\int_0^{\infty} e^{-J/\varepsilon} dJ d\phi_0 = 1$$

Initial beam distribution $\longrightarrow$ initial α, β, γ

$$\langle x^2 \rangle = \frac{1}{2\pi\varepsilon} \iint 2J\beta \sin\phi_0^2 e^{-J/\varepsilon} dJ d\phi_0 = \varepsilon\beta \qquad \longrightarrow \qquad \langle x'^2 \rangle = \varepsilon\gamma$$

$$\langle x x' \rangle = -\frac{1}{2\pi\varepsilon} \iint 2J\alpha \sin\phi_0^2 e^{-J/\varepsilon} dJ d\phi_0 = \varepsilon\alpha$$

$$\varepsilon = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2}$$ is called the emittance.

Propagation of Twiss Parameters

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$$(x_0, x_0')\begin{pmatrix} \gamma_0 & \alpha_0 \\ \alpha_0 & \beta_0 \end{pmatrix}\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = 2J$$

$$(x, x')\begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix}\begin{pmatrix} x \\ x' \end{pmatrix} = 2J = (x_0, x_0')\underline{M}^T\begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix}\underline{M}\begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

$$\begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix} = \underline{M}^{-T}\begin{pmatrix} \gamma_0 & \alpha_0 \\ \alpha_0 & \beta_0 \end{pmatrix}\underline{M}^{-1}$$

$$\begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} = \underline{M}\begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix}\underline{M}^T$$

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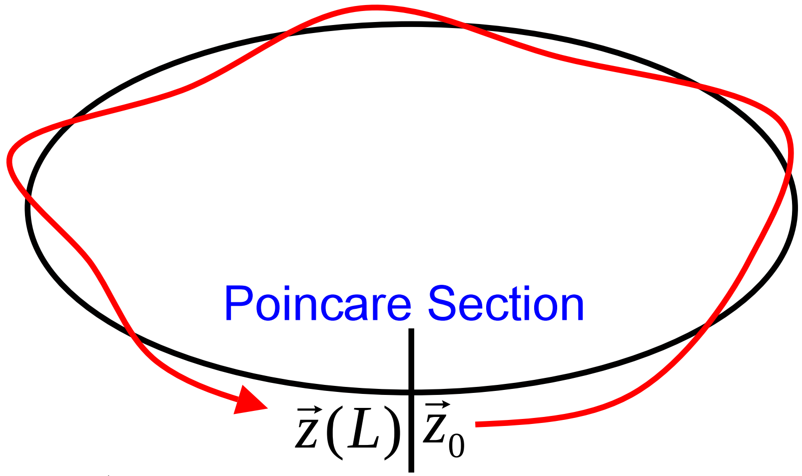
The One Turn Matrix for a Ring

$$\vec{z}(s) = \underline{M}(s,0)\vec{z}(0)$$

$$\vec{z}(L) = \underline{M}(L,0)\vec{z}(0)$$

$$\vec{z}(s+L) = \underline{M}_0(s)\vec{z}(s) \quad , \quad \underline{M}_0 = \underline{M}(s+L,s)$$

$$\vec{z}(s+nL) = \underline{M}_0^n(s)\vec{z}(s)$$

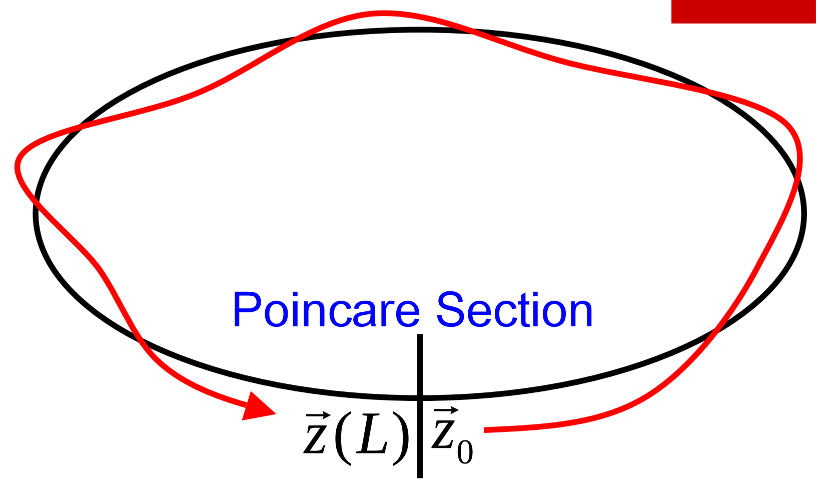


The Periodic Beta Function

If the particle distribution in a ring is stable, it is periodic from turn to turn.

$$\rho(x,x',s+L)=\rho(x,x',s)$$

To be matched to such a beam, the Twiss parameters α, β, γ must be the same after every turn.



$$\underline{M}(s,0)=\begin{pmatrix}\sqrt{\frac{\beta}{\beta_0}}[\cos\psi+\alpha_0\sin\psi] & \sqrt{\beta_0\beta}\sin\psi \\ \sqrt{\frac{1}{\beta_0\beta}}[(\alpha_0-\alpha)\cos\psi-(1+\alpha_0\alpha)\sin\psi] & \sqrt{\frac{\beta_0}{\beta}}[\cos\psi-\alpha\sin\psi]\end{pmatrix}$$

$$\underline{M}_0(s)=\begin{pmatrix}\cos\mu+\alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu-\alpha\sin\mu\end{pmatrix}=\cos\mu+\begin{pmatrix}\alpha & \beta \\ -\gamma & -\alpha\end{pmatrix}\sin\mu$$

$$\mu=\psi(s+L)-\psi(s)$$

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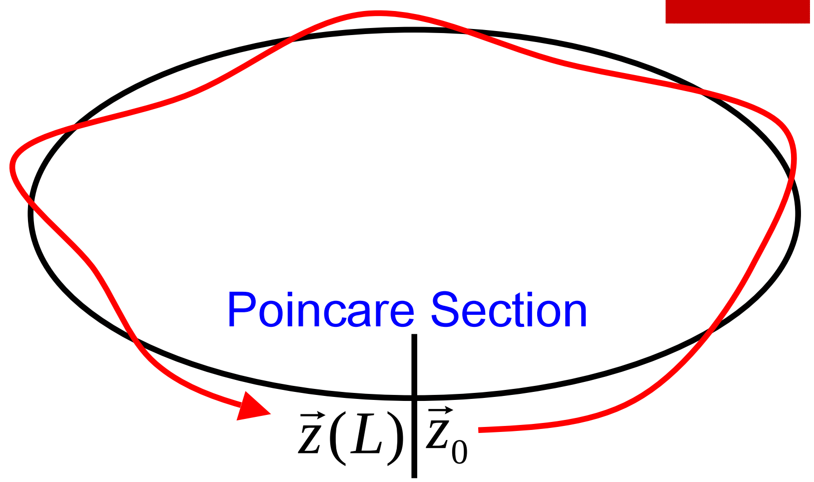
The Tune

The betatron phase advance per turn devided by 2π is called the TUNE.

$$\mu = 2\pi\nu = \psi(s+L) - \psi(s)$$

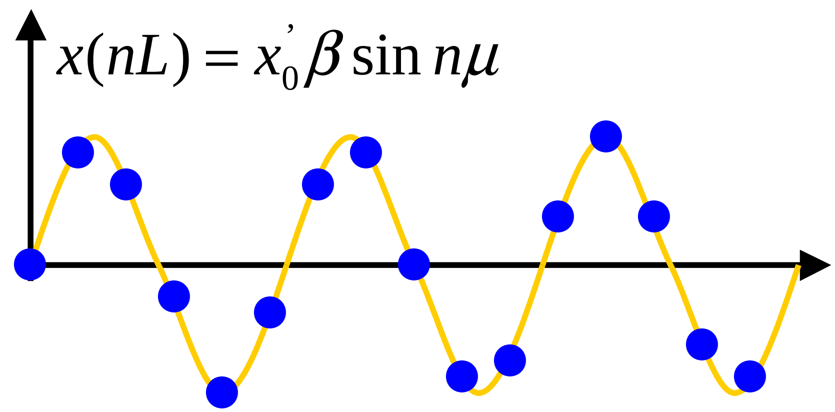
It is a property of the ring and does not depend on the azimuth s.



$$\underline{M}_0(s) = \cos\mu + \begin{pmatrix} -\alpha(s) & \beta(s) \\ \gamma(s) & \alpha(s) \end{pmatrix} \sin\mu$$

$$2\cos\mu(s) = \mathrm{Tr}[\underline{M}_0(s)] = \mathrm{Tr}[\underline{M}(s,0)\underline{M}_0(0)\underline{M}_0^{-1}(s,0)]$$

$$= \mathrm{Tr}[\underline{M}_0(0)] = 2\cos\mu(0)$$



$$\underline{M}_0^n = \cos n\mu + \begin{pmatrix} -\alpha & \beta \\ \gamma & \alpha \end{pmatrix} \sin n\mu$$

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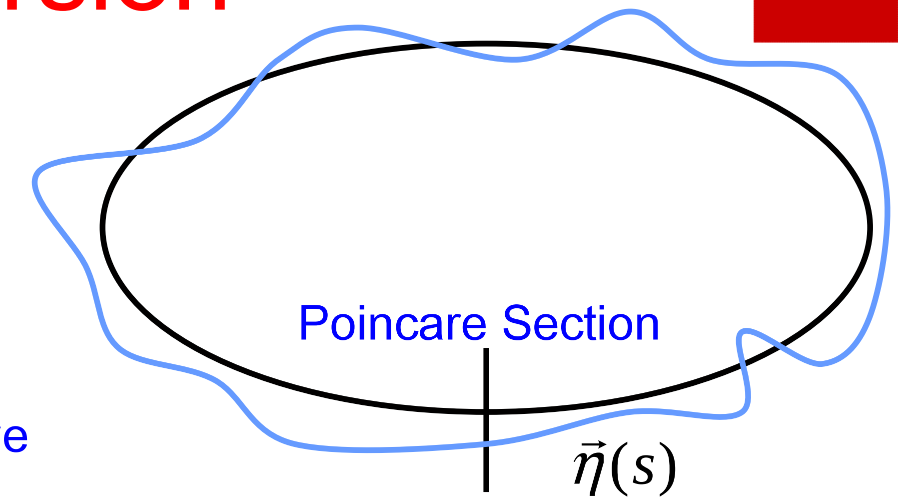
The Periodic Dispersion

$$\begin{pmatrix} \vec{D}(L)\delta \\ M_{56}\delta \\ \delta \end{pmatrix} = \begin{pmatrix} \underline{M}_{0x} & \vec{0} & \vec{D}(L) \\ \vec{T}^T & 1 & M_{56} \\ \vec{0}^T & 0 & 1 \end{pmatrix} \begin{pmatrix} \vec{0} \\ 0 \\ \delta \end{pmatrix}$$



The periodic orbit for particles with relative energy deviation δ is

$$\vec{\eta}(L) = \underline{M}_{0x}\vec{\eta}(0) + \vec{D}(L) \quad \text{with} \quad \vec{\eta}(L) = \vec{\eta}(0)$$

$$\vec{\eta}(0) = \underline{M}_0\vec{\eta}(0) + \vec{D}(L)$$

$$\Downarrow$$

$$\vec{\eta}(0) = [\underline{1} - \underline{M}_0(0)]^{-1}\vec{D}(L)$$

Particles with energy deviation δ oscillates around this periodic orbit.

$$\vec{z} = \vec{z}_\beta + \delta\vec{\eta}$$

$$\vec{z}_\beta(L) + \delta\vec{\eta}(L) = \vec{z}(L) = \underline{M}_0\vec{z}(0) + \vec{D}(L)\delta = \underline{M}_0[\vec{z}_\beta(0) + \delta\vec{\eta}(0)] + \vec{D}(L)\delta$$

$$= \underline{M}_0\vec{z}_\beta(0) + \delta\vec{\eta}(L)$$

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Dynamical systems

$\vec{z}(s) = \vec{M}(s; s_0, \vec{z}_0)$

dynamical variable $\vec{z}$
Flow, transport map $\vec{M}$

By referring to a reference trajectory, transport maps in accelerators become origin preserving: $\vec{M}(s; s_0, \vec{0}) = \vec{0}$

Flows build a group under concatenation:

$$\vec{M}(s; s_1, \cdot) \circ \vec{M}(s_1; s_0, \vec{z}_0) = \vec{M}(s; s_1, \vec{M}(s_1; s_0, \vec{z}_0)) = \vec{M}(s; s_0, \vec{z}_0)$$

1) Identity element: $\vec{M}(s; s_0, \vec{z}) = \vec{z}$

2) Inverse element of $\vec{M}(s; s_0, \vec{z}) = \vec{M}^{-1}(s; s_0, \vec{z})$ is $\vec{M}(s_0; s, \vec{z})$

In physics, the flow is often given as a solution of a first order ODE $\frac{d}{ds}\vec{z} = \vec{f}(\vec{z}, s)$

(Note that an nth order ODE can be rewritten as an n-dimensional first order ODE.)

Uniqueness

Note that not all ODEs $\frac{d}{ds}\vec{z} = \vec{f}(\vec{z}, s)$

have a unique solution $\vec{z}(s)$

through a given point $\vec{z}(0) = \vec{z}_0$

Picard-Lindeloef:

A unique solution through $(\vec{z}_0, s_0)$ exists for $\frac{d}{ds}\vec{z} = \vec{f}(\vec{z}, s)$ if $\vec{f}(\vec{z}, s)$ is Lipschitz continuous and bounded,

i.e. it is continuous, bounded, and there is a number N such that

$$|\vec{f}(\vec{z}_1, s) - \vec{f}(\vec{z}_2, s)| < N |\vec{z}_1 - \vec{z}_2|$$

Example: $H = \frac{1}{2}p^2 + V(q)\,,\quad V(q) = -8\sqrt{|q|}^3 \quad\Rightarrow\quad \dot{q} = p\,,\quad \dot{p} = 12\sqrt{|q|}$

There are two solutions through the point (q,p,t)=(0,0,0)

1. $(q(t), p(t)) = (0,0)$
2. $(q(t), p(t)) = (t^4, 4t^3) \quad\Rightarrow\quad (\dot{q}, \dot{p}) = (4t^3, 12t^2)$

In our following treatments we do require uniqueness of solutions.

Linear systems

Linear ODEs in N dimesions $\frac{d}{ds}\vec{z} = \vec{f}(\vec{z},s)$ have $\vec{f}(\lambda\vec{z},s) = \lambda\vec{f}(\vec{z},s)$

$$\frac{d}{ds}\vec{z} = \underline{L}(s)\vec{z}$$

There are N linearly independent solutions. For example $\vec{z}_n(s)$ going through $(\vec{z}_0, s_0)$
With $z_{0i} = 0$ for $i \neq n$ and $z_{0n} = 1$

$$\vec{z}_0 = (0,\dots,1,\dots,0)^T \quad \Rightarrow \quad \vec{z}_n(s)$$

One speaks of N fundamental solutions.

Superposition for linear ODEs:

If z1 is a solution and z2 is a solution, then
any linear combination Az1 +Bz2 is also a solution

$$\frac{d}{ds}\vec{z}_1 = \underline{L}(s)\vec{z}_1 \quad \& \quad \frac{d}{ds}\vec{z}_2 = \underline{L}(s)\vec{z}_2 \quad \Rightarrow \quad \frac{d}{ds}(A\vec{z}_1 + B\vec{z}_2) = \underline{L}(s)(A\vec{z}_1 + B\vec{z}_2)$$

Therefore any solution through $(\vec{z}_0, s_0)$ can be written as $\vec{z}(s) = \sum_{n=1}^{N} \vec{z}_n(s) z_{0n}$

$$\vec{z}(s) = \vec{M}(s;s_0,\vec{z}_0) = \underline{M}(s,s_0)\vec{z}_0 \quad \text{with} \quad \underline{M}(s,s_0) = (\vec{z}_1(s),\dots,\vec{z}_N(s))$$

Nonlinear systems

Noninear ODEs in N dimesions $\frac{d}{ds}\vec{z} = \vec{f}(\vec{z}, s)$

Have no fundamental solutions. Each solution has to be determined separately for each initial condition.

Examples: Plasma, Galaxies

$$H(\ldots, \vec{r}_j, \ldots, \ldots, \vec{p}_j, \ldots) = \sum_j \frac{\vec{p}_j^{\,2}}{2m_j} + \sum_{k \neq j} \frac{q_j q_k}{|\vec{r}_j - \vec{r}_k|}$$

Finding a general solution, flow, or transport map can be very hard.
This has not even been possible for the 3 body problem.

$$\vec{z}(s) = \vec{M}(s; s_0, \vec{z}_0)$$

Weakly nonlinear systems

Weakly nonlinear ODEs $\frac{d}{ds}\vec{z} = \vec{f}(\vec{z}, s)$

Have a right hand side that can be

approximated well by a truncated Taylor expansion

$$\vec{f}(\vec{z}, s) \approx \underline{L}(s)\vec{z} + \sum_{j,k} \vec{f}_{jk} z_j z_k + \sum_{j,k,l} \vec{f}_{jkl} z_j z_k z_l + \ldots + \sum_{\vec{k},\text{order O}} \vec{f}_{\vec{k}} \vec{z}^{\vec{k}} + \ldots$$

$$\vec{z}^{\vec{k}} = \prod_{n=1}^{N} z_n^{k_n}, \quad \sum_{\vec{k},\text{order O}} \ldots = \sum_{n=1}^{N} \sum_{k_n} \ldots \quad \text{with} \quad \sum_{n=1}^{N} k_n = O$$

By solving the Taylor expanded ODE one tries to find a Taylor expansion of the transport map: $\vec{M}(s; s_0, \vec{z}_0) \approx \underline{M}(s, s_0)\vec{z}_0 + \ldots + \sum_{\vec{k},\text{order O}} \vec{M}_{\vec{k}} \vec{z}_0^{\vec{k}} + \ldots$

Note:

While this approach is usually chosen, it is not certain that a transport map of the Taylor expanded ODE is a Taylor expansion of the transport map of the original ODE. One therefore often speaks of “formally” finding the Taylor expansion of the transport map.

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Aberrations and sensitivities

$$\vec{M}(s;s_0,\vec{z}_0) \approx \underline{M}(s,s_0)\vec{z}_0 + \ldots + \sum_{\vec{k},\text{order } O} \vec{M}_{\vec{k}} \vec{z}_0^{\vec{k}} + \ldots$$

The Taylor coefficients are called aberrations of order O and are denoted by

$$(z_i, z_1^{k_1} \ldots z_6^{k_6}) \equiv M_{\vec{k},i} \,, \quad \text{order} \quad O = \sum_{n=1}^{6} k_n$$

Parameter dependences lead to sensitivities:

$$\tfrac{d}{ds}\vec{z} = \vec{f}(\vec{z},s,\varepsilon) \quad \Rightarrow \quad \vec{z}(s) = \vec{M}(s,\varepsilon;s_0,\vec{z}_0)$$

$$\vec{M}(s,\varepsilon;s_0,\vec{z}_0) \approx \underline{M}(s,s_0)\vec{z}_0 + \underline{M}^1(s,s_0)\vec{z}_0\varepsilon + \ldots + \sum_{\vec{k},n,\text{order } O} \vec{M}_{\vec{k}}^{\,n} \vec{z}_0^{\vec{k}} \varepsilon^n$$

$$(z_i, z_1^{k_1} \ldots z_6^{k_6}\varepsilon^n) \equiv M_{\vec{k},i}^{\,n} \,, \quad \text{order} \quad O = n + \sum_{j=1}^{6} k_j$$

How can all these Taylor coefficients be computed?

