Projection probabilities and ket vectors

$$\Psi(x,t)=\Phi_1 e^{-i\frac{E_1}{\hbar}t}A_1+\Phi_2 e^{-i\frac{E_2}{\hbar}t}A_2$$

Measurement:

E_1 with probability $|A_1|^2$

x with probability $|\Psi(x,t)|^2$

Projection amplitude:

$$e^{-i\frac{E_1}{\hbar}t}A_1=\langle 1|\Psi\rangle$$

$$=\int_{-\infty}^{\infty}\Phi_1(x)^*\Psi(x,t)dx$$

$$\Psi(x,t)=\langle x|\Psi\rangle$$

$$\Phi_n(x)=\langle x|n\rangle$$

Ket vector:

$$|\Psi\rangle=|1\rangle e^{-i\frac{E_1}{\hbar}t}A_1+|2\rangle e^{-i\frac{E_2}{\hbar}t}A_2$$

$$|\Psi\rangle=\sum_{\text{all }x}|x\rangle\langle x|\Psi\rangle$$

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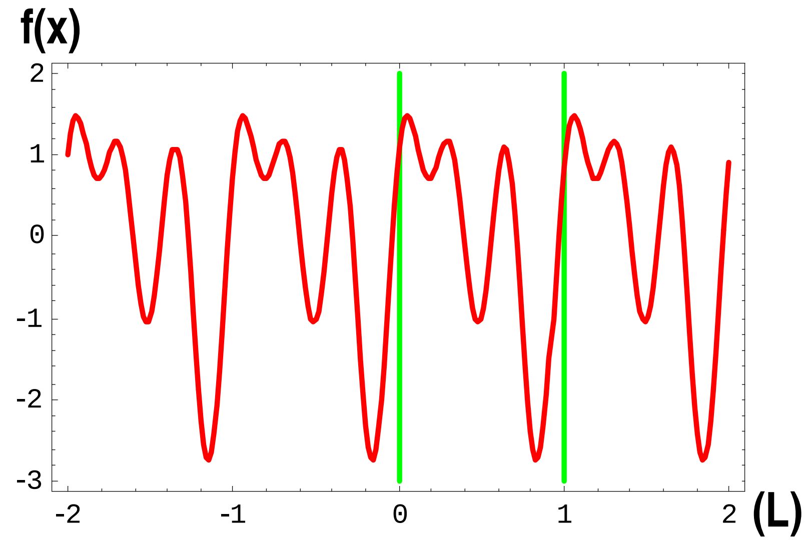
Fourier Series

Any “well behaved” periodic function **f(x)=f(x+L)** can be written as the sum of elementary periodic functions



$$e^{in\frac{2\pi}{L}x} = \cos(n\tfrac{2\pi}{L}x) + i\sin(n\tfrac{2\pi}{L}x)$$

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{in\frac{2\pi}{L}x}$$

$$f(x)e^{-im\frac{2\pi}{L}x} = \sum_{n=-\infty}^{\infty} a_n e^{i(n-m)\frac{2\pi}{L}x} \longrightarrow \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)e^{-im\frac{2\pi}{L}x}dx = a_m$$

Real functions f(x): $a_{-m} = a_m^*$

$$f(x) = a_0 + \sum_{n=1}^{\infty} 2\,\mathrm{Re}[a_n e^{in\frac{2\pi}{L}x}]$$

$$= a_0 + \sum_{n=1}^{\infty} 2\,\mathrm{Re}[a_n]\cos(n\tfrac{2\pi}{L}x) - \sum_{n=1}^{\infty} 2\,\mathrm{Im}[a_n]\sin(n\tfrac{2\pi}{L}x)$$

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Fourier Transforms (Fourier Series for infinite L)

