Normalization and probability density

Probability to find a particle in the volume element $d^3\vec{x}$ is given by $|\Psi(\vec{x},t)|^2\, d^3\vec{x}$

Probability to find the particle somewhere is one: $\int |\Psi(\vec{x},t)|^2\, d^3\vec{x} = 1$

Examples:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\Psi(\vec{x},t)|^2\, dxdydz = 1$$

$$\int_0^{\infty}\int_0^{\pi}\int_0^{2\pi} |\Psi(r,\vartheta,\varphi,t)|^2\, r^2 \sin\vartheta\, d\varphi\, d\vartheta\, dr = 1$$

Spherical symmetric wave functions: $4\pi \int_0^{\infty} |\Psi(r,t)|^2\, r^2\, dr = 1$

Probability to find a particle with a radius between **r** and **r+dr :** $4\pi\, |\Psi(r,t)|^2\, r^2\, dr$

For a stationary state this is $4\pi\, |r\,\Phi(r)|^2\, dr = 4\pi\, |u(r)|^2\, dr$

$u(r)$ is not normalized to **1** but to **1/4π :** $\int_0^{\infty} |u(r)|^2\, dr = \frac{1}{4\pi}$

Georg.Hoffstaetter@Cornell.edu

04/18/2005

Expectation values

The most probable value of **r** is given by $\frac{\partial}{\partial r} u(r) = 0$

The average radius after many measurements in identically prepared states:

After N measurements one might have measured a radius $\mathbf{r_1}$, $\mathbf{n_1}$ time $\mathbf{r_2}$, $\mathbf{n_2}$ times, etc.

The average radius is then

$$\langle r \rangle = \tfrac{1}{N}(n_1 r_1 + n_2 r_2 + n_3 r_3 + \ldots), \quad N = n_1 + n_2 + n_3 + \ldots$$

$$= p_1 r_1 + p_2 r_2 + p_3 r_3 + \ldots \quad , \quad \text{propability } p_j \text{ to measure } r_j$$

$$= 4\pi \int_0^\infty r\,|u(r)|^2\, dr$$

$$= \int_0^\infty \int_0^\pi \int_0^{2\pi} r\,|\Psi(r,\vartheta,\varphi,t)|^2\, r^2 \sin\vartheta\, d\varphi\, d\vartheta\, dr$$

Similarly in one dimension: $\langle x \rangle = \int_{-\infty}^{\infty} x\,|\Psi(x,t)|^2\, dx$

Georg.Hoffstaetter@Cornell.edu

A limit of the Bohr model

$$E = E_1 \qquad\qquad E = \tfrac{1}{4} E_1$$

$$r_{\max} = a_0 \qquad\qquad r_{\max} = (3+\sqrt{5})a_0 > 4a_0$$

The Bohr radius is not the most likely radius.

$$\langle r \rangle_{E_1} = \tfrac{3}{2} a_0 \qquad\qquad \langle r \rangle_{E_2} = 6a_0$$

The Bohr radius is also not the expectation value.

Georg.Hoffstaetter@Cornell.edu