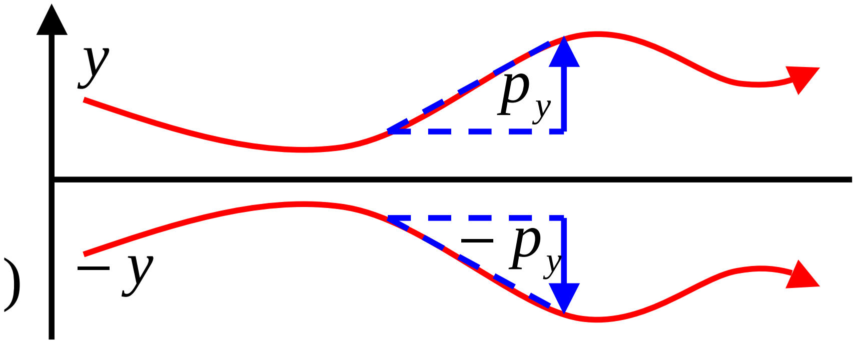
Horizontal midplane symmetry

This is the most important symmetry in nearly all accelerators.

$$\vec{r}^{\oplus} = (x, -y, z)$$

$$\vec{p}^{\oplus} = (p_x, -p_y, p_z)$$

$$\tfrac{d}{dt}\vec{p} = \vec{F}(\vec{r}, \vec{p}) \quad \Rightarrow \quad \tfrac{d}{dt}\vec{p}^{\oplus} = \vec{F}(\vec{r}^{\oplus}, \vec{p}^{\oplus})$$



$$\vec{z} \;= (x, a, y, b, \tau, \delta) \qquad \vec{z}(s) \;= \vec{M}(s, \vec{z}_0)$$

$$\vec{z}^{\oplus} = (x, a, -y, -b, \tau, \delta) \qquad \vec{z}^{\oplus}(s) = \vec{M}(s, \vec{z}_0^{\oplus})$$

$$M_i(s, \vec{z}_0^{\oplus}) = \quad M_i(s, \vec{z}_0) \quad \text{for} \quad i \in \{1,2,5,6\}$$

$$M_i(s, \vec{z}_0^{\oplus}) = -M_i(s, \vec{z}_0) \quad \text{for} \quad i \in \{3,4\}$$

$(x, x_0^{k_1} \ldots \delta_0^{k_6}) = 0 \quad \text{for} \quad k_3 + k_4$ is odd — similarly for a, τ and δ

$(y, x_0^{k_1} \ldots \delta_0^{k_6}) = 0 \quad \text{for} \quad k_3 + k_4$ is even — similarly for b

Double midplane symmetry

In addition to midplane symmetry, some elements are symmetric around the vertical plane, e.g. quadrupoles, glass lenses

$$\vec{z} = (x, a, y, b, \tau, \delta) \qquad \vec{z}(s) = \vec{M}(s, \vec{z}_0)$$

$$\vec{z}^{\oplus} = (x, a, -y, -b, \tau, \delta) \qquad \vec{z}^{\oplus}(s) = \vec{M}(s, \vec{z}_0^{\oplus})$$

$$\vec{z}^{\otimes} = (-x, -a, y, b, \tau, \delta) \qquad \vec{z}^{\otimes}(s) = \vec{M}(s, \vec{z}_0^{\otimes})$$

$$M_i(s, \vec{z}_0^{\oplus}) = M_i(s, \vec{z}_0) \quad \text{for} \quad i \in \{1,2,5,6\}$$

$$M_i(s, \vec{z}_0^{\oplus}) = -M_i(s, \vec{z}_0) \quad \text{for} \quad i \in \{3,4\}$$

$$M_i(s, \vec{z}_0^{\otimes}) = -M_i(s, \vec{z}_0) \quad \text{for} \quad i \in \{1,2\}$$

$$M_i(s, \vec{z}_0^{\otimes}) = M_i(s, \vec{z}_0) \quad \text{for} \quad i \in \{3,4,5,6\}$$

$(x, x_0^{k_1} \ldots \delta_0^{k_6}) = 0 \quad \text{for} \quad k_1 + k_2$ is even or $k_3 + k_4$ is odd   similarly for a, τ and δ

$(y, x_0^{k_1} \ldots \delta_0^{k_6}) = 0 \quad \text{for} \quad k_1 + k_2$ is odd or $k_3 + k_4$ is even   similarly for b

Rotational symmetry

Some optical elements are completely rotationally symmetric in the x-y plane, e.g. solenoid magnets, many glass lenses

$$w = x + iy\,,\quad \alpha = a + ib$$

$$\vec{z} = (w, \bar{w}, \alpha, \bar{\alpha}, \tau, \delta)$$

$$\vec{z}^{\oplus} = (e^{i\varphi} w, e^{-i\varphi}\bar{w}, e^{i\varphi}\alpha, e^{-i\varphi}\bar{\alpha}, \tau, \delta)$$

$$\vec{z}(s) = \vec{M}(s, \vec{z}_0)$$

$$\vec{z}^{\oplus}(s) = \vec{M}(s, \vec{z}_0^{\oplus})$$

$$M_i(s, \vec{z}_0^{\oplus}) = e^{i\varphi} M_i(s, \vec{z}_0) \quad \text{for} \quad i \in \{1,3\}$$

$$M_i(s, \vec{z}_0^{\oplus}) = e^{-i\varphi} M_i(s, \vec{z}_0) \quad \text{for} \quad i \in \{2,4\}$$

$$M_i(s, \vec{z}_0^{\oplus}) = M_i(s, \vec{z}_0) \quad \text{for} \quad i \in \{5,6\}$$

$(w, w^{k_1} \dots \delta^{k_6}) = 0 \quad \text{for} \quad k_1 - k_2 + k_3 - k_4 \neq 1$ similarly for α

$(\bar{w}, w^{k_1} \dots \delta^{k_6}) = 0 \quad \text{for} \quad k_1 - k_2 + k_3 - k_4 \neq -1$ similarly for α^*

$(\tau, w^{k_1} \dots \delta^{k_6}) = 0 \quad \text{for} \quad k_1 - k_2 + k_3 - k_4 \neq 0$ similarly for δ

$(w, |w_0|^2 w_0), (\bar{\alpha}, |w_0|^2 \bar{w}_0), (\alpha, w_0^2 \bar{\alpha}_0), (\tau, |w_0|^2)$ can all be non-zero

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C_n symmetry

Some optical elements have C_n symmetric in the x-y plane, e.g. C_2 for quadrupole, C_3 for sextupoles, etc.

$$w = x + iy\,, \quad \alpha = a + ib$$

$$\vec{z} = (w, \overline{w}, \alpha, \overline{\alpha}, \tau, \delta)$$

$$\vec{z}^{\oplus} = (e^{i\frac{2\pi}{n}} w, e^{-i\frac{2\pi}{n}} \overline{w}, e^{i\frac{2\pi}{n}} \alpha, e^{-i\frac{2\pi}{n}} \overline{\alpha}, \tau, \delta)$$

$$\vec{z}(s) = \vec{M}(s, \vec{z}_0)$$

$$\vec{z}^{\oplus}(s) = \vec{M}(s, \vec{z}_0^{\oplus})$$

$$M_i(s, \vec{z}_0^{\oplus}) = e^{i\frac{2\pi}{n}} M_i(s, \vec{z}_0) \quad \text{for} \quad i \in \{1,3\}$$

$$M_i(s, \vec{z}_0^{\oplus}) = e^{-i\frac{2\pi}{n}} M_i(s, \vec{z}_0) \quad \text{for} \quad i \in \{2,4\}$$

$$M_i(s, \vec{z}_0^{\oplus}) = M_i(s, \vec{z}_0) \quad \text{for} \quad i \in \{5,6\}$$

$$(w, w^{k_1} \dots \delta^{k_6}) = 0 \quad \text{for} \quad k_1 - k_2 + k_3 - k_4 \neq jn + 1$$ similarly for α

$$(\overline{w}, w^{k_1} \dots \delta^{k_6}) = 0 \quad \text{for} \quad k_1 - k_2 + k_3 - k_4 \neq jn - 1$$ similarly for α^*

$$(\tau, w^{k_1} \dots \delta^{k_6}) = 0 \quad \text{for} \quad k_1 - k_2 + k_3 - k_4 \neq jn$$ similarly for δ

$$(w, \overline{w}_0), (\overline{\alpha}, |w_0|^2 w_0), (\alpha, \overline{w}_0^2 \alpha_0), (\tau, |w_0|^2)$$ can all be non-zero for C_2

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Symplecticity

$$[\vec{\partial}\vec{M}^T]^T \underline{J}[\vec{\partial}\vec{M}^T] = \underline{J}$$

Symplecticity leads to the requirement that sums over certain products of aberrations must be either 0 or 1.

Separation into linear an nonlinear part of the map: $\vec{M}(\vec{z}) = \underline{M}_1(\vec{z} + \vec{N}(\vec{z}))$

$$\underline{M}(\vec{z}) = [\vec{\partial}\vec{M}^T]^T = [(\underline{1} + \vec{\partial}\vec{N}^T)\underline{M}_1^T]^T = \underline{M}_1(\underline{1} + \underline{N}(\vec{z}))$$

$$(1+\underline{N})^T \underline{M}_1^T \underline{J}\underline{M}_1(1+\underline{N}) = \underline{J} \quad \Rightarrow \quad \underline{M}_1^T \underline{J}\underline{M}_1 = \underline{J}\,, \quad \underline{N}^T\underline{J} + \underline{J}\underline{N} = -\underline{N}^T\underline{J}\underline{N}$$

For the leading order n-1 (the first order that appears in N): $\underline{N}^T\underline{J} + \underline{J}\underline{N} = 0 + O^n$

N is a Hamiltonian matrix up to order n and can thus be written up to order n as: $\vec{N}(\vec{z}) = \underline{J}\vec{\partial} f(\vec{z}) + O^{n+1}$

$$w(s_f) = (x, x_0)\partial_a f + (x, y_0)\partial_b f + i[(y, x_0)\partial_a f + (y, y_0)\partial_b f]$$

$$= (w, x_0)\partial_a f + (w, y_0)\partial_b f$$

$$= \tfrac{1}{2}[(w, w_0) + (w, \overline{w}_0)][\partial_\alpha f + \partial_{\overline{\alpha}} f] - \tfrac{1}{2}[(w, w_0) - (w, \overline{w}_0)][\partial_\alpha f - \partial_{\overline{\alpha}} f]$$

$$= (w, w_0)\partial_{\overline{\alpha}} f + (w, \overline{w}_0)\partial_\alpha f = (w, w_0)\partial_{\overline{\alpha}} f$$

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Special aberrations

Some aberrations and sensitivities have special names:

Dispersion (for δ as parameter of 4-dimensional motion) $\vec{z} = \underline{M}(s)\,\vec{z}_0 + \vec{D}(s)\delta$

Chromatic aberrations $(x,\ldots\delta^n)\,,\quad n \neq 0$

Geometric aberrations $(x, x^{k_1}a^{k_2}y^{k_3}b^{k_4}\ldots)\,,\quad \sum_{i=1}^{4} k_i \neq 0$

Purely Geometric aberrations $(x,\ldots\delta^n)\,,\quad n = 0$

Opening aberrations $(x, x^{k_1}\ldots y^{k_2}\ldots)\,,\quad k_1 + k_2 = 0$

Field aberrations $(x, x^{k_1}\ldots y^{k_2}\ldots)\,,\quad k_1 + k_2 \neq 0$

Spherical imaging systems: $(w,\alpha) = 0$

Spherical aberration for rotational symmetry $(w, \alpha|\alpha|^2)$

Coma line $(w, w|\alpha|^2)$

Coma circle $(w, \overline{w}\alpha^2)$

Astigmatism $(w, w^2\overline{\alpha})$

Curvature of Image $(w, |w|^2\alpha)$

Distortion $(w, w|w|^2)$

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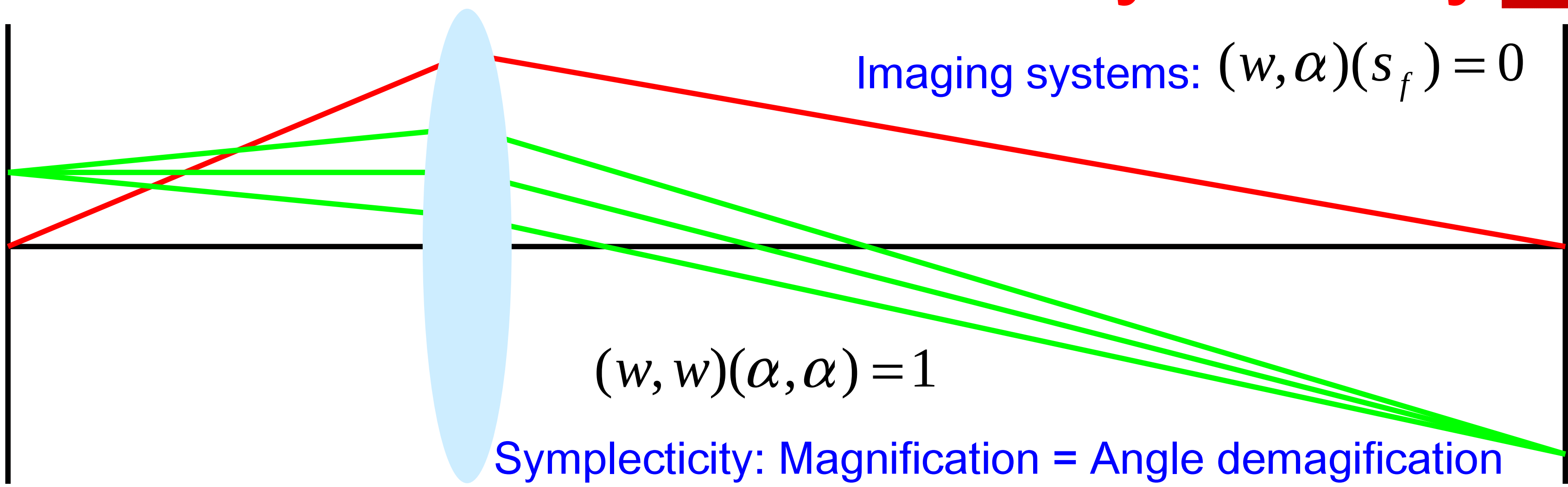
Aberrations for rotational symmetry

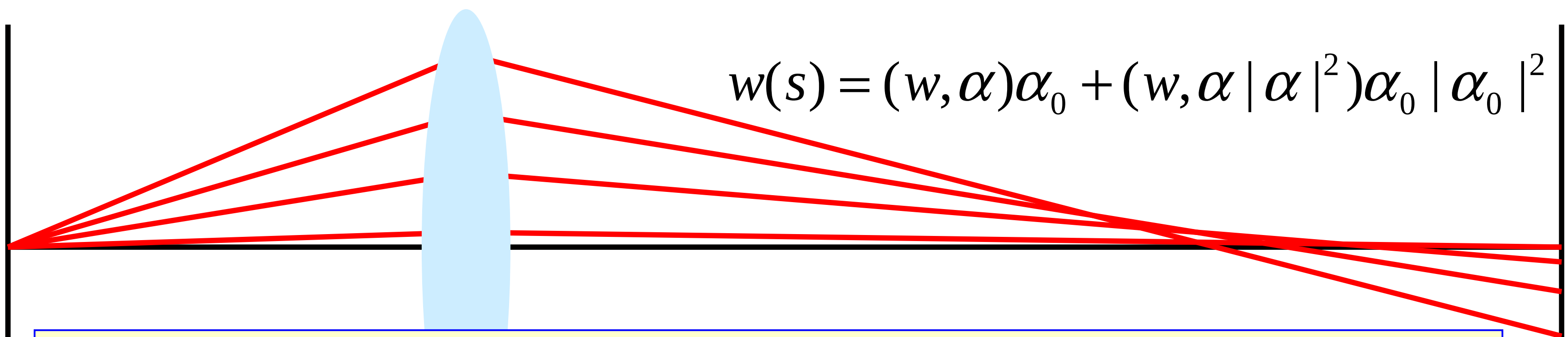


Imaging systems: $(w,\alpha)(s_f)=0$

$(w,w)(\alpha,\alpha)=1$

Symplecticity: Magnification = Angle demagification

Spherical aberration for rotational symmetry $(w,\alpha|\alpha|^2)$



$w(s)=(w,\alpha)\alpha_0+(w,\alpha|\alpha|^2)\alpha_0|\alpha_0|^2$

Scherzer Theorem:

For rotationally symmetric electro-magnetic systems, the focal length for larger angels is always shorter.

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Koma line and Koma circle

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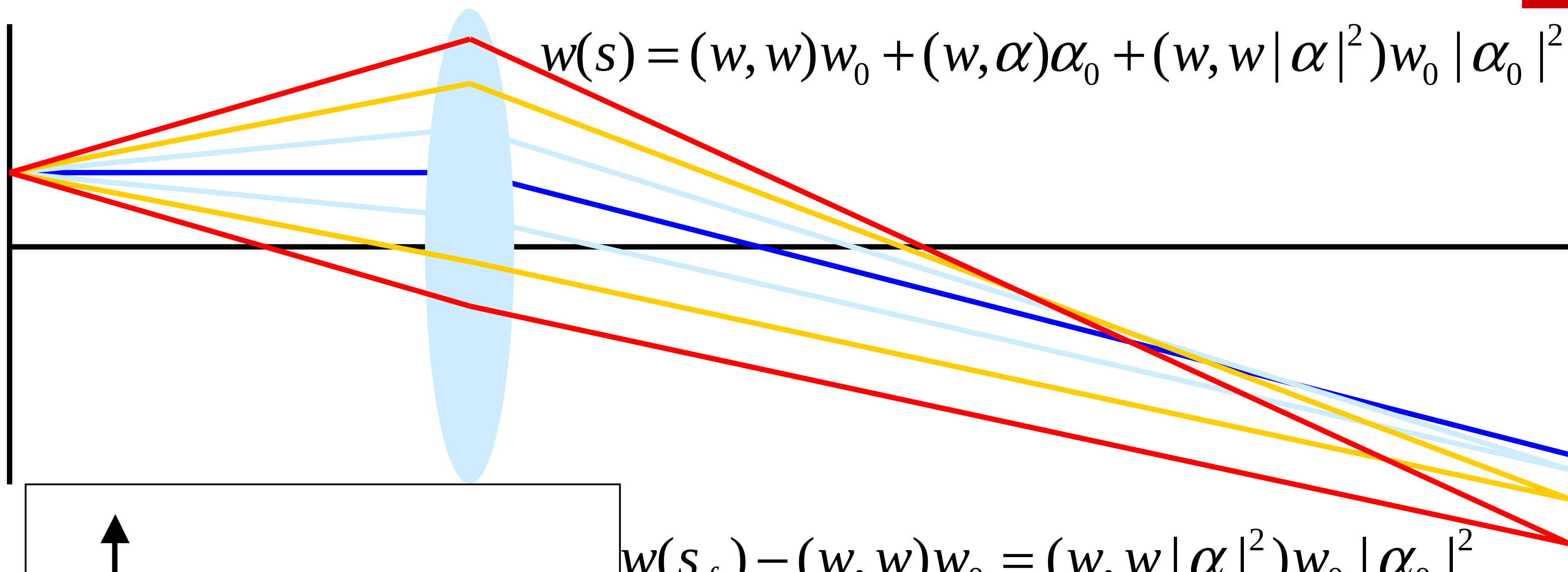
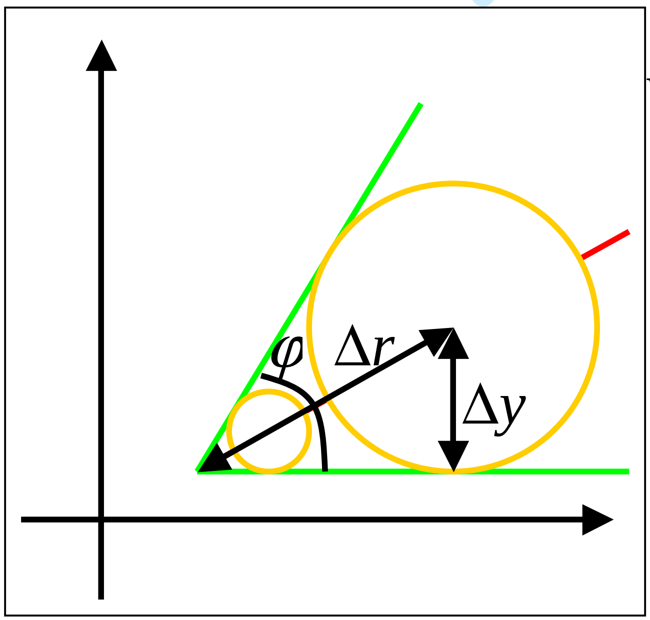
$$w(s)=(w,w)w_0+(w,\alpha)\alpha_0+(w,w|\alpha|^2)w_0|\alpha_0|^2$$





$$w(s_f)-(w,w)w_0=(w,w|\alpha|^2)w_0|\alpha_0|^2$$
$$+(w,\overline{w}\alpha^2)\overline{w}_0\alpha_0^2$$

$$\varphi=2\arcsin(\tfrac{\Delta y}{\Delta r})=2\arcsin(\tfrac{(w,\overline{w}\alpha^2)}{(w,w|\alpha|^2)})$$