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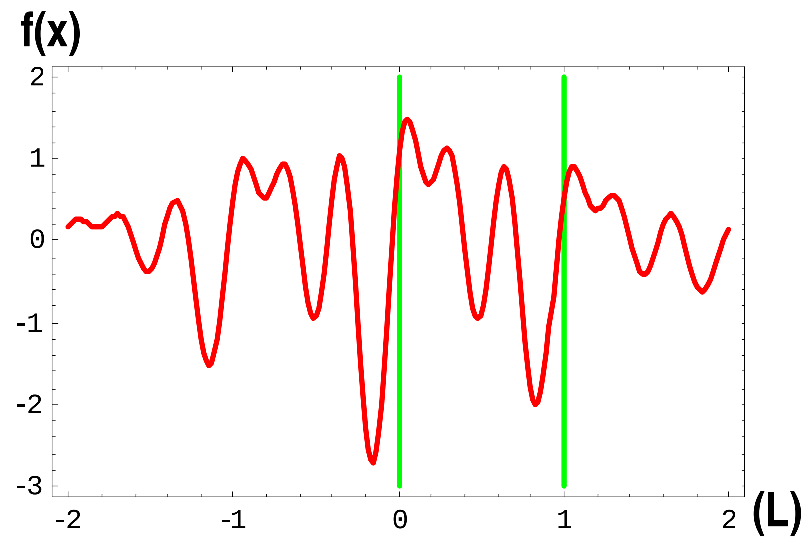
Any "well behaved" non-periodic function **f(x)** can be written as an integral over elementary periodic functions:



$$f(x) = \lim_{L\to\infty} \sum_{n=-\infty}^{\infty} a_n e^{in\frac{2\pi}{L}x} \Delta n$$

$$= \lim_{L\to\infty} \tfrac{L}{2\pi} \sum_{n=-\infty}^{\infty} a_n e^{ik_n x} \Delta k, \quad \Delta k = \tfrac{2\pi}{L} \Delta n$$

$$= \int_{-\infty}^{\infty} F(k) \frac{e^{ikx}}{\sqrt{2\pi}} dk \qquad \textbf{with} \qquad a_n = \tfrac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) e^{-in\frac{2\pi}{L}x} dx$$

$$F(k_n) = \sqrt{2\pi} \lim_{L\to\infty} \tfrac{L}{2\pi} a_n = \int_{-\infty}^{\infty} f(x) \frac{e^{-ik_n x}}{\sqrt{2\pi}} dx$$

F(k) is called the Fourier transform of f(x). f(x) is the Fourier back-transform of F(k).

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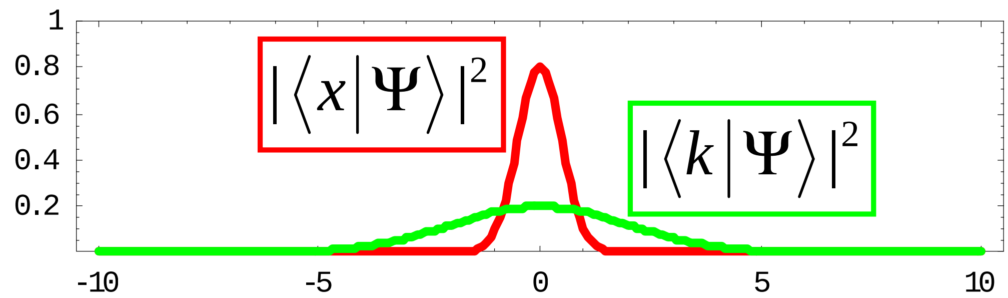
Free particle packet states

$$\Psi(x,0) = \frac{1}{\sqrt{\sqrt{\pi}\sigma}} e^{-\frac{x^2}{2\sigma^2}} = f(x)$$



$$\Psi(x,0) = \frac{1}{\sqrt{\sqrt{\pi}\sigma^{-1}}} \int_{-\infty}^{\infty} e^{-\frac{k^2}{2\sigma^{-2}}} \frac{e^{ikx}}{\sqrt{2\pi}} dk$$

Since:

$$F(k) = \frac{1}{\sqrt{\sqrt{\pi}\sigma}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} \frac{e^{-ikx}}{\sqrt{2\pi}} dx = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\sqrt{\pi}\sigma}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(x+ik\sigma^2)^2} e^{-k^2\frac{\sigma^2}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\sqrt{\pi}\sigma}} e^{-k^2\frac{\sigma^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\xi^2} d\xi = \frac{1}{\sqrt{2\pi}} \frac{\sqrt{2}\sigma}{\sqrt{\sqrt{\pi}\sigma}} e^{-k^2\frac{\sigma^2}{2}} \int_{-\infty}^{\infty} e^{-\zeta^2} d\zeta = \frac{1}{\sqrt{\sqrt{\pi}\sigma^{-1}}} e^{-\frac{k^2}{2\sigma^{-2}}}$$

Uncertainty of position and momentum measurements:

For a Gaussian probability distribution:

$$\Delta x = \frac{\sigma}{\sqrt{2}}, \quad \Delta k = \frac{\sigma^{-1}}{\sqrt{2}} \quad \rightarrow \quad \Delta x \Delta p = \frac{\hbar}{2}$$

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