# Macroscopic Fields in Accelerators



$$\frac{d}{dt}\vec{p} = q(\vec{E} + \vec{v} \times \vec{B})$$

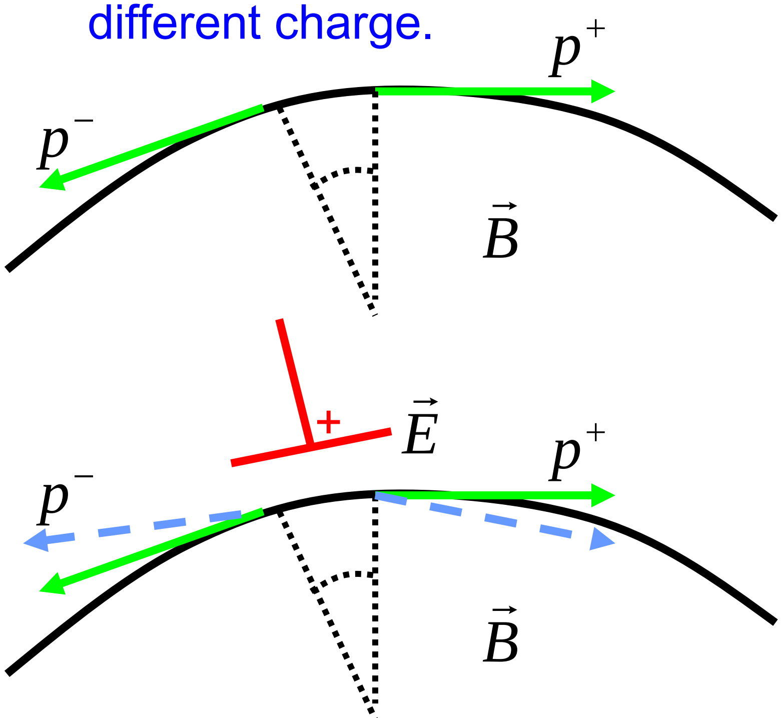
E has a similar effect as v B.

For relativistic particles B = 1T has a similar effect as

E = cB = 3 $10^8$ V/m , such an

Electric field is beyond technical limits.

- Electric fields are only used for very low energies or
- For separating two counter rotating beams with different charge.




Electrostatic separators at CESR

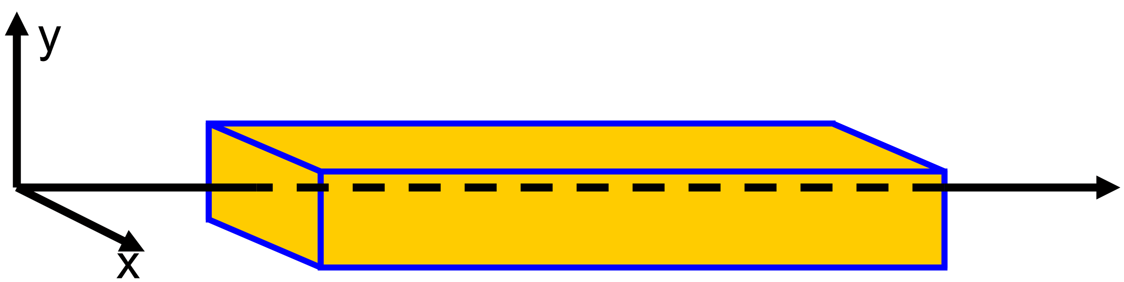
# Magnetic Fields in Accelerators



Static magnetic fileds: $\partial_t \vec{B} = 0\,;\quad \vec{E} = 0$ Charge free space: $\vec{j} = 0$

$$\vec{\nabla} \times \vec{B} = \mu_0(\vec{j} + \varepsilon_0 \partial_t \vec{E}) = 0 \quad \Rightarrow \quad \vec{B} = -\vec{\nabla}\psi(\vec{r})$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \Rightarrow \quad \vec{\nabla}^2 \psi(\vec{r}) = 0$$



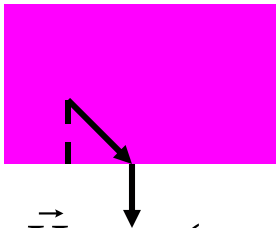

(x=0,y=0) is the beam's design curve

For finite fields on the design curve,
Ψ can be power expanded in x and y:

$$\psi(x,y,z) = \sum_{n,m=0}^{\infty} b_{nm}(z) x^n y^m$$

# Surfaces of Equal Potential