Symplecticity yields:

$$(w,w|\alpha|^2)=2(w,\overline{w}\alpha^2)\quad\Rightarrow\quad\varphi=60^\circ$$

Since:

$$w(s_f)=(w,w_0)\partial_{\overline{\alpha}}[\ldots+\mathrm{Re}\{Kw\alpha\overline{\alpha}^2\}]=(w,w_0)[\ldots+Kw\alpha\overline{\alpha}+\tfrac{1}{2}\overline{K}\overline{w}\alpha^2]$$

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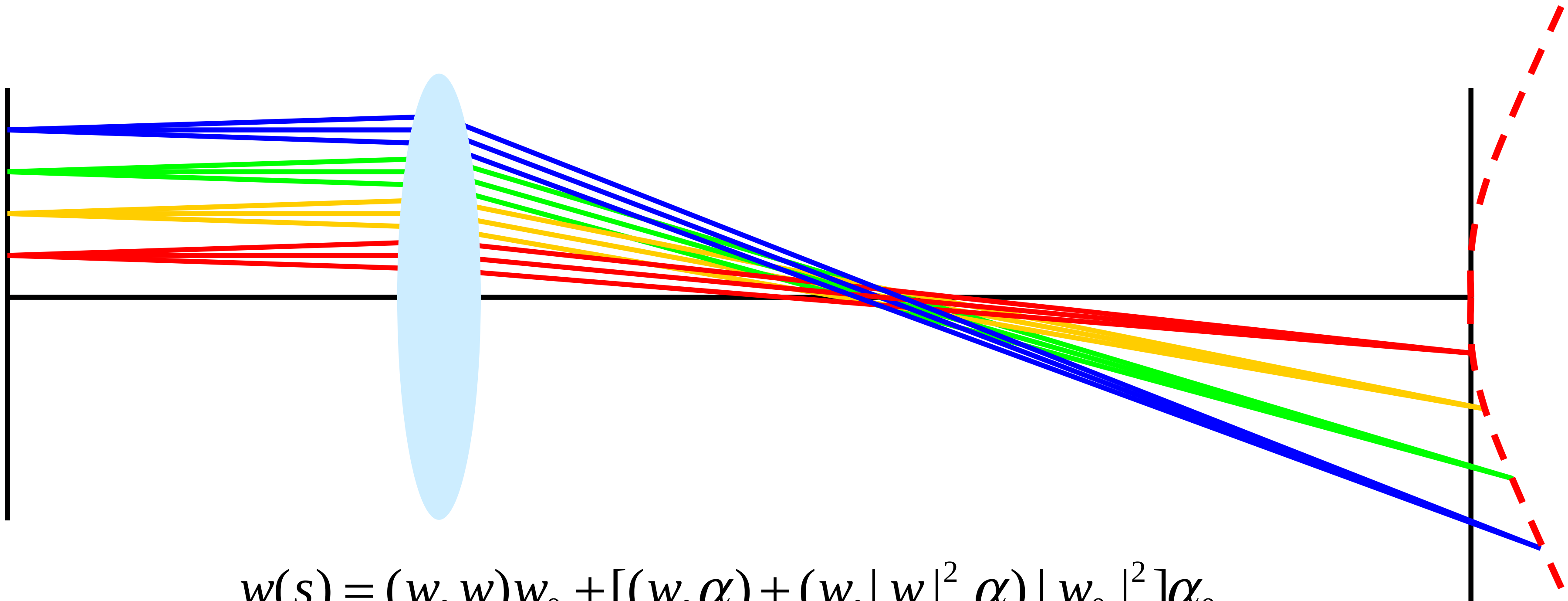
Curvature of image



$$w(s) = (w,w)w_0 + [(w,\alpha) + (w,|w|^2\alpha)|w_0|^2]\alpha_0$$

The focus occurs at $(w,\alpha)(s_f) + (w,|w|^2\alpha)(s_f)|w_0|^2 = 0$

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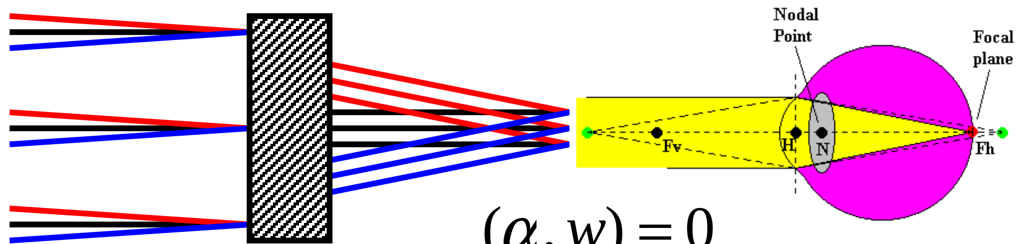
Other special systems

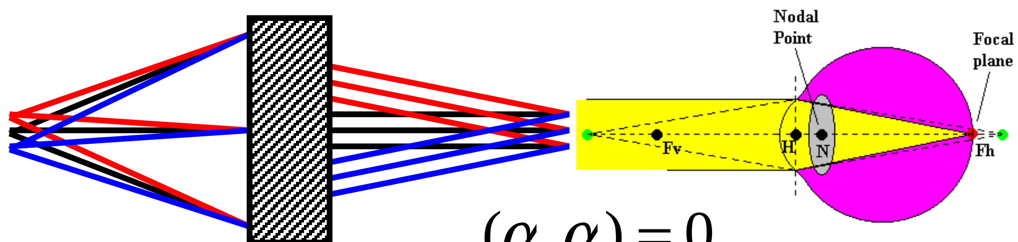
Telescope:
parallel to parallel system



$(\alpha, w) = 0$

Nonlinearly corrected telescope: $(\alpha, w^n) = 0$

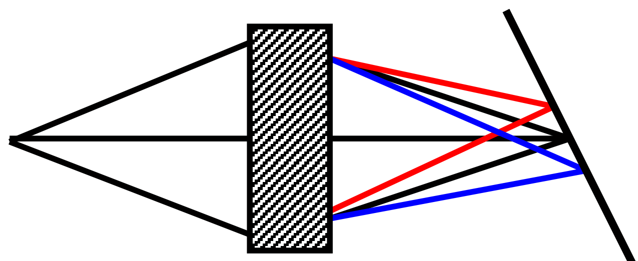
Microscope:
point to parallel system



$(\alpha, \alpha) = 0$

Nonlinearly corrected microscope: $(\alpha, \alpha^n) = 0$

Spectrograph:
point to parallel system



(x, δ) large
$(x, a) = 0$
(x, x) small

Nonlinearly corrected spectrograph: $(x, a^n b^m) = 0$

Tilt of focal plane: $(x, a\delta) \neq 0$ the focus is at $(x,a)(s_f) + (x, a\delta)(s_f)\delta = 0$

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Variation of Constants

$$\vec{z}\,' = \vec{f}(\vec{z}, s)$$

$$\vec{z}\,' = \underline{L}(s)\vec{z} + \Delta\vec{f}(\vec{z}, s)$$

Field errors, nonlinear fields, etc can lead to $\Delta\vec{f}(\vec{z}, s)$

$$\vec{z}\,'_H = \underline{L}(s)\vec{z}_H \quad \Rightarrow \quad \vec{z}_H(s) = \underline{M}(s)\vec{z}_{H0} \quad \text{with} \quad \underline{M}\,'(s)\vec{a} = \underline{L}(s)\underline{M}(s)\vec{a}$$

$$\vec{z}(s) = \underline{M}(s)\vec{a}(s) \quad \Rightarrow \quad \vec{z}\,'(s) = \underline{M}\,'(s)\vec{a} + \underline{M}(s)\vec{a}\,'(s) = \underline{L}(s)\vec{z} + \Delta\vec{f}(\vec{z}, s)$$

$$\vec{a}(s) = \vec{z}_0 + \int_0^s \underline{M}^{-1}(\hat{s})\Delta\vec{f}(\vec{z}(\hat{s}), \hat{s})\,d\hat{s}$$

$$\vec{z}(s) = \underline{M}(s)\left\{\vec{z}_0 + \int_0^s \underline{M}^{-1}(\hat{s})\Delta\vec{f}(\vec{z}(\hat{s}), \hat{s})\,d\hat{s}\right\}$$

$$= \vec{z}_H(s) + \int_0^s \underline{M}(s, \hat{s})\Delta\vec{f}(\vec{z}(\hat{s}), \hat{s})\,d\hat{s}$$

Perturbations are propagated from s to s’

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Iteration for Aberrations

$$\vec{z}(s) = \vec{z}_H(s) + \int_0^s \underline{M}(s,\hat{s})\Delta\vec{f}(\vec{z}(\hat{s}),\hat{s})\,d\hat{s}$$

$$\vec{z}_1(s) = \vec{z}_H(s)$$

$$\vec{z}_2(s) = \vec{z}_H(s) + \int_0^s \underline{M}(s,\hat{s})\Delta\vec{f}(\vec{z}_1(\hat{s}),\hat{s})\,d\hat{s}$$

$$\vdots$$

$$\vec{z}_n(s) = \vec{z}_H(s) + \int_0^s \underline{M}(s,\hat{s})\Delta\vec{f}(\vec{z}_{n-1}(\hat{s}),\hat{s})\,d\hat{s}$$

Taylor expansions: $\Delta\vec{f}(\vec{z},s) = \Delta\vec{f}_2(\vec{z},s) + \Delta\vec{f}_3(\vec{z},s) + \ldots\ ,\quad \Delta f_O = \sum_{\vec{k},\text{order } O} \vec{f}_{\vec{k}}\,\vec{z}^{\,\vec{k}}$

$$\vec{z}_1(s) = \underline{M}(s)\vec{z}_0$$

$$\vec{z}_2(s) = \underline{M}(s)\vec{z}_0 + \int_0^s \underline{M}(s,\hat{s})\Delta\vec{f}_2(\vec{z}_1(\hat{s}),\hat{s})\,d\hat{s}$$

$$\vec{z}_3(s) = \underline{M}(s)\vec{z}_0 + \int_0^s \underline{M}(s,\hat{s})\{[\Delta\vec{f}_2(\vec{z}_2(\hat{s}),\hat{s})]_3 + \Delta\vec{f}_3(\vec{z}_1(\hat{s}),\hat{s})\}\,d\hat{s}$$

$$\vdots$$

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Poisson Bracket

The Poisson Bracket is defined as

$$[f(\vec{z}),g(\vec{z})]=\sum_i \partial_{q_j} f\,\partial_{p_j} g-\partial_{p_j} f\,\partial_{q_j} g=\vec{\partial}^T f\,\underline{J}\,\vec{\partial} g$$

The Poisson Bracket can be viewed as a product on the vector space of phase space functions. It is:

1) Linear: $[f,ag]=[af,g]=a[f,g],\quad a\in IR$

2) Distributive: $[f,g+h]=[f,g]+[f,h]$

This turns the vector space into an algebra.

The multiplication is furthermore:

1) Anti-commutative: $[f,g]=-[g,f]$

2) Has a Jacobi-identity: $[f,[g,h]]+[g,[h,f]]+[h,[f,g]]=0$

as can be proven by the product rule: $[f,gh]=g[f,h]+[f,g]h$

This turns the algebra into a Lie algebra.

Example: $\vec{a}\times\vec{b}$ turns IR^3 into a Lie algebra.

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Map computation by Lie Algebra

The Poisson-Bracket operator of f, $:g:$ is defined as $:g:h=[g,h]$

$$:H:g=[H,g]=-[g,H]=-\vec{\partial}^T g \underline{J}\, \vec{\partial} H=-\vec{\partial}^T g \tfrac{d}{ds}\vec{z}=-\tfrac{d}{ds}g(\vec{z})$$

$$\tfrac{d}{ds}\vec{z}=\vec{f}(\vec{z},s) \quad\Rightarrow\quad -:H:z_j=\tfrac{d}{ds}z_j=f_j(\vec{z},s)\,,\quad -:H:f_j=\tfrac{d}{ds}f_j-\tfrac{\partial}{\partial s}f_j$$

In the main field region where $\tfrac{d}{ds}\vec{z}=\vec{f}(\vec{z}) \quad\Rightarrow\quad -:H:f_j=\tfrac{d}{ds}f_j=\tfrac{d^2}{ds^2}z_j$

$$\text{If } g(\vec{z})=\tfrac{d^n}{ds^n}z_j \text{ then } -:H:g=\tfrac{d}{ds}g=\tfrac{d^{n+1}}{ds^{n+1}}z_j \quad\Rightarrow\quad (-:H:)^n z_j=\tfrac{d^n}{ds^n}z_j$$

Propagator: $e^{-\Delta s:H:}\vec{z}=\sum_{n=0}^{\infty}\frac{(-\Delta s:H:)^n}{n!}\vec{z}=\sum_{n=0}^{\infty}\frac{\Delta s^n}{n!}\frac{d^n}{ds^n}\vec{z}=\vec{M}(s+\Delta s;s,\vec{z})$

$$\vec{M}_2\circ\vec{M}_1(\vec{z}_0)=\vec{M}_2(\Delta s_2,\vec{z}(\Delta s_1))=\sum_{n=1}^{\infty}\frac{(-\Delta s_1:H_1:)^n}{n!}\vec{M}_2(\Delta s_2,\vec{z}_0)$$

$$=e^{-\Delta s_1:H_1(\vec{z}_0):}e^{-\Delta s_2:H_2(\vec{z}_0):}\vec{z}_0$$

$$\boxed{\vec{M}(s;\vec{z}_0)=\vec{M}_n\circ\ldots\circ\vec{M}_2\circ\vec{M}_1(\vec{z}_0)=e^{-\Delta s_1:H_1:}\ldots e^{-\Delta s_n:H_n:}\vec{z}_0}$$

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Poisson Bracket invariants

The Poisson Bracket is invariant under a symplectic transfer map

$$[f(\vec{M}(\vec{z})), g(\vec{M}(\vec{z}))] = \vec{\partial}^T f\Big|_{\vec{M}} \underline{M}\,\underline{J}\,\underline{M}^T \vec{\partial} g\Big|_{\vec{M}} = [f(\vec{z}), g(\vec{z})]\Big|_{\vec{M}(\vec{z})}$$

For nonlinear expansions, one writes the transport map as a linear matrix

and a nonlinear Lie exponent,

$$\vec{M}_1(\vec{z}) = \underline{M}_1 e^{:H_1(\vec{z}):^n}\vec{z} = \underline{M}_1 \sum_{n=0}^{\infty} \frac{:H_1^n:}{n!}\vec{z}$$

since a linear Lie exponent requires infinitely many terms in the power sum, but the nonlinear exponent terminates when a finite order expansion is sought.

$$(\underline{M}_2 e^{:H_2^n(\vec{z}):}\vec{z}) \circ (\underline{M}_1 e^{:H_1^n(\vec{z}):}\vec{z}) = \underline{M}_2 e^{:H_2^n(\underline{M}_1 e^{:H_1^n(\vec{z}):}\vec{z}):}\underline{M}_1 e^{:H_1^n(\vec{z}):}\vec{z}$$

$$= \underline{M}_2\underline{M}_1 e^{:H_2^n(\underline{M}_1 e^{:H_1^n(\vec{z}):}\vec{z}):} e^{:H_1^n(\vec{z}):}\vec{z} = \underline{M}_2\underline{M}_1 e^{:H_1^n(\vec{z}):} e^{:H_2^n(\underline{M}_1\vec{z}):}\vec{z}$$

When these equations are used to compute and manipulate transfer maps, one speaks of the Lie algebraic method.

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Computing Taylor expansions

$$f(x) \approx \sum_{n=1}^{\text{order } O} \frac{x^n}{n!} \partial^n f\Big|_0$$

But taking this approach for complicated functions would be very cumbersome:

1.

$$f(x) = \frac{1}{1+\sin x} - 1, \quad f(0) = 0, \quad \partial f\big|_0 = \frac{-\cos x}{(1+\sin x)^2}\Bigg|_0 = -1,$$

$$\partial^2 f(x) = \frac{\sin x(1+\sin x) + 2\cos^2 x}{(1+\sin x)^3}\Bigg|_0 = 2, \quad f(x) \approx -x + x^2 + O^3$$

2.

$$f(x) = \frac{1}{1+\sin x} - 1,$$

$$f(x) \approx \frac{1}{1+x-\frac{1}{6}x^3+O^4} - 1$$

$$\approx -(x-\tfrac{1}{6}x^3) + (x-\tfrac{1}{6}x^3)^2 - (x-\tfrac{1}{6}x^3)^3 + O^4$$

$$\approx -x + x^2 - \tfrac{5}{6}x^3 + O^4$$

This approach is formalized in the field of automatic differentiation using a Differential Algebra.

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Computations with TPS(n)

Computation of a function in *IR* is done by a finite number of elementary operations (+,-,x) and elementary function evaluations (sin, cos, exp, 1/x, …).

$$f(x) = \frac{1}{1+\sin x} - 1$$

1. $x \in IR$
2. $\sin$
3. $1+$
4. $1/$
5. -1

If $g_n(x)$ is the truncated power series of order n of g(x) and $h_n(x)$ is that of h(x) we can look for elementary operations ("+","-","x") so that

g_n"+"h_n is the TPS(n) of g+h

g_n"-"h_n is the TPS(n) of g-h

g_n"x"h_n is the TPS(n) of gxh

Similarly we can look for elementary functions ("sin","cos","exp","1/x",…) so that "sin"(g_n) is the TPS(n) of sin(g), "exp"(g_n) is the TPS(n) of exp(g), etc.

Evaluating all elementary operations and elementary functions in f(x) in terms of "+","…" starting with the TPS(n) of x, leads to the TPS(n) of f(x).

Automatic differentiation with TPSA(n)

Example: computing the TPS(3) of $f(x)=\frac{1}{1+\sin x}-1$

1. TPS(3) of x is x
2. TPS(3) of "sin" x is $x-\frac{1}{6}x^3$
3. $1"+"x-\frac{1}{3}x^3=1+x-\frac{1}{6}x^3$
4. $i(x)=\frac{1}{1+x}, "i"(x)=1-x+x^2-x^3, "i"(x-\frac{1}{6}x^3)=1-x+x^2-\frac{5}{6}x^3$
5. $1-x+x^2-\frac{5}{6}x^3"-"1=-x+x^2-\frac{5}{6}x^3$

This automatically (i.e. with a computer) leads to derivatives of f(x):

$$f(0)=0,\ f'(0)=-1,\ f''(0)=2,\ f'''(0)=-4$$

Truncated power series can be added "+" and multiplied "x" and there is a neutral element of multiplication (i.e.1). Therefore the vector space of TPS(n) forms an algebra. It is called the Truncated Power Series Algebra TPSA(n).

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The Algebra ${}_1D_1$

An addition and multiplication with a scalar leads to a vector space over IR^2:

$$\{a_0,a_1\},\{b_0,b_1\}\in IR^2, t\in IR$$

$$\{a_0,a_1\}+\{b_0,b_1\}=\{a_0+b_0,a_1+b_1\}$$

$$t\{a_0,a_1\}=\{ta_0,ta_1\}$$

The introduction of a multiplication $\{a_0,a_1\}\{b_0,b_1\}=\{a_0b_0,a_0b_1+a_1b_0\}$ leads to an algebra if it is:

1) Distribut. $\{a_0,a_1\}(\{b_0,b_1\}+\{c_0,c_1\})=\{a_0,a_1\}\{b_0,b_1\}+\{a_0,a_1\}\{c_0,c_1\}$

2) Has a neutral element: $\{a_0,a_1\}\{1,0\}=\{a_0,a_1\}$

and additionally to a ring if it is

3) Commutative: $\{a_0,a_1\}\{b_0,b_1\}=\{b_0,b_1\}\{a_0,a_1\}$

4) Associative: $\{a_0,a_1\}(\{b_0,b_1\}\{c_0,c_1\})=(\{a_0,a_1\}\{b_0,b_1\})\{c_0,c_1\}$

All these properties are clearly given, since first order power expansion have this multiplication: $(a_0+a_1x)(b_0+b_1x)=a_0b_0+(a_0b_1+a_1b_0)x+O^2$

The Differential Algebra $_1D_1$

By the introduced addition and multiplication we created an algebra, since the multiplication is commutative and associative we also created a ring, but not a field. Complex numbers are a field since there is a multiplicative inverse for all numbers except 0.

$$\{a_0,a_1\}\{b_0,b_1\}=\{a_0b_0,a_0b_1+a_1b_0\} \quad \Rightarrow \quad \{a_0,a_1\}\{\tfrac{1}{a_0},-\tfrac{a_1}{a_0^2}\}=\{1,0\}$$

We further introduce a differentiation: $\partial\{a_0,a_1\}=\{a_1,0\}$

It is a differentiation since it satisfies a product rule:

$$\partial(\{a_0,a_1\}\{b_0,b_1\})=\{a_0b_1+a_1b_0,0\}=(\partial\{a_0,a_1\})\{b_0,b_1\}+\{a_0,a_1\}(\partial\{b_0,b_1\})$$

By adding a differentiation we have created a Differential Algebra (DA).

Differentiation of Polynomials: $f(x)=2+x^2 \quad \Rightarrow \quad f'(x)=2x$

$$f(\{2,1\})=\{2,0\}+\{4,4\}=\{6,4\}=\{f(2),f'(2)\}$$

Since $\{f,f'\}+\{g,g'\}=\{(f+g),(f+g)'\}$, $\{f,f'\}\{g,g'\}=\{(fg),(fg)'\}$

Every polynomial: $P(\{f,f'\})=\{P(f),[P(f)]'\}$ and $\boxed{P(\{x,1\})=\{P(x),P'(x)\}}$

Elementary functions in ${}_1D_1$

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$$e(a_0 + a_1 x) = e(a_0) + e'(a_0) a_1 x + O^2$$

leads to

$$e(\{a_0, a_1\}) = \{e(a_0), a_1 e'(a_0)\}$$

$$\sin(\{a_0, a_1\}) = \{\sin a_0, a_1 \cos a_0\}$$

$$\cos(\{a_0, a_1\}) = \{\cos a_0, -a_1 \sin a_0\}$$

Since $\{f, f'\} + \{g, g'\} = \{(f+g), (f+g)'\}$, $\{f, f'\}\{g, g'\} = \{(fg), (fg)'\}$

and $e(\{f, f'\}) = \{e(f), [e(f)]'\}$

Therefore $F(\{f, f'\}) = \{F(f), [F(f)]'\}$ and $F(\{x, 1\}) = \{F(x), F'(x)\}$

So that automatic differentiation works not only for Polynomials but for any function that is constructed from a finite number of operations and elementary functions.

Computer programs that have differential algebra elements as data types can evaluate any function or algorithm in this data type and obtain derivatives of the function or derivatives of the algorithm.

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The differential algebra $_nD_v$

The concept of $_1D_1$ can be extended to truncated power series of order n and to v variables. This leads to the differential algebra $_nD_v$. For each coefficient in the nth order expansion there is one dimension in the vectors of $_nD_v$.

Power expansions for v variables have extremely many expansion coefficients:

A polynomial of order n in v variables has $\dim({}_nD_v) = \frac{(n+v)!}{n!v!}$ coefficients since

$$\underbrace{\dim({}_nD_v) - \dim({}_{n-1}D_v)}_{z_1^{k_1}\dots z_v^{k_v},\ \sum_{j=1}^{v} k_j = n} = \underbrace{\dim({}_nD_{v-1})}_{z_1^{k_1}\dots z_{v-1}^{k_{v-1}},\ \sum_{j=1}^{v} k_j \le n} \quad , \quad \frac{(n+v)!}{n!v!} - \frac{(n-1+v)!}{(n-1)!v!} = \frac{(n+v-1)!}{n!(v-1)!}$$

and iteration of $_nD_v$ starts with the correct conditions:

$$\dim({}_nD_1) = n+1 = \tfrac{(n+1)!}{n!}$$

$$\dim({}_0D_v) = 1 = \tfrac{v!}{v!}$$

Example: $\dim({}_{10}D_6) = 8008$

Computer programs that have differential algebra elements as data types produce the nth order power expansion of v-dimensional functions or algorithms automatically.

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Equivalence classes and ${}_nD_v$

A TPS(n) of a function f(x) defines the equivalence class of all functions that have the same TPS(n).

$$\text{Def}: \quad f =_n g \quad \text{if} \quad \vec{\partial}^{\vec{k}} f(0) = \vec{\partial}^{\vec{k}} g(0) \quad \forall\, \vec{k} \text{ with order} \le n$$

$=_n$ is an equivalence relation since it has

1) the identity property $f =_n f \quad \forall\, f$
2) the symmetry property $f =_n g \quad if \quad g =_n f$
3) the transitivity property $f =_n h \quad if \quad f =_n g \; and \; g =_n h$

Equivalence classes: $\text{Def}: \quad [f]_n = \{g \mid g =_n f\}$

Arithmetic of equivalence class:

$$[f]_n + [g]_n \equiv [f+g]_n$$
$$[f]_n [g]_n \equiv [f\,g]_n$$
$$t[f]_n \equiv [t\,f]_n$$
$$\partial_j [f]_n \equiv [\partial_j f]_{n-1}$$
$$e([f]_n) \equiv [e(f)]_n$$

Those operations generate a differential algebra.

Composition of Maps

$$f(x),\, g(x) \quad \text{and} \quad [f]_n, [g_n] \in_n D_1$$

$$[f(g(x))]_0 = [f(g(0))]_0$$

$$[f(g(x))]_1 = [f(g(0)) + g'(0) f'(g(0)) x]_1$$

The composition of two TPS(n) can only be computed if the first one is origin preserving, then $[f]_n \circ [g]_n \equiv [f(g(x))]_n$

If two maps that are know to order n and the first one is origin preserving, then the composition of the maps is known to order n.

$$[\vec{M}_1]_n, [\vec{M}_2]_n \in_n D_v^v$$

$$[\vec{M}_2]_n \circ [\vec{M}_1]_n = [\vec{M}_2(\vec{M}_1(\vec{z}))]_n$$

Therefore the reference trajectory is always chosen as origin for the maps accelerator elements.

Inversion of Maps

The nth order inverse of an origin preserving function can be computed within the differential algebra (DA):

$$\vec{M}(\vec{z}) = \underline{M}_1\vec{z} + \vec{N}(\vec{z})$$

$$\vec{M} \circ \vec{M}^{-1}(\vec{z}) = \underline{M}_1\vec{M}^{-1} + \vec{N} \circ \vec{M}^{-1} = \vec{z}$$

$$\vec{M}^{-1} = \underline{M}_1^{-1}(\vec{z} - \vec{N} \circ \vec{M}^{-1})$$

$$[\vec{M}^{-1}]_n = \underline{M}_1^{-1}[\vec{z} - \vec{N} \circ \vec{M}^{-1}]_n = \underline{M}_1^{-1}(\vec{z} - [\vec{N}]_n \circ [\vec{M}^{-1}]_{n-1})$$

Iterative computation of the inverse:

$$[\vec{M}^{-1}]_1 = [\underline{M}_1^{-1}\vec{z}]_1$$

$$[\vec{M}^{-1}]_2 = \underline{M}_1^{-1}(\vec{z} - [\vec{N}]_2 \circ [\underline{M}_1^{-1}\vec{z}]_1)$$

$$[\vec{M}^{-1}]_3 = \underline{M}_1^{-1}(\vec{z} - [\vec{N}]_3 \circ (\underline{M}_1^{-1}(\vec{z} - [\vec{N}]_2 \circ [\underline{M}_1^{-1}\vec{z}]_1)))$$

$$\vdots$$

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Generating Functions

The motion of particles can be represented by Generating Functions

Each flow or transport map: $\vec{z}(s) = \vec{M}(s, \vec{z}_0)$

With a Jacobi Matrix : $M_{ij} = \partial_{z_{0j}} M_i$ or $\underline{M} = (\vec{\partial}_0 \vec{M}^T)^T$

That is Symplectic: $\underline{M}\,\underline{J}\,\underline{M}^T = \underline{J}$

Can be represented by a Generating Function:

$$F_1(\vec{q}, \vec{q}_0, s) \quad \text{with} \quad \vec{p} = -\vec{\partial}_q F_1 \quad , \quad \vec{p}_0 = \vec{\partial}_{q_0} F_1$$

$$F_2(\vec{p}, \vec{q}_0, s) \quad \text{with} \quad \vec{q} = \vec{\partial}_p F_2 \quad , \quad \vec{p}_0 = \vec{\partial}_{q_0} F_2$$

$$F_3(\vec{q}, \vec{p}_0, s) \quad \text{with} \quad \vec{p} = -\vec{\partial}_q F_3 \quad , \quad \vec{q}_0 = -\vec{\partial}_{p_0} F_3$$

$$F_4(\vec{p}, \vec{p}_0, s) \quad \text{with} \quad \vec{q} = \vec{\partial}_q F_4 \quad , \quad \vec{q}_0 = -\vec{\partial}_{p_0} F_4$$

6-dimensional motion needs only one function ! But to obtain the transport map this has to be inverted.

Computation of Generating Functions

For any map for which the TPS(n) is know, a TPS(n+1) of a generating function that produces this map can be computed. For example, looking for

$$F_1(\vec q,\vec q_0,s) \quad \text{with} \quad \vec p = -\vec\partial_q F_1(\vec q,\vec q_0,s) \quad , \quad \vec p_0 = \vec\partial_{q_0} F_1(\vec q,\vec q_0,s)$$

$\vec z = \vec M(\vec z_0)$ is given as TPS(n)

$$\begin{pmatrix}\vec q\\ \vec q_0\end{pmatrix} = \begin{pmatrix}\vec M_q(\vec z_0)\\ \vec q_0\end{pmatrix} = \vec l(\vec z_0)\,, \quad \begin{pmatrix}\vec p_0\\ \vec p\end{pmatrix} = \begin{pmatrix}\vec p_0\\ \vec M_p(\vec z_0)\end{pmatrix} = \vec h(\vec z_0) = \underline{J}\left[\vec\partial F_1(\vec q,\vec q_0)\right]_{\vec l(\vec z_0)}$$

$$\vec\partial F_1 = -\underline{J}\vec h\circ\vec l^{-1} \quad \Rightarrow F_1 = -\underline{J}\int_0^{(\vec q,\vec q_0)} \vec h\circ\vec l^{-1}(\vec Q)\,d\vec Q$$

$$[\vec M]_n \quad \Rightarrow [\vec l]_n, [\vec h]_n \quad \Rightarrow [\vec l^{-1}]_n, [\vec h]_n\circ[\vec l^{-1}]_n$$

$$[F_1]_{n+1} = -\underline{J}\int_0^{(\vec q,\vec q_0)} [\vec h]_n\circ[\vec l^{-1}]_n\,d\vec Q$$

Particle coordinates (q0,p0) are propagated by such generating functions when zeros of the following equations are found numerically:

$$\vec p + \vec\partial_q F_1(\vec q,\vec q_0,s) = \vec 0 \quad \text{and} \quad \vec p_0 - \vec\partial_{q_0} F_1(\vec q,\vec q_0,s) = \vec 0$$

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F ↦ SP(2N)

Generating Functions produce symplectic tranport maps

$$F_1(\vec{q},\vec{q}_0,s) \quad \text{with} \quad \vec{p}=-\vec{\partial}_q F_1(\vec{q},\vec{q}_0,s) \quad , \quad \vec{p}_0 = \vec{\partial}_{q_0} F_1(\vec{q},\vec{q}_0,s)$$

$$\left.\begin{aligned} \vec{z} &= \begin{pmatrix} \vec{q} \\ \vec{p} \end{pmatrix} = \begin{pmatrix} \vec{q} \\ -\vec{\partial}_q F_1(\vec{q},\vec{q}_0,s) \end{pmatrix} = \vec{f}(\vec{Q},s) \\ \vec{z}_0 &= \begin{pmatrix} \vec{q}_0 \\ \vec{p}_0 \end{pmatrix} = \begin{pmatrix} \vec{q}_0 \\ \vec{\partial}_{q_0} F_1(\vec{q},\vec{q}_0,s) \end{pmatrix} = \vec{g}(\vec{Q},s) \end{aligned}\right\} \quad \begin{aligned} \vec{z} &= \vec{f}(\vec{g}^{-1}(\vec{z}_0,s),s) \\ \vec{M} &= \vec{f}\circ\vec{g}^{-1} \end{aligned}$$

(function concatenation)

Jacobi matrix of concatenated functions:

$$\vec{C}(\vec{z}_0) = \vec{A}\circ\vec{B}(\vec{z}_0)$$

$$C_{ij} = \partial_j C_i = \sum_k \partial_{z_{0j}} B_k(\vec{z}_0)\left[\partial_{z_k} A_i(\vec{z})\right]_{\vec{z}=\vec{B}(\vec{z}_0)} \quad \Rightarrow \quad \underline{C} = \underline{A}(\vec{B})\underline{B}$$

$$\vec{M}\circ\vec{g} = \vec{f} \quad \Rightarrow \quad \underline{M}(\vec{g}) = \underline{F}\,\underline{G}^{-1}$$

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$$\vec{f}(\vec{Q},s)=\begin{pmatrix}\vec{q}\\ -\vec{\partial}_q F_1(\vec{q},\vec{q}_0,s)\end{pmatrix} \Rightarrow F=\begin{pmatrix}\underline{1} & \underline{0}\\ -\vec{\partial}_q\vec{\partial}_q^T F_1 & -\vec{\partial}_q\vec{\partial}_{q_0}^T F_1\end{pmatrix}$$

$$\vec{g}(\vec{Q},s)=\begin{pmatrix}\vec{q}_0\\ \vec{\partial}_{q_0} F_1(\vec{q},\vec{q}_0,s)\end{pmatrix} \Rightarrow G=\begin{pmatrix}\underline{0} & \underline{1}\\ \vec{\partial}_{q_0}\vec{\partial}_q^T F_1 & \vec{\partial}_{q_0}\vec{\partial}_{q_0}^T F_1\end{pmatrix}$$

$$F=\begin{pmatrix}1 & 0\\ -F_{11} & -F_{12}\end{pmatrix},\quad G=\begin{pmatrix}0 & 1\\ F_{21} & F_{22}\end{pmatrix} \Rightarrow G^{-1}=\begin{pmatrix}-F_{21}^{-1}F_{22} & F_{21}^{-1}\\ 1 & 0\end{pmatrix}$$

$$\underline{M}(\vec{g})=FG^{-1}=\begin{pmatrix}-F_{21}^{-1}F_{22} & F_{21}^{-1}\\ F_{11}F_{21}^{-1}F_{22}-F_{12} & -F_{11}F_{21}^{-1}\end{pmatrix}$$

$$\underline{M}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\underline{M}^T$$

→ The map from a generating function is symplectic.

$$=\begin{pmatrix}-F_{21}^{-1} & -F_{21}^{-1}F_{22}\\ F_{11}F_{21}^{-1} & F_{11}F_{21}^{-1}F_{22}-F_{12}\end{pmatrix}\begin{pmatrix}-F_{22}F_{12}^{-1} & F_{22}F_{12}^{-1}F_{11}-F_{21}\\ F_{12}^{-1} & -F_{12}^{-1}F_{11}\end{pmatrix}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$

SP(2N) $\mapsto$ F

Symplectic tranport maps have a Generating Functions

$$\vec{z} = \vec{M}(\vec{z}_0)$$

$$\begin{pmatrix} \vec{q} \\ \vec{q}_0 \end{pmatrix} = \begin{pmatrix} \vec{M}_1(\vec{z}_0) \\ \vec{q}_0 \end{pmatrix} = \vec{l}(\vec{z}_0)\,, \quad \begin{pmatrix} \vec{p}_0 \\ \vec{p} \end{pmatrix} = \begin{pmatrix} \vec{p}_0 \\ \vec{M}_2(\vec{z}_0) \end{pmatrix} = \vec{h}(\vec{z}_0) = \underline{J}\left[\vec{\partial} F_1(\vec{q}, \vec{q}_0)\right]_{\vec{l}(\vec{z}_0)}$$

$$\vec{\partial} F_1 = -\underline{J}\vec{h} \circ \vec{l}^{\,-1} = \vec{F}$$

For F_1 to exist it is necessary and sufficient that $\partial_i F_j = \partial_j F_i \quad \Rightarrow \quad \underline{F} = \underline{F}^T$

$$-\underline{J}\vec{h} = \vec{F} \circ \vec{l} \quad \Rightarrow \quad -\underline{J}\,\underline{h} = \underline{F}(\vec{l}\,)\,\underline{l}$$

Is $\underline{J}$ $\underline{h}$ $\underline{l}^{-1}$ symmetric ? Yes since:

$$\underline{J}\,\underline{h}\,\underline{l}^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \vec{\partial}^T_{q_0} \vec{M}_2 & \vec{\partial}^T_{p_0} \vec{M}_2 \end{pmatrix} \begin{pmatrix} \vec{\partial}^T_{q_0} \vec{M}_1 & \vec{\partial}^T_{p_0} \vec{M}_1 \\ 1 & 0 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} M_{21} & M_{22} \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ M_{12}^{-1} & -M_{12}^{-1} M_{11} \end{pmatrix} = \begin{pmatrix} M_{22} M_{12}^{-1} & M_{21} - M_{22} M_{12}^{-1} M_{11} \\ M_{12}^{-1} & M_{12}^{-1} M_{11} \end{pmatrix}$$

$$\underline{J}\,\underline{h}\,\underline{l}^{-1} = \begin{pmatrix} M_{22}M_{12}^{-1} & M_{21} - M_{22}M_{12}^{-1}M_{11} \\ M_{12}^{-1} & M_{12}^{-1}M_{11} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$$\vec{M}(\vec{z}_0) = \begin{pmatrix} \vec{M}_1(\vec{q}_0, \vec{p}_0) \\ \vec{M}_2(\vec{q}_0, \vec{p}_0) \end{pmatrix}, \quad \underline{M} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

$$\underline{M}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\underline{M}^T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \begin{pmatrix} -M_{12} & M_{11} \\ -M_{22} & M_{21} \end{pmatrix}\begin{pmatrix} M_{11}^T & M_{21}^T \\ M_{12}^T & M_{22}^T \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$M_{12}M_{11}^T = M_{11}M_{12}^T \qquad \Rightarrow \qquad (M_{12}^{-1}M_{11})^T = [M_{12}^{-1}M_{11}M_{12}^T]M_{12}^{-T} = M_{12}^{-1}M_{11}$$

$$M_{21}M_{22}^T = M_{22}M_{21}^T$$

$$D = D^T$$

$$M_{11}M_{22}^T - M_{12}M_{21}^T = 1$$

$$M_{22}M_{11}^T - M_{21}M_{12}^T = 1$$

$$A = A^T$$

$$(M_{22}M_{12}^{-1})^T = [M_{22}M_{11}^T M_{12}^{-T} - M_{21}]M_{22}^T = M_{22}[M_{12}^{-1}M_{11}M_{22}^T - M_{21}^T] = M_{22}M_{12}^{-1}$$

$$M_{21} - M_{22}M_{12}^{-1}M_{11} = M_{21} - M_{22}M_{11}M_{12}^{-T} = M_{12}^{-T} \longrightarrow B = C^T$$

Symplectic Representations

Hamiltonian

$$\vec{z}\,' = \underline{J}\,\vec{\partial} H(\vec{z}, s)$$

Generating Functions

$$(\vec{p}, \vec{p}_0) = -\underline{J}\,\vec{\partial} F_1(\vec{q}, \vec{q}_0, s)$$

Symplectic transport map

$$\underline{M}\,\underline{J}\,\underline{M}^T = \underline{J}$$

Advantages of Symplecticity

- Determinant of the transfer matrix of linear motion is 1:

$$\vec{z}(s) = \underline{M}(s) \cdot \vec{z}_0 \quad \text{with} \quad \det(\underline{M}(s)) = +1$$

- One function suffices to compute the total nonlinear transfer map:

$$F_1(\vec{q}, \vec{q}_0, s) \quad \text{with} \quad \vec{p} = -\vec{\partial}_q F_1(\vec{q}, \vec{q}_0, s) \quad , \quad \vec{p}_0 = \vec{\partial}_{q_0} F_1(\vec{q}, \vec{q}_0, s)$$

$$\left.\begin{aligned} \vec{z} &= \begin{pmatrix} \vec{q} \\ \vec{p} \end{pmatrix} = \begin{pmatrix} \vec{q} \\ -\vec{\partial}_q F_1(\vec{q}, \vec{q}_0, s) \end{pmatrix} = \vec{f}(\vec{Q}, s) \\ \vec{z}_0 &= \begin{pmatrix} \vec{q}_0 \\ \vec{p}_0 \end{pmatrix} = \begin{pmatrix} \vec{q}_0 \\ \vec{\partial}_{q_0} F_1(\vec{q}, \vec{q}_0, s) \end{pmatrix} = \vec{g}(\vec{Q}, s) \end{aligned}\right\} \quad \begin{aligned} \vec{z} &= \vec{f}(\vec{g}^{-1}(\vec{z}_0, s), s) \\ \vec{M} &= \vec{f} \circ \vec{g}^{-1} \end{aligned}$$

- Therefore Taylor Expansion coefficients of the transport map are related.
- Computer codes can numerically approximate $\vec{M}(s, \vec{z}_0)$ with exact symplectic symmetry.
- Liouville's Theorem for phase space densities holds.

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Eigenvalues of a Symplectic Matrix

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For matrices with real coefficients:

If there is an eigenvector and eigenvalue: $\underline{M}\vec{v}_i = \lambda_i \vec{v}_i$

then the complex conjugates are also eigenvector and eigenvalue: $\underline{M}\vec{v}_i^{\,*} = \lambda_i^* \vec{v}_i^{\,*}$

For symplectic matrices:

If there are eigenvectors and eigenvalues: $\underline{M}\vec{v}_i = \lambda_i \vec{v}_i$ with $\underline{J} = \underline{M}^T \underline{J}\,\underline{M}$

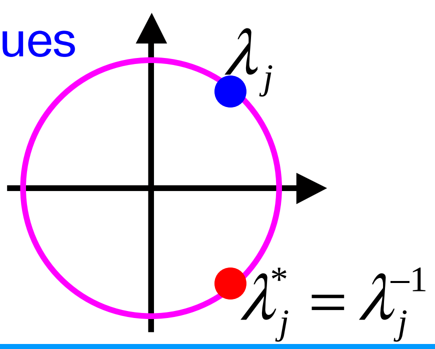
then $\vec{v}_i^{\,T} \underline{J}\vec{v}_j = \vec{v}_i^{\,T} \underline{M}^T \underline{J}\,\underline{M}\vec{v}_j = \lambda_i \lambda_j \vec{v}_i^{\,T} \underline{J}\vec{v}_j \quad \Rightarrow \quad \vec{v}_i^{\,T} \underline{J}\vec{v}_j (\lambda_i \lambda_j - 1) = 0$

Therefore $\underline{J}\vec{v}_j$ is orthogonal to all eigenvectors with eigenvalues that are not $1/\lambda_j$. Since it cannot be orthogonal to all eigenvectors, there is at least one eigenvector with eigenvalue $1/\lambda_j$

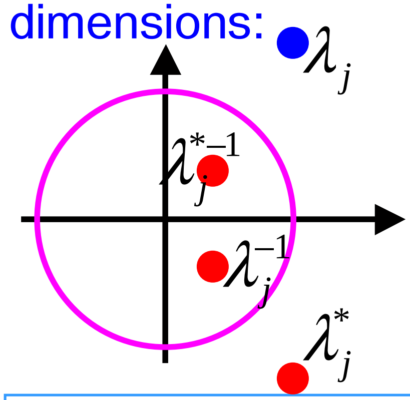
Two dimensions: λ_j is eigenvalue

Then $1/\lambda_j$ and λ_j^* are eigenvalues

$1/\lambda_j = \lambda_j^* \quad \Rightarrow \quad |\lambda_j| = 1$



Four dimensions:



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Perturbations to linear motion

$$\begin{pmatrix} x' \\ a' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -K & 0 \end{pmatrix} \begin{pmatrix} x \\ a \end{pmatrix} + \begin{pmatrix} 0 \\ \Delta f \end{pmatrix}$$

$$\begin{pmatrix} x \\ a \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \sin(\psi+\phi_0) \\ \cos(\psi+\phi_0) \end{pmatrix} = \sqrt{2J}\, \underline{\beta}\, \vec{S}$$

This would be a solution with constant J and ϕ when $\Delta f=0$.

Variation of constants:

$$\frac{J'}{\sqrt{2J}} \underline{\beta}\, \vec{S} + \sqrt{2J}\, \phi_0' \begin{pmatrix} 0 & \sqrt{\beta} \\ -\frac{1}{\sqrt{\beta}} & -\frac{\alpha}{\sqrt{\beta}} \end{pmatrix} \vec{S} = \begin{pmatrix} 0 \\ \Delta f \end{pmatrix}$$

$$\frac{J'}{\sqrt{2J}} \vec{S} + \sqrt{2J}\, \phi_0' \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \vec{S} = \underline{\beta}^{-1} \begin{pmatrix} 0 \\ \Delta f \end{pmatrix} \quad \text{with} \quad \underline{\beta}^{-1} = \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix}$$

$$\frac{J'}{\sqrt{2J}} = \cos(\psi+\phi_0)\sqrt{\beta}\Delta f \quad , \quad \sqrt{2J}\, \phi_0' = -\sin(\psi+\phi_0)\sqrt{\beta}\Delta f$$

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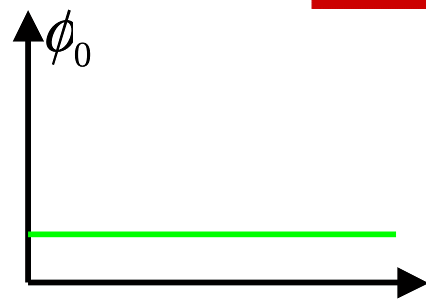
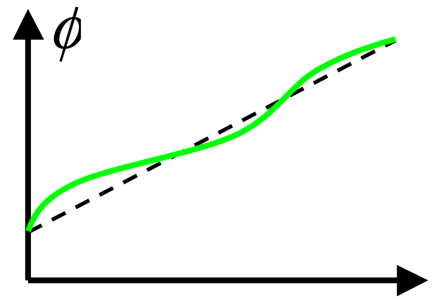
Simplification of linear motion

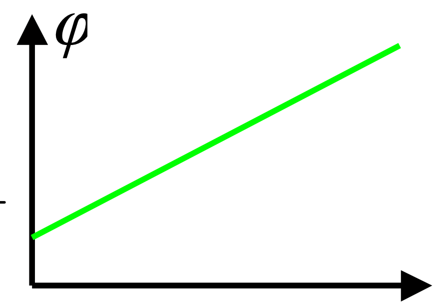
$$\begin{pmatrix} x \\ a \end{pmatrix} = \sqrt{2J}\begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix}\begin{pmatrix} \sin(\psi+\phi_0) \\ \cos(\psi+\phi_0) \end{pmatrix} \Rightarrow \begin{matrix} J'=0 \\ \phi_0'=0 \end{matrix}$$



$$\begin{pmatrix} x \\ a \end{pmatrix} = \sqrt{2J}\begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix}\begin{pmatrix} \sin\phi \\ \cos\phi \end{pmatrix} \Rightarrow \begin{matrix} J'=0 \\ \phi'=\frac{1}{\beta} \end{matrix}$$



$$\begin{pmatrix} x \\ a \end{pmatrix} = \sqrt{2J}\begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix}\begin{pmatrix} \sin(\psi-\mu\frac{s}{L}+\varphi) \\ \cos(\psi-\mu\frac{s}{L}+\varphi) \end{pmatrix} \Rightarrow \begin{matrix} J'=0 \\ \varphi'=\mu\frac{1}{L} \end{matrix}$$



$$\tilde{\psi} = \psi - \mu\tfrac{s}{L} \Rightarrow \tilde{\psi}(s+L) = \tilde{\psi}(s)$$

Corresponds to Floquet's Theorem

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Quasi-periodic Perturbation

$$J' = \cos(\psi+\phi)\sqrt{2J\beta}\Delta f \quad , \quad \phi' = -\sin(\psi+\phi)\sqrt{\tfrac{\beta}{2J}}\Delta f$$

$$J' = \cos(\tilde{\psi}+\varphi)\sqrt{2J\beta}\Delta f \quad , \quad \varphi' = \mu\tfrac{1}{L} - \sin(\tilde{\psi}+\varphi)\sqrt{\tfrac{\beta}{2J}}\Delta f$$

New independent variable $\vartheta = 2\pi\frac{s}{L}$

$$\tfrac{d}{d\vartheta}J = \cos(\tilde{\psi}+\varphi)\sqrt{2J\beta}\Delta f\,\tfrac{L}{2\pi} \quad , \quad \tfrac{d}{d\vartheta}\varphi = \upsilon - \sin(\tilde{\psi}+\varphi)\sqrt{\tfrac{\beta}{2J}}\Delta f\,\tfrac{L}{2\pi}$$

$$\Delta f(x) = \Delta f(\sqrt{2J\beta}\sin(\tilde{\psi}+\varphi))$$

The perturbations are 2π periodic in ϑ and in φ

φ is approximately $\varphi \approx \upsilon\cdot\vartheta$

For irrational ν, the perturbations are quasi-periodic.

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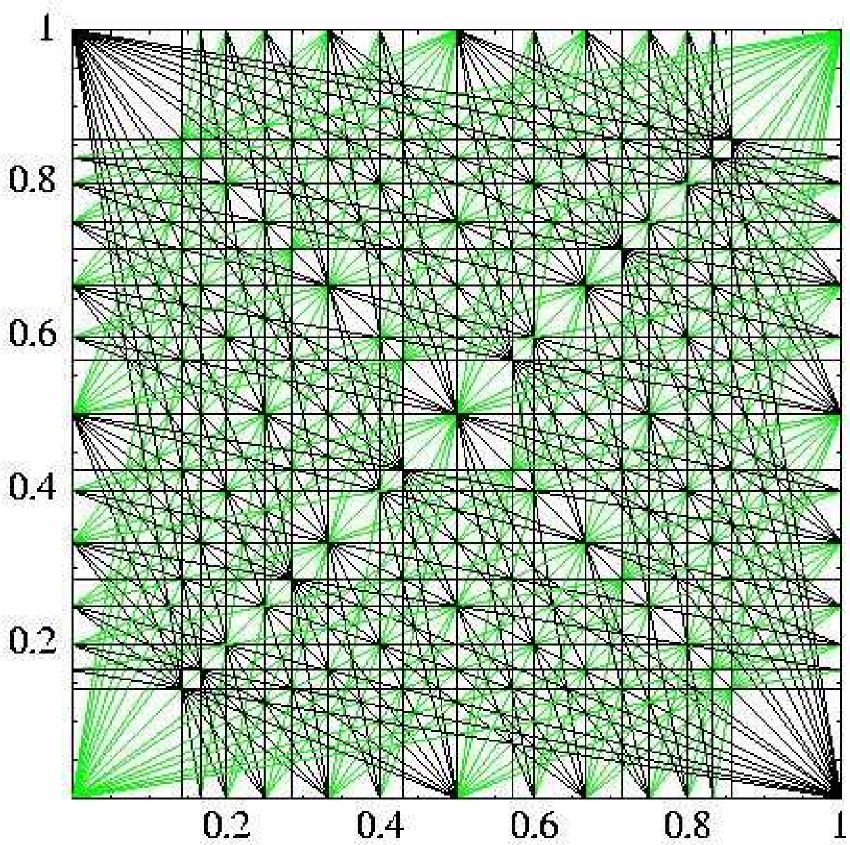
Resonances Diagram

$n + m_x \upsilon_x + m_y \upsilon_y \approx 0$ means that oscillations in y can drive oscillations in x in

$$x'' = -K\, x + \Delta f(x, s)$$



All these resonances have to be avoided by their respective resonance width.

The position of an accelerator in the tune plane s called its Working Point.

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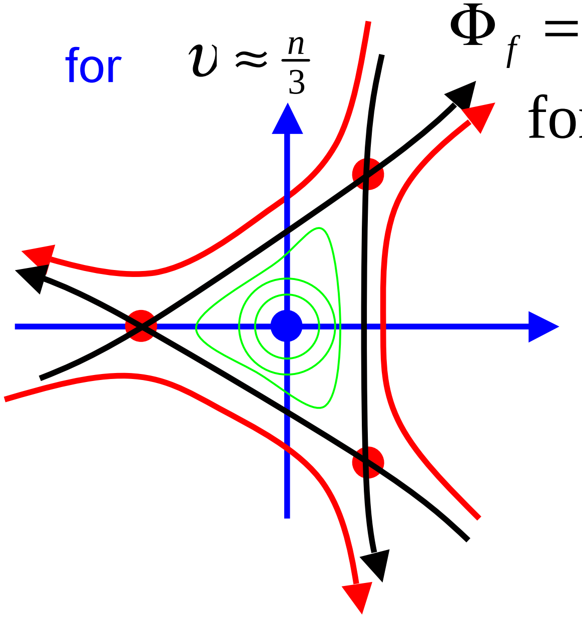
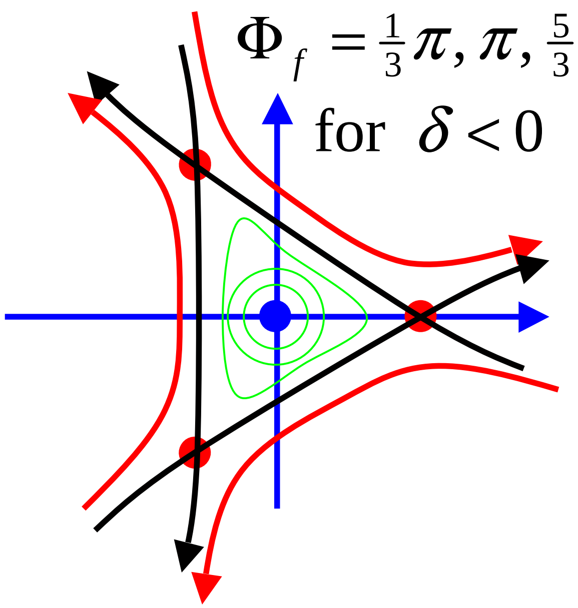
Resonances

for $\upsilon \approx \frac{n}{3}$

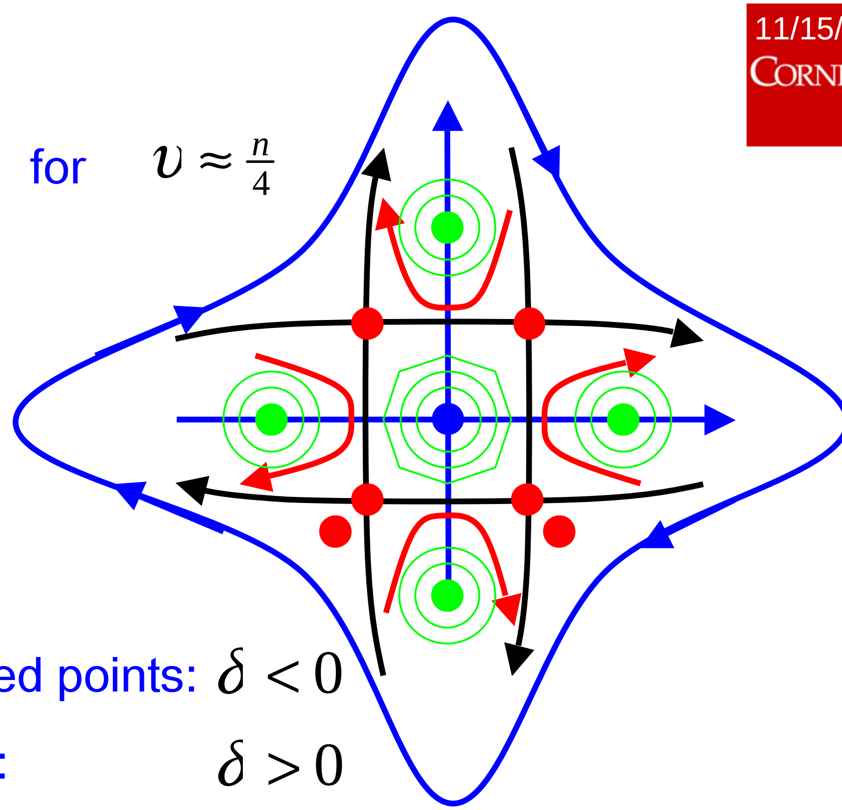
$\Phi_f = \frac{1}{3}\pi, \pi, \frac{5}{3}\pi$ for $\delta > 0$



$\Phi_f = \frac{1}{3}\pi, \pi, \frac{5}{3}\pi$ for $\delta < 0$



for $\upsilon \approx \frac{n}{4}$



Either 8 fixed points: $\delta < 0$

or none for: $\delta > 0$

How can the motion inside the fixed points be simplified for a real accelerator ?

→ Normal From Theory

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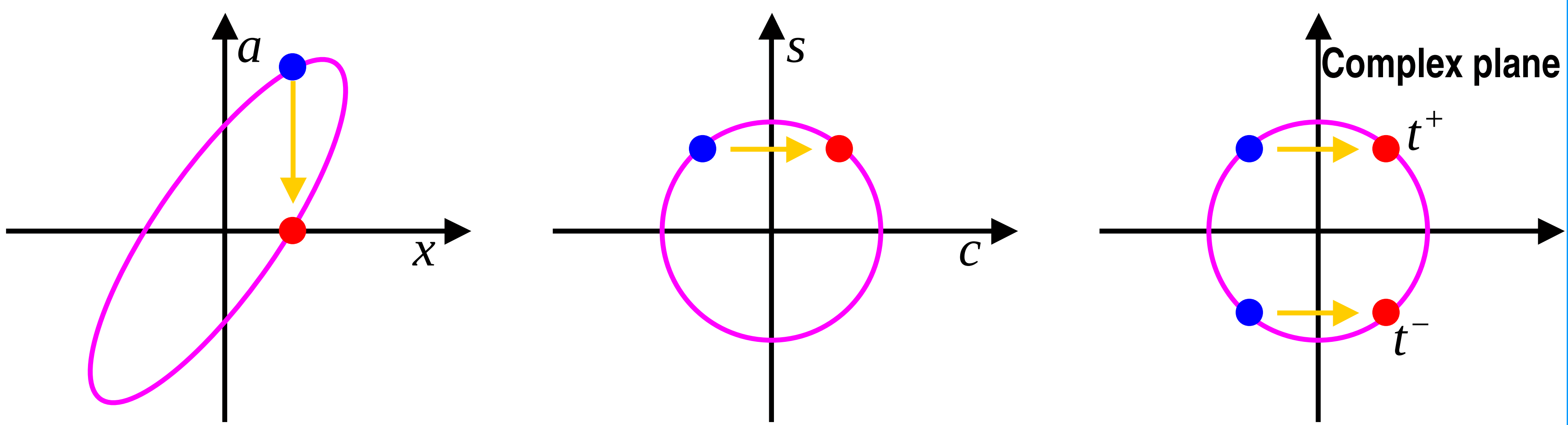
Linear Normal Form Theory

$$\begin{pmatrix} x \\ a \end{pmatrix} = \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix} \begin{pmatrix} x_0 \\ a_0 \end{pmatrix}, \quad \begin{pmatrix} s \\ c \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix} \begin{pmatrix} x_0 \\ a_0 \end{pmatrix}$$

$$\begin{pmatrix} t^+ \\ t^- \end{pmatrix} = \frac{1}{\sqrt{2i}} \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} \begin{pmatrix} s \\ c \end{pmatrix}$$

$$\begin{pmatrix} t^+ \\ t^- \end{pmatrix} = \frac{1}{2i} \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix} \begin{pmatrix} 1 & -1 \\ i & i \end{pmatrix} \begin{pmatrix} t_0^+ \\ t_0^- \end{pmatrix} = \begin{pmatrix} e^{i\mu} & 0 \\ 0 & e^{-i\mu} \end{pmatrix} \begin{pmatrix} t_0^+ \\ t_0^- \end{pmatrix} = \underline{D}\,\vec{t}$$



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Non-linear Normal Form Theory

In nonlinear normal form theory, one tries to remove as many of the higher order terms in the transport map by nonlinear coordinate transformations.

$$\begin{pmatrix} t^+ \\ t^- \end{pmatrix} = \begin{pmatrix} e^{-i\Delta\psi} & 0 \\ 0 & e^{i\Delta\psi} \end{pmatrix} \begin{pmatrix} \frac{\alpha_0 - i}{\sqrt{\beta_0}} & \sqrt{\beta_0} \\ -\frac{\alpha_0 + i}{\sqrt{\beta_0}} & -\sqrt{\beta_0} \end{pmatrix} \begin{pmatrix} t^+ \\ t^- \end{pmatrix}$$

$$\text{Map}: \underline{D}\vec{t}_0 + \vec{N}(\vec{t}_0)\,, \quad \vec{t}_n = \vec{t}_{n-1} + \vec{A}_n(\vec{t}_{n-1})\,, \quad \vec{A}_n(\vec{t}_{n-1}) =_{n-1} \vec{0}$$

$$\text{Map}: \underline{D}\vec{t}_n + \vec{N}_n(\vec{t}_n)$$

$$\underline{D}\vec{t}_n + \vec{N}_n(\vec{t}_n) =_n [\vec{1} + \vec{A}_n] \circ [\vec{D} + \vec{N}_{n-1}] \circ [\vec{1} - \vec{A}_n] = \vec{D} + \vec{N}_{n-1} + [\vec{A}_n, \vec{D}]$$

$\vec{A}_n$ is chosen so that as many nth order terms in $\vec{N}_{n-1}$ are eliminated as possible.

$$\underline{D}\vec{t}_n + \vec{N}_n(\vec{t}_n) =_n [\vec{1} + \vec{A}_n] \circ [\vec{D} + \vec{N}_{n-1}] \circ [\vec{1} - \vec{A}_n] =_n \vec{D} + \vec{N}_{n-1} + [\vec{A}_n, \vec{D}]$$

$$\vec{N}_n(\vec{t}_n) =_n \vec{N}_{n-1} + \vec{A}_n(\underline{D}\vec{t}_n) - \underline{D}\vec{A}_n(\vec{t}_n)$$

$$N^{\pm}_{n,j,\vec{k}} = N^{\pm}_{n-1,j,\vec{k}} + A^{\pm}_{n,j,\vec{k}} e^{i(\vec{\mu}\cdot\vec{k}^+ - \vec{\mu}\cdot\vec{k}^-)} - e^{\pm i\mu_j} A^{\pm}_{n,j,\vec{k}}$$

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The Normal Form

$$N^{\pm}_{n,j,\vec{k}} = N^{\pm}_{n-1,j,\vec{k}} + A^{\pm}_{n,j,\vec{k}} e^{i(\vec{\mu}\cdot\vec{k}^{+} - \vec{\mu}\cdot\vec{k}^{-})} - e^{\pm i\mu_j} A^{\pm}_{n,j,\vec{k}}$$

All terms for which the denominator is not very small are eliminated by the choice

$$A^{\pm}_{n,j,\vec{k}} = \frac{N^{\pm}_{n-1,j,\vec{k}}}{e^{\pm i\mu_j} - e^{i(\vec{\mu}\cdot\vec{k}^{+} - \vec{\mu}\cdot\vec{k}^{-})}}$$

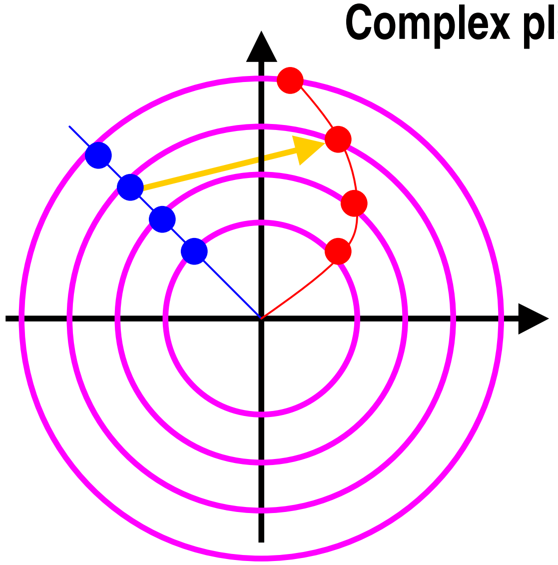
Terms which are not eliminated are:

1) Amplitude dependent tune shift terms:

$$\left.\begin{array}{l} t^{+}_{j}(t^{+}_{1}t^{-}_{1})^{k^{+}_{1}} \dots (t^{+}_{N}t^{-}_{N})^{k^{+}_{N}} \\ t^{-}_{j}(t^{+}_{1}t^{-}_{1})^{k^{+}_{1}} \dots (t^{+}_{N}t^{-}_{N})^{k^{+}_{N}} \end{array}\right\} \quad t^{\pm}_{j} \mapsto t^{\pm}_{j} f_j(|t^{+}_{m}|^2)$$

2) Resonance terms:

$$\vec{\mu}\cdot\vec{m} = m_0$$

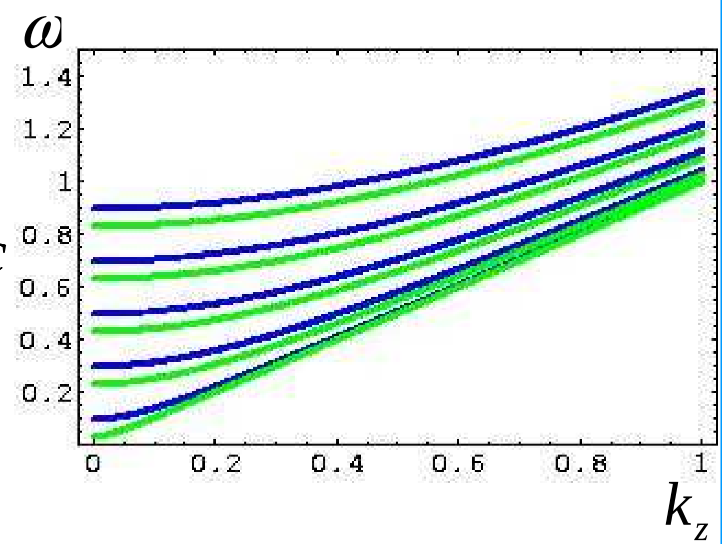


Dispersion relation in waveguides

$$\omega(k_z) = c\sqrt{A_n^{\,2} + k_z^2}$$

Phase velocity $v_{ph} = \omega / k_z = c\sqrt{1+\left(\frac{A_n}{k_z}\right)^2} > c$

Group velocity $v_{gr} = d\omega / dk_z = c / \sqrt{1+\left(\frac{A_n}{k_z}\right)^2} < c$

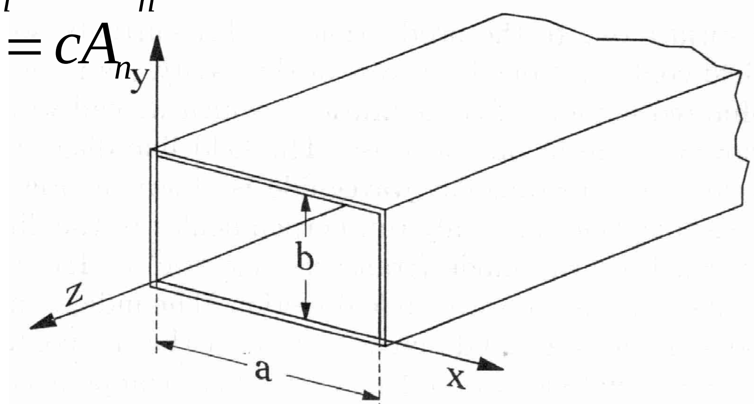


For each excitation frequency ω one obtains a propagation in the wave guide of

$$e^{ik_z z}\ , \quad k_z = \sqrt{(\tfrac{\omega}{c})^2 - A_n^2}$$

Transport for ω above the cutoff frequency $\omega > \omega_n = cA_n$

Damping for ω below the cutoff frequency $\omega < \omega_n = cA_n$



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Rectangular TE and TM Modes

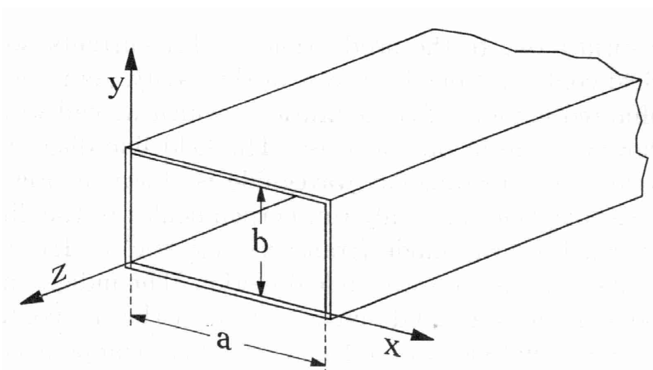
TE Modes

$$\vec{B}(\vec{x}) = B_z \begin{pmatrix} \frac{n\pi}{a}\frac{k_z}{k_{nm}^{(E)2}}\sin(\frac{n\pi}{a}x)\cos(\frac{m\pi}{b}y)\sin(k_z z-\omega t) \\ \frac{m\pi}{b}\frac{k_z}{k_{nm}^{(E)2}}\cos(\frac{n\pi}{a}x)\sin(\frac{m\pi}{b}y)\sin(k_z z-\omega t) \\ \cos(\frac{n\pi}{a}x)\cos(\frac{m\pi}{b}y)\cos(k_z z-\omega t) \end{pmatrix}$$

$$\vec{E}(\vec{x}) = \frac{\omega}{k_{nm}^{(E)2}} B_z \begin{pmatrix} \frac{m\pi}{b}\cos(\frac{n\pi}{a}x)\sin(\frac{m\pi}{b}y)\sin(k_z z-\omega t) \\ -\frac{n\pi}{a}\sin(\frac{n\pi}{a}x)\cos(\frac{m\pi}{b}y)\sin(k_z z-\omega t) \\ 0 \end{pmatrix}$$

TM Modes: Exchange of E and B

Notation: TE_{nm} Mode

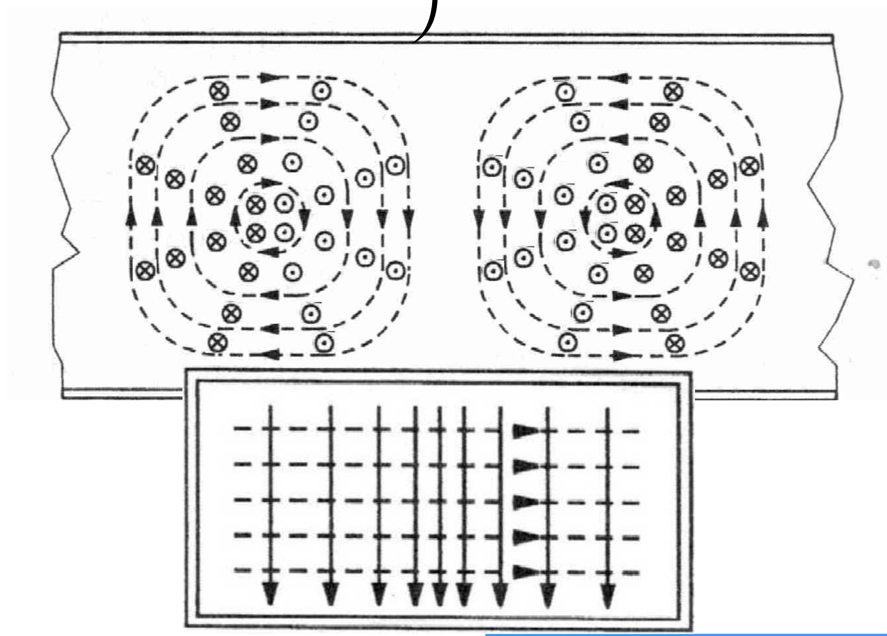


$\vec{E}$ ⟶

$\vec{B}$ - - →

$n=1$

$m=0$



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Fundamental Mode

Mode for particle acceleration: TM_{01}

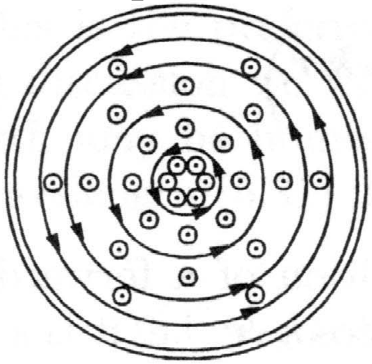
$$E_z(\vec{x}) = E_z J_0(\tfrac{r}{r_0})\cos(k_z z - \omega t)$$

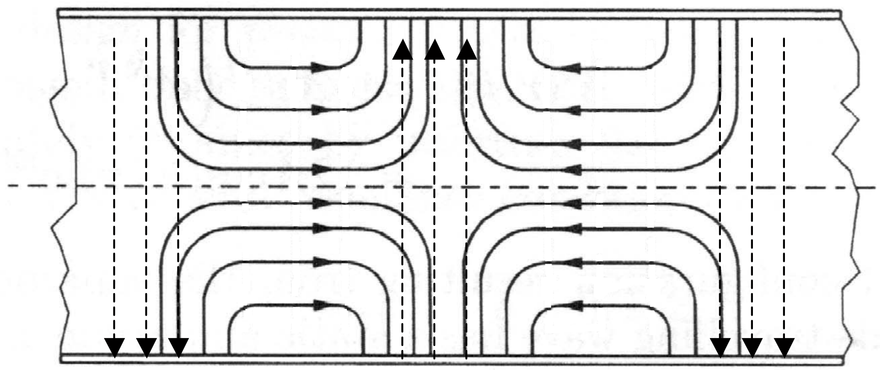
$$E_r(\vec{x}) = -E_z r_1 k_z J_0'(\tfrac{r}{r_1})\sin(k_z z - \omega t)$$

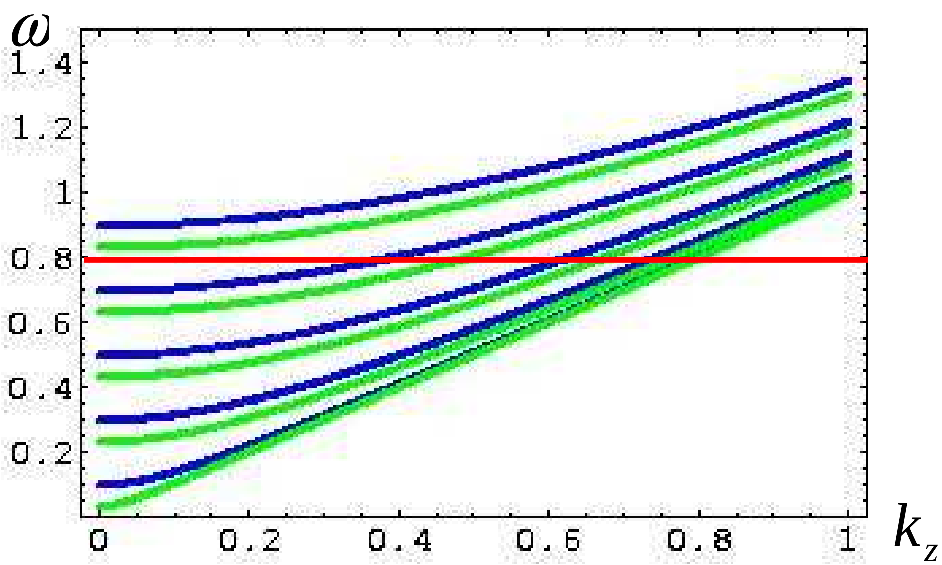
$$E_\varphi(\vec{x}) = 0$$

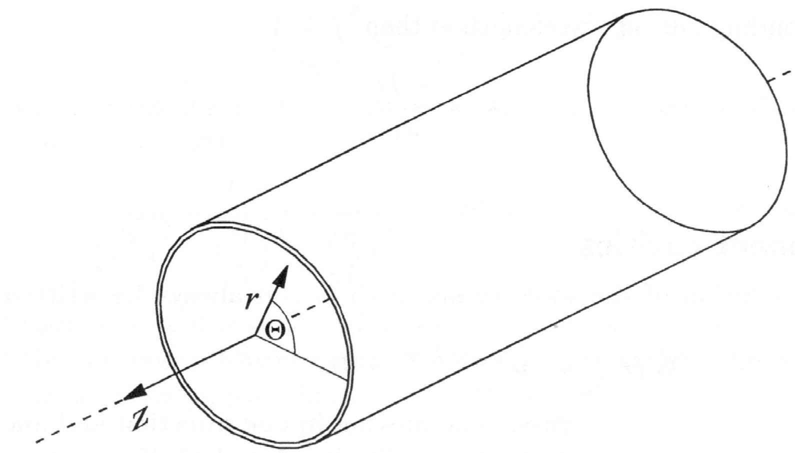
$$B_r(\vec{x}) = 0$$

$$B_\varphi(\vec{x}) = -E_z r_1 \tfrac{\omega}{c^2} J_0'(\tfrac{r}{r_1})\sin(k_z z - \omega t)$$









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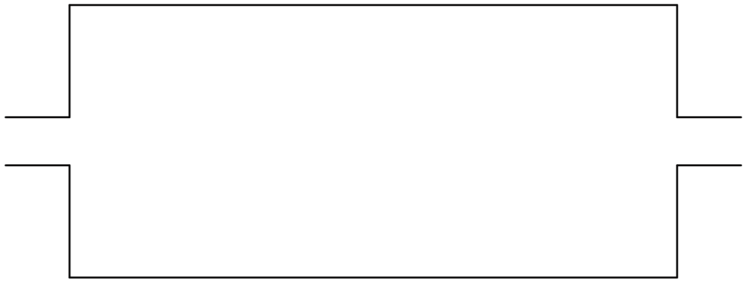
Resonant Cavities

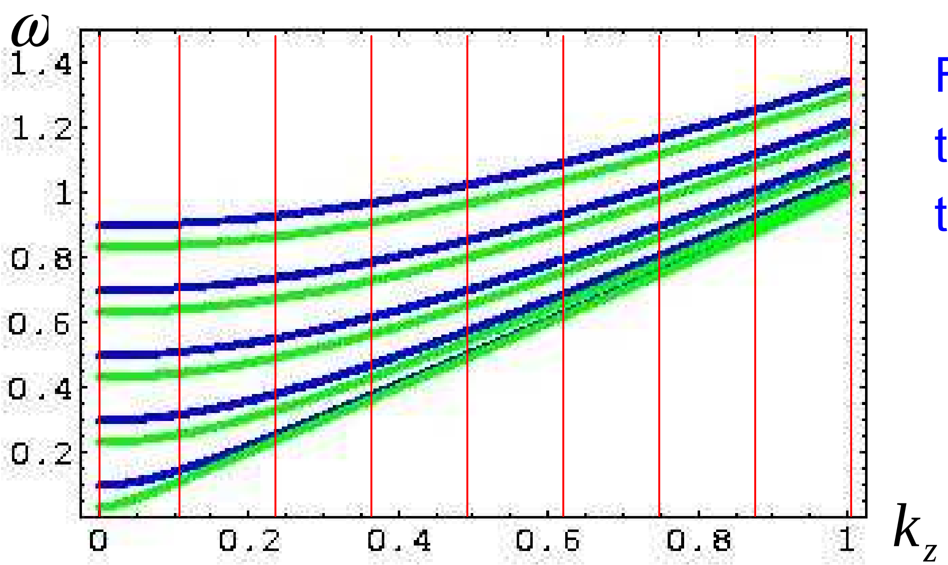


TE Modes: Standing waves with nodes

$$B_z(\vec{x}) \propto \sin(k_z z)\sin(\omega t)\,, \quad k_z = \tfrac{l\pi}{L}$$
$$l > 0$$

TM Modes: Standing waves with nodes

$$E_z(\vec{x}) \propto \cos(k_z z)\cos(\omega t)\,, \quad k_z = \tfrac{l\pi}{L}$$
$$l \geq 0$$

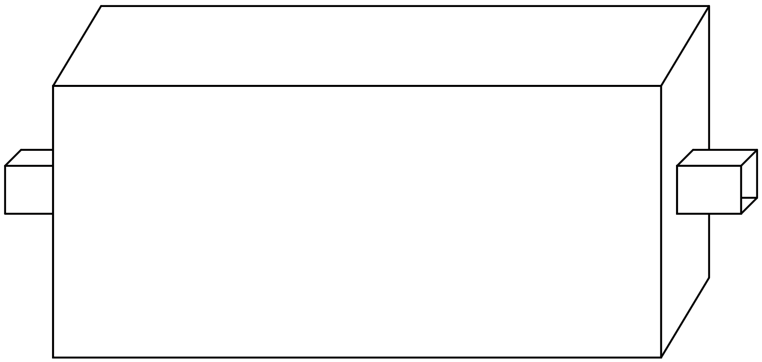


For each mode TE_{nm} or TM_{nm} there is a discrete set of frequencies that can be excited.

$$\omega_{nm}^{(E/B)} = c\sqrt{k_{nm}^{(E/B)2} + \left(\tfrac{l\pi}{L}\right)^2}$$

Resonant Cavities Examples

Rectangular cavity:

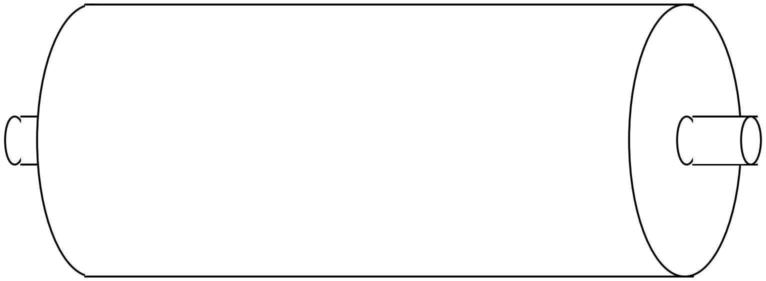


$$\omega_{nm}^{(E/B)} = c\sqrt{\left(\frac{n\pi}{L_x}\right)^2 + \left(\frac{m\pi}{L_y}\right)^2 + \left(\frac{l\pi}{L_z}\right)^2}$$

Fundamental acceleration mode: $\omega_{11}^{(B)} = c\frac{\pi}{L}\sqrt{2}$

$$L_x = L_y = 22cm \quad \Rightarrow \quad f_{110}^{(B)} = 1.0\text{GHz}$$

Pill Box cavity:



$$\omega_{nm}^{(E/B)} = c\sqrt{k_{nm}^{(E/B)2} + \left(\frac{l\pi}{L}\right)^2}$$

$k_{nm}^{(B)}r$ is the mth 0 of the nth Bessel function.

$k_{nm}^{(E)}r$ is the mth extremeum of J_n

Fundamental acceleration mode: $\omega_{01}^{(E)} = c\frac{2.4}{r}$

$$r = 11cm \quad \Rightarrow \quad f_{010}^{(M)} = 1.0\text{GHz}$$

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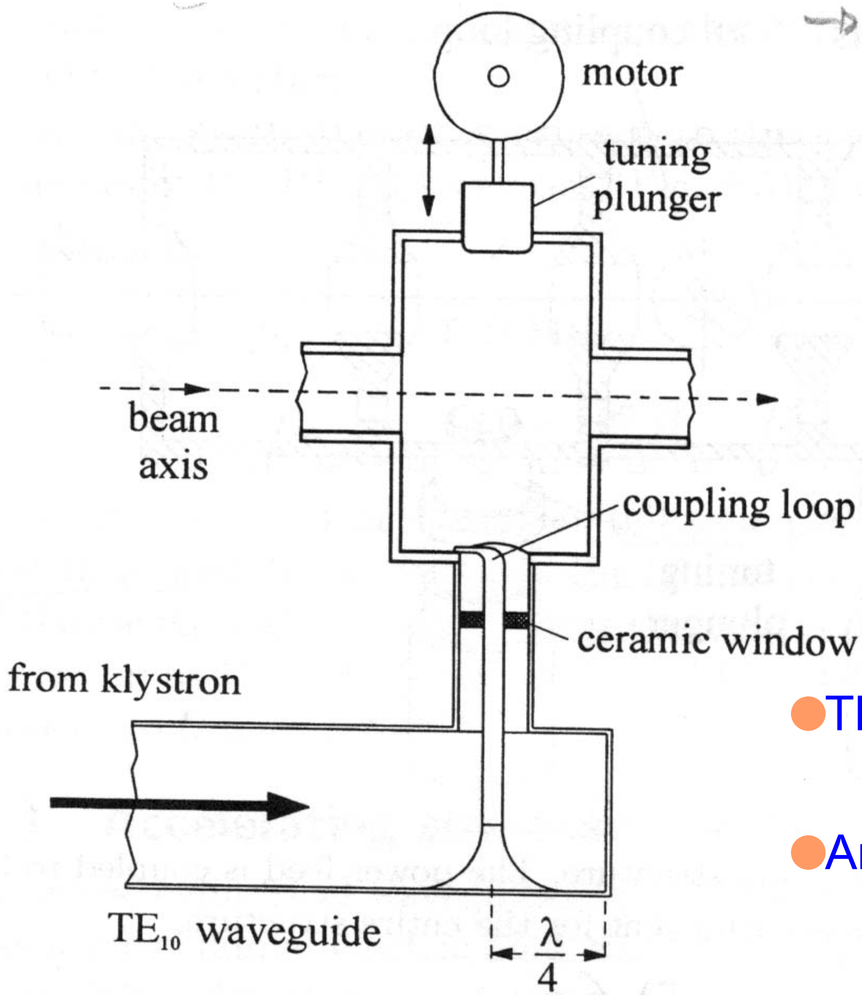
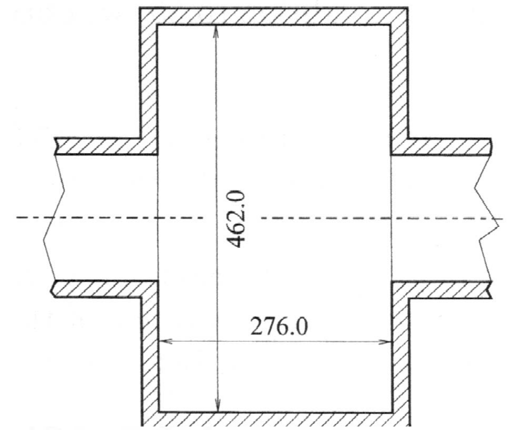
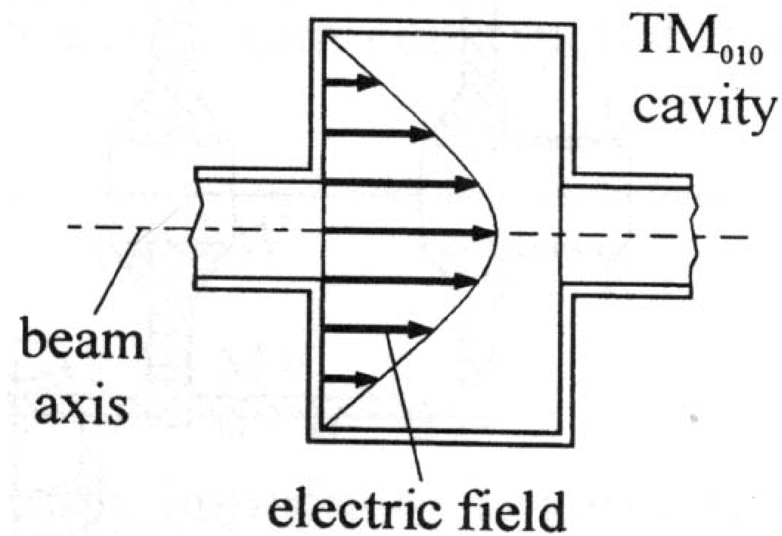
Cavity Operation

500MHz Cavity of DORIS:

$$r = 23.1cm \quad \Rightarrow \quad f_{010}^{(M)} = 0.4967\text{GHz}$$

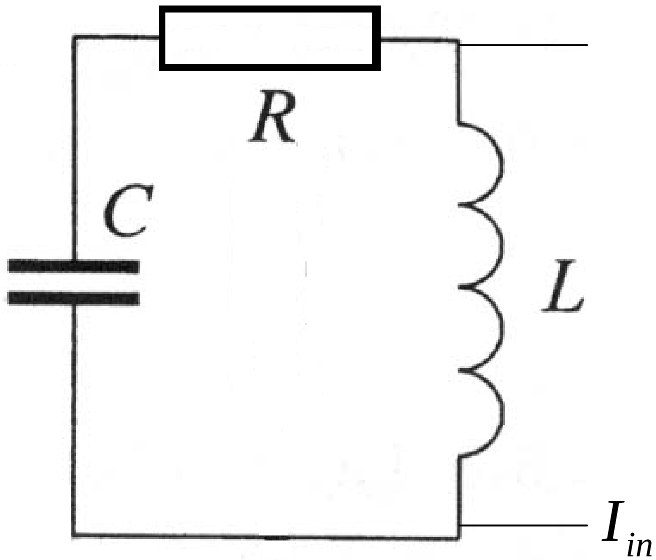






- The frequency is increased and tuned by a tuning plunger.
- An inductive coupling loop excites the magnetic field at the equator of the cavity.

RF systems for accelerators



L and C: determined by the cavity geometry

R : determined by the surface resistance

$$U_C = \int \frac{I_C}{C} dt \to -i \frac{I_C}{C\omega}$$

$$L(\dot{I}_{in} - \dot{I}_C) = RI_C + \int \frac{I_C}{C} dt$$

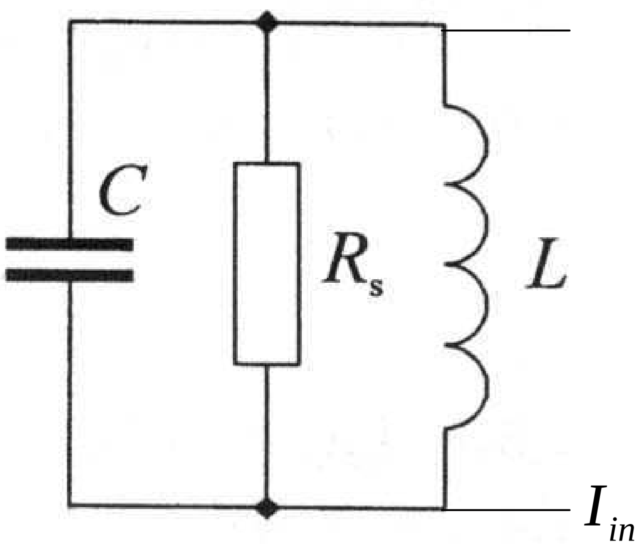
$$I_C = \left(R - i\frac{1}{C\omega} + iL\omega \right)^{-1} iL\omega I_{in}$$

$$\hat{U}_C = \frac{1}{\sqrt{R^2 + \left(\frac{1}{C\omega} - L\omega\right)^2}} \frac{L}{C} \hat{I}_{in} \quad \to \quad \hat{U}_{Cres} = \frac{L}{RC} \hat{I}_{in} \,, \quad \omega_{res} = \frac{1}{\sqrt{LC}}$$

$$P_{RF} = \langle U_L I_{in} \rangle = \left\langle (R + \tfrac{1}{iC\omega}) R \tfrac{1}{iL\omega} I_C^2 \right\rangle = \left\langle (iC\omega R + 1) R \tfrac{C}{L} U_C^2 \right\rangle = \tfrac{1}{2} \tfrac{C}{L} R \sqrt{\tfrac{C}{L} R^2 + 1} \, \hat{U}_C^2$$

(An alternative circuit diagram leads to simplified formulas)

RF systems for accelerators



L and C: determined by the cavity geometry

R_s : shunt impedance, related to surface res. R

$$I_{in} = \left(\frac{1}{R_s} + iC\omega + \frac{1}{iL\omega} \right) U_C$$

$$\hat{U}_C = \frac{1}{\sqrt{\frac{1}{R_s^2} + \left(\frac{1}{L\omega} - C\omega \right)^2}} \hat{I}_{in} \to \hat{U}_{Cres} = R_s \hat{I}_{in}$$

$$P_{RF} = \langle U_L I_{in} \rangle = \tfrac{1}{2} \frac{1}{R_s} \hat{U}_C^2 \qquad \hat{U}_C = \sqrt{2 R_s P_{RF}}$$

Quallity factor: $$Q = 2\pi \frac{E}{\Delta E} = 2\pi \frac{\frac{1}{2} C U_C^2}{T P_{RF}} = \omega R_s C = R_s \sqrt{\tfrac{C}{L}}$$

Geometry factor: $$\frac{R_s}{Q} = \sqrt{\frac{L}{C}}$$

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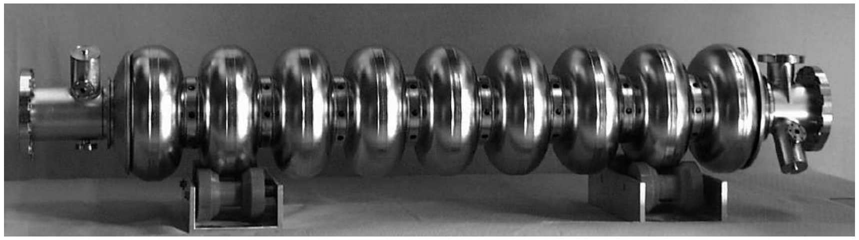
Superconducting Cavities



Q = 10^{10}

E = 20MV/m

A bell with this Q would ring for a year.

- Very low wall losses.
- Therefore continuous operation is possible.
- Energy recovery becomes possible.

Normal conducting cavities

- Significant wall losses.
- Cannot operate continuously with appreciable fields.
- Energy recovery was therefore not possible.

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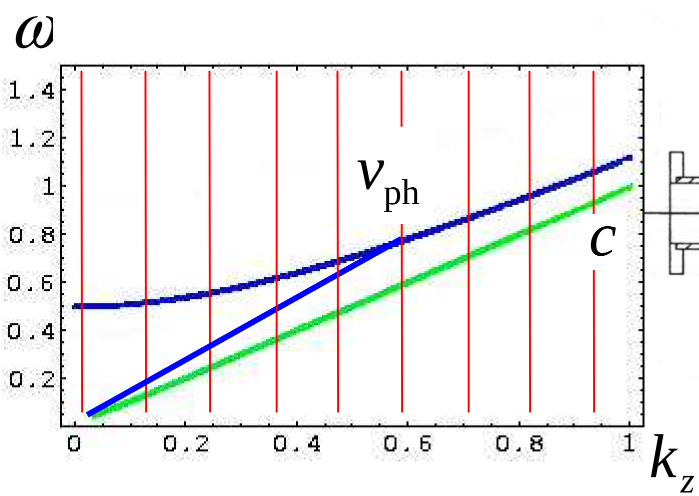
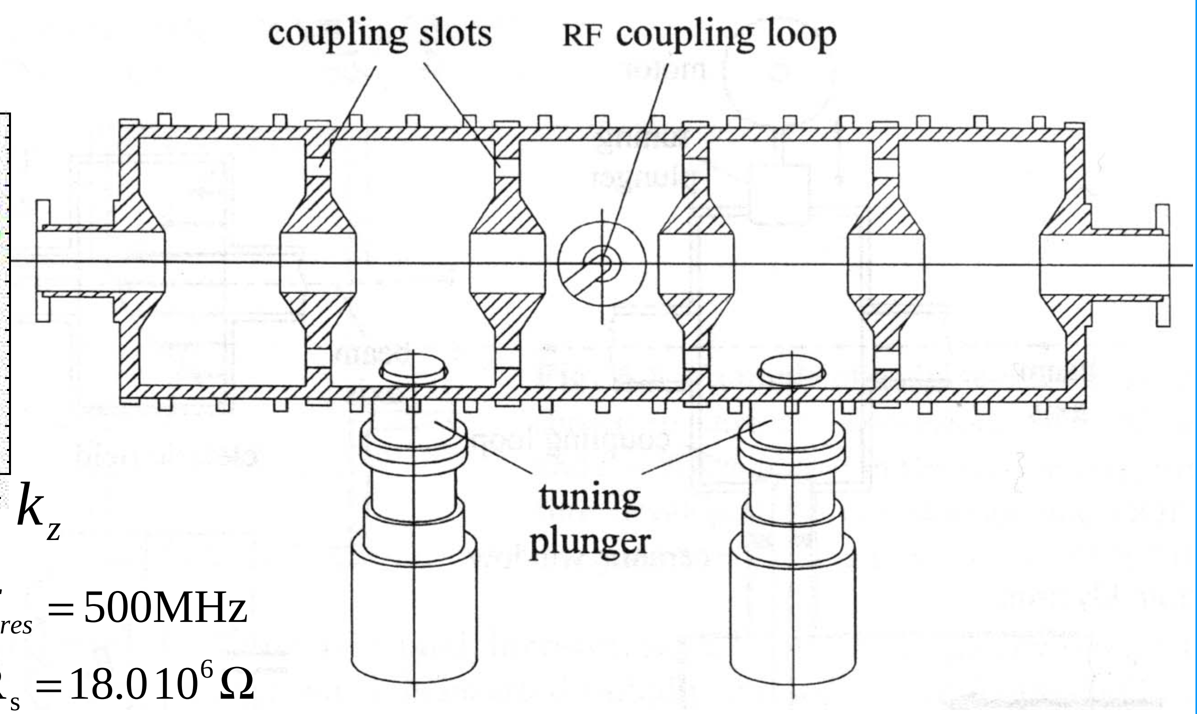
Multicell Cavities

The filed in many cells can be excited by a single power source and a single input coupler in order to have the voltage of several cavities available.





Example: PETRA Cavity $f_{res} = 500\text{MHz}$

$$R_s = 18.0\,10^6\,\Omega$$

$$125\text{kW} \rightarrow 2.12\text{MV}$$

Without the walls: Long single cavity with too large wave velocity. $v_{\text{ph}} = \dfrac{\omega}{k}$

Thick walls: shield the particles from regions with decelerating phase.

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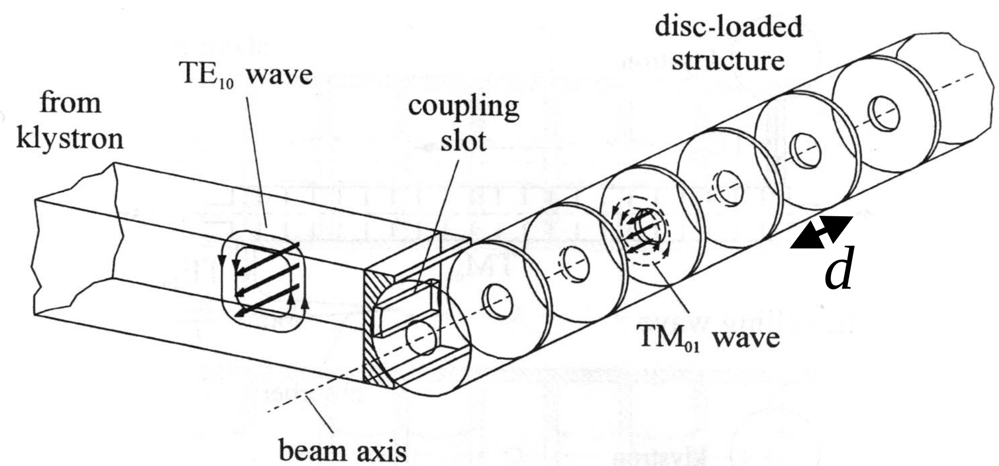
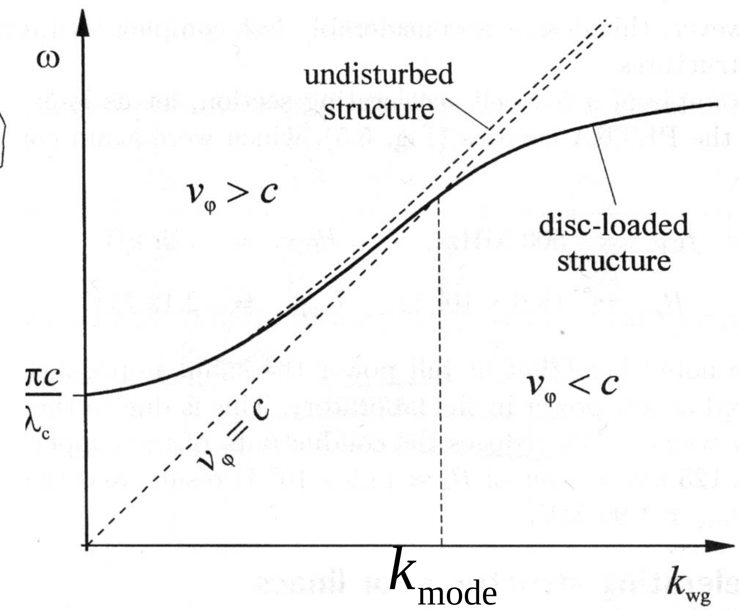
Disc Loaded Waveguides

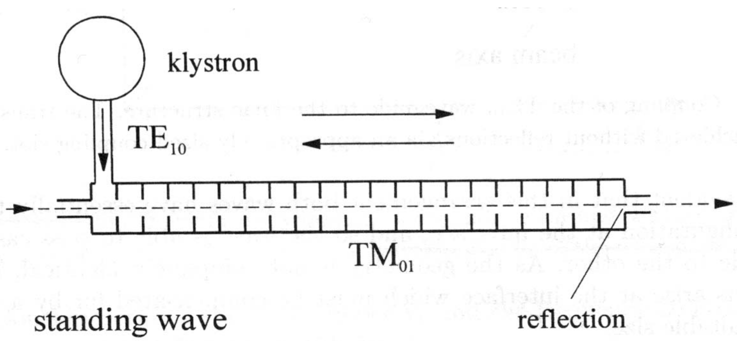
The iris size is chosen to let the phase velocity equal the particle velocity.



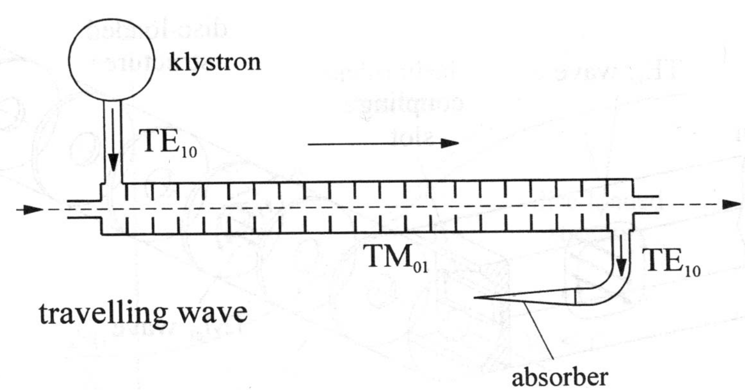


Loss free propergation: $k = \frac{2\pi}{nd}$

Standing wave cavity.

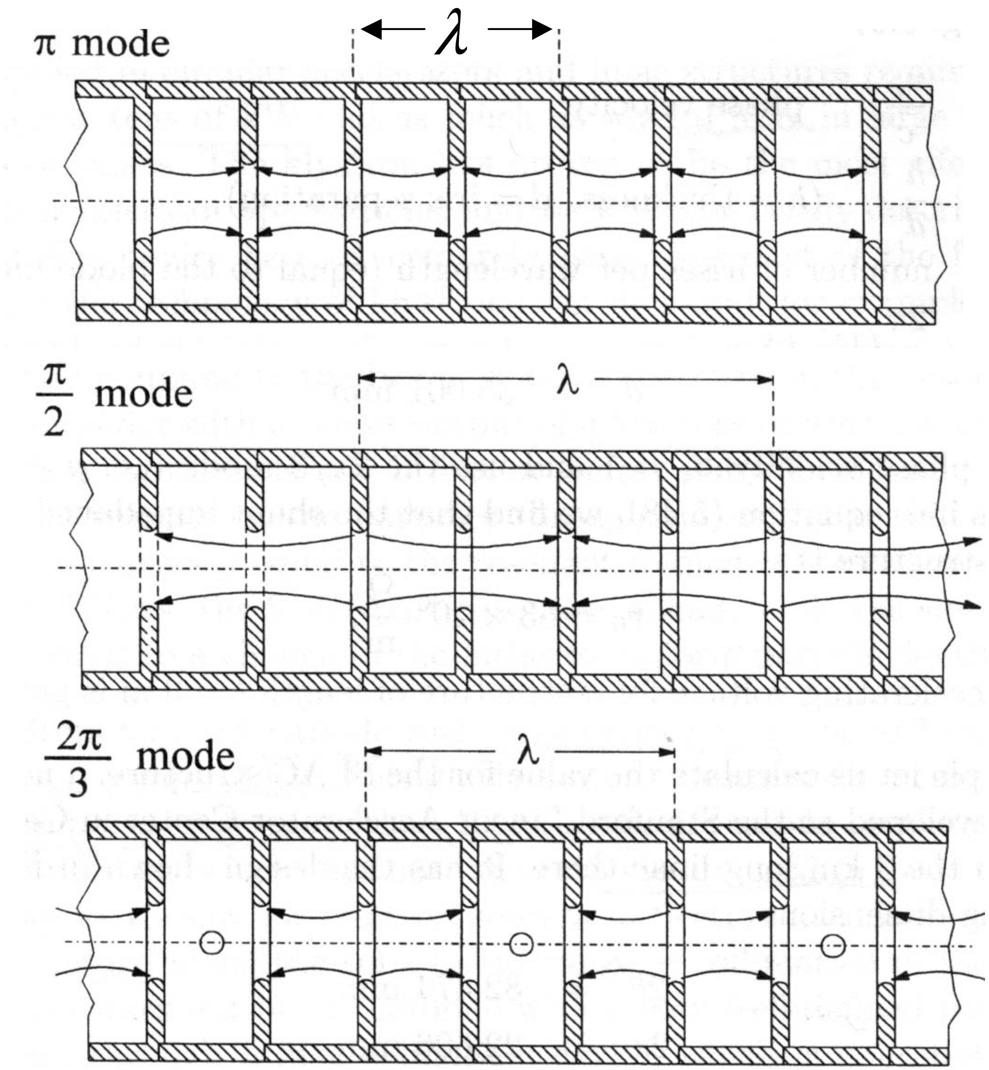


Traveling wave cavity (wave guide).



Modes in Waveguides

The iris size is chosen to let the phase velosity equal the particle velocity.



Long initial settling or filling time, not good for pulsed operation.

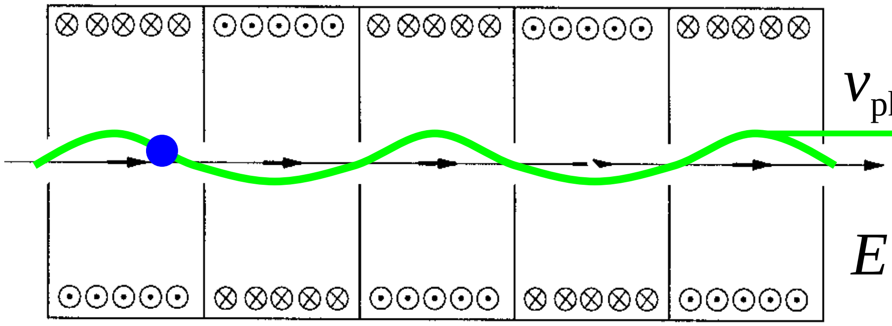
Small shunt impedance per length.

Common compromise.

Accelerating cavities

- 1933: J.W. Beams uses resonant cavities for acceleration

Traveling wave cavity:

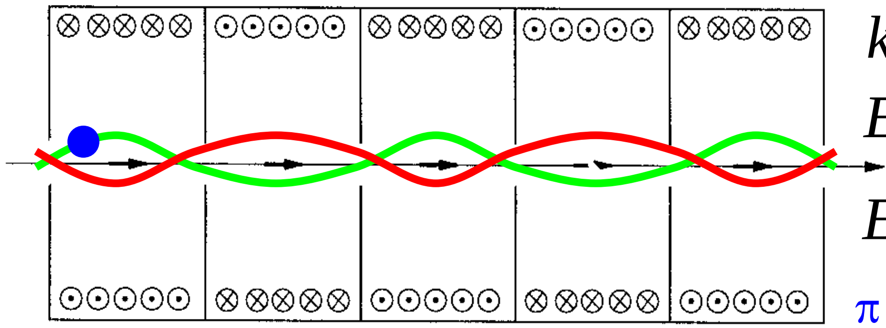


$$v_{\text{phase}} = \frac{\omega}{k} = v_{\text{particle}}$$

Here v=c for electrons

$$E(t,s) \approx E_{\max} \sin(\omega t - ks)$$

Standing wave cavity:



$$\frac{\omega}{k} = v_{\text{particle}}$$

$$E(t,s) \approx E_{\max} \sin(\omega t) \sin(ks)$$

$$E(\tfrac{s}{v_{\text{particle}}}, s) \approx E_{\max} \sin^2(ks)$$

π or the $1/2\beta\lambda$ mode

Transit factor (for this example): $\langle E \rangle = \frac{1}{\lambda_{RF}} \int_0^{\lambda_{RF}} E(\tfrac{s}{v_{\text{particle}}}, s)\, ds = \tfrac{1}{2} E_{\max}$

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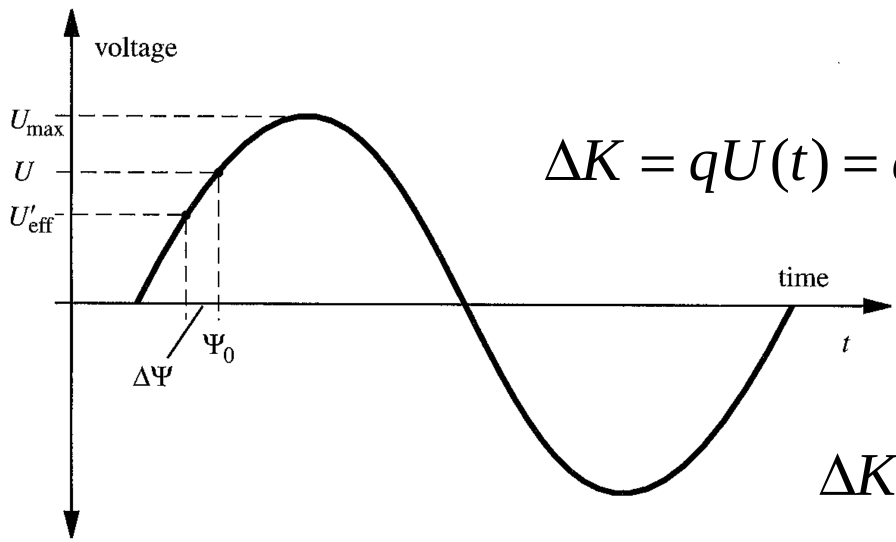
Phase focusing

- 1945: Veksler (UDSSR) and McMillan (USA) realize the importance of phase focusing



$$\Delta K = qU(t) = qU_{max}\sin(\omega(t - t_0) + \psi_0)$$

Longitudinal position in the bunch:

$$\sigma = s - s_0 = -v_0(t - t_0)$$

$$\Delta K(\sigma) = qU_{max}\sin(-\tfrac{\omega}{v_0}(s - s_0) + \psi_0)$$

$$\Delta K(0) > 0$$ (Acceleration)

$$\Delta K(\sigma) < \Delta K(0) \text{ for } \sigma > 0 \Rightarrow \tfrac{d}{d\sigma}\Delta K(\sigma) < 0$$ (Phase focusing)

$$\left.\begin{aligned} qU(t) > 0 \\ q\tfrac{d}{dt}U(t) > 0 \end{aligned}\right\} \quad \psi_0 \in (0, \frac{\pi}{2})$$

Phase focusing is required in any RF accelerator.

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Longitudinal stability

Reference particle: $$\frac{dE_0}{ds} = \hat{E}\cos\Phi_0$$

Other particles: $$\frac{dE}{ds} = \hat{E}\cos\Phi$$

$$\phi = \Phi - \Phi_0 = \omega(t - t_0)$$

$$\frac{d\delta}{ds} = \tfrac{\hat{E}}{E_0}\big(\cos(\Phi_0 + \phi) - \cos\Phi_0\big) \approx -\phi\tfrac{\hat{E}}{E_0}\sin\Phi_0$$

$$\frac{d\phi}{ds} = \omega(\frac{1}{v} - \frac{1}{v_0}) \approx \omega(\frac{1}{v_0 + \frac{dv}{d\delta}\big|_0\delta} - \frac{1}{v_0}) \approx -\omega\frac{\frac{dv}{d\delta}\big|_0}{v_0^2}\delta = -\omega\frac{c^2}{v_0^3\gamma_0^2}\delta$$

$$\frac{d^2\phi}{ds^2} \approx -\omega\frac{c^2}{v_0^3\gamma_0^2}\frac{d\delta}{ds} \approx \underline{\underline{\tfrac{\hat{E}}{E_0}\sin\Phi_0\omega\frac{c^2}{v_0^3\gamma_0^2}}}\phi$$

Stability for small phases when the factor on the right hand side is negative.

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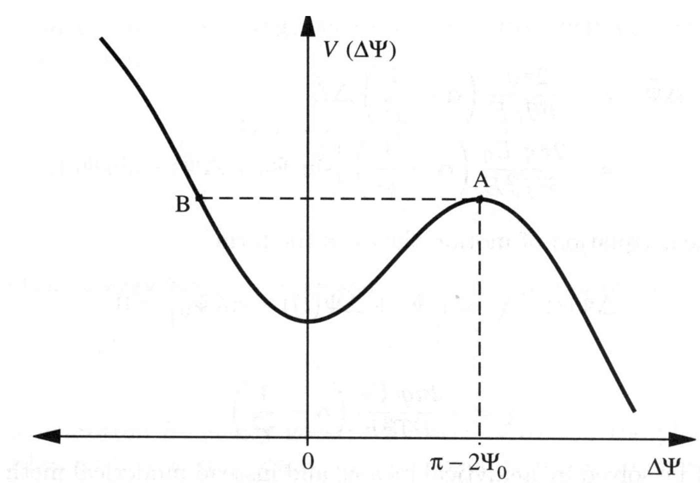
Effective longitudinal potential

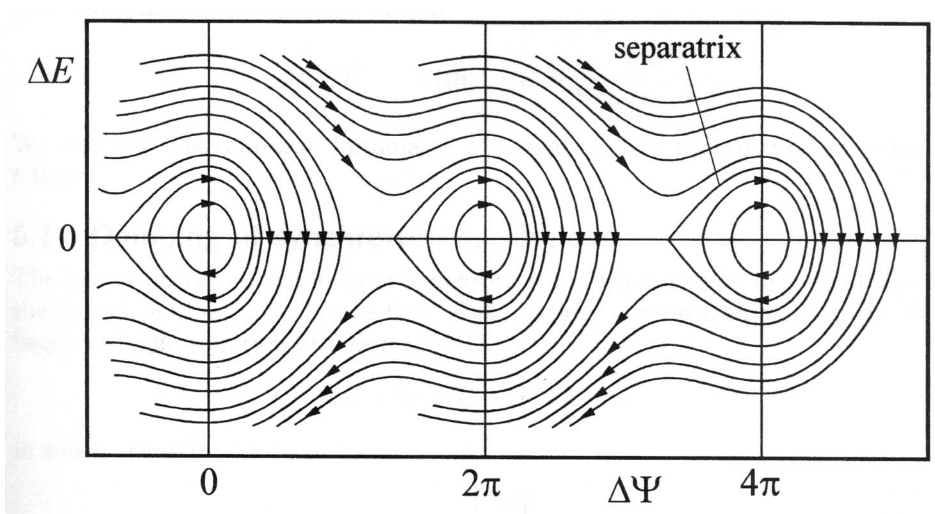
For not very small phases one cannot linearize.

$$\frac{d\delta}{ds} = \tfrac{\hat{E}}{E_0}\left(\cos(\Phi_0 + \phi) - \cos\Phi_0\right) \qquad \frac{d\phi}{ds} \approx -\omega\frac{c^2}{v_0^3\gamma_0^2}\delta$$

$$H(\phi,\delta) = -\tfrac{q\overline{E_s}}{K_0}\left(\sin(\Phi_0+\phi) - \phi\cos\Phi_0\right) - \omega\frac{c^2}{v_0^3\gamma_0^2}\delta^2$$

$$\tfrac{d}{dt}\phi = \tfrac{\partial}{\partial\delta}H \ , \quad \tfrac{d}{dt}\delta = -\tfrac{\partial}{\partial\phi}H$$





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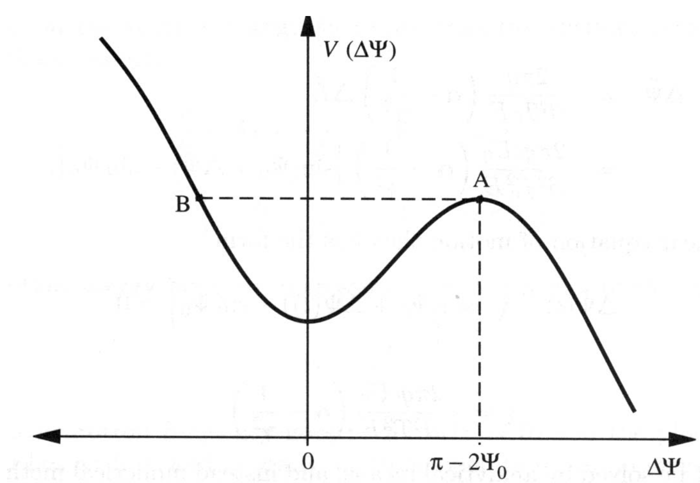
Hamiltonian for longitudinal motion

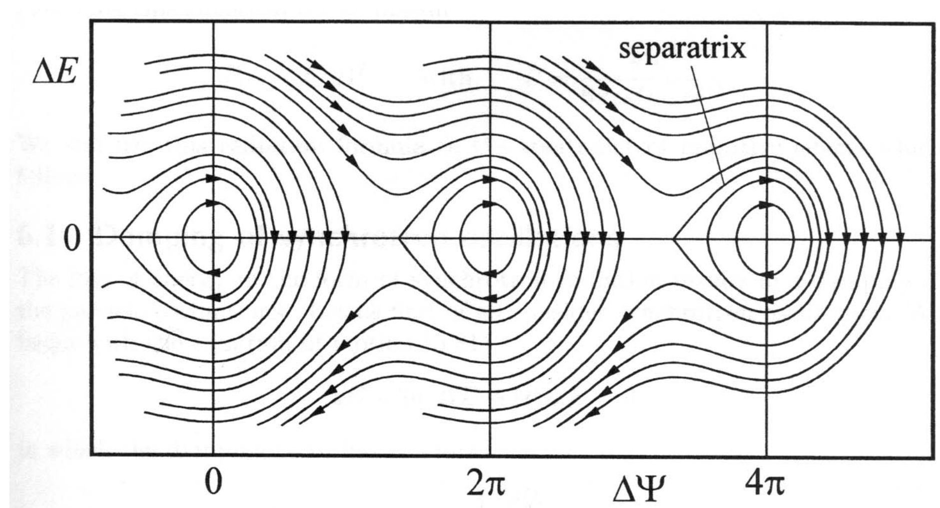
$$\frac{d\delta}{ds} = \frac{q\bar{E}_s}{K_0}\left(\cos(\Phi_0+\phi)-\cos\Phi_0\right) \approx -\phi\frac{q\bar{E}_s}{K_0}\sin\Phi_0$$

$$\frac{d^2\phi}{ds^2} \approx -\omega\frac{c^2}{v_0^3\gamma_0^2}\frac{d\delta}{ds} = -\omega\frac{c^2}{v_0^3\gamma_0^2}\frac{q\bar{E}_s}{K_0}\left(\cos(\Phi_0+\phi)-\cos\Phi_0\right)$$

$$= -\frac{d}{d\phi}\,\omega\frac{c^2}{v_0^3\gamma_0^2}\frac{q\bar{E}_s}{K_0}\left(\sin(\Phi_0+\phi)-\phi\cos\Phi_0\right)$$

Effective potential





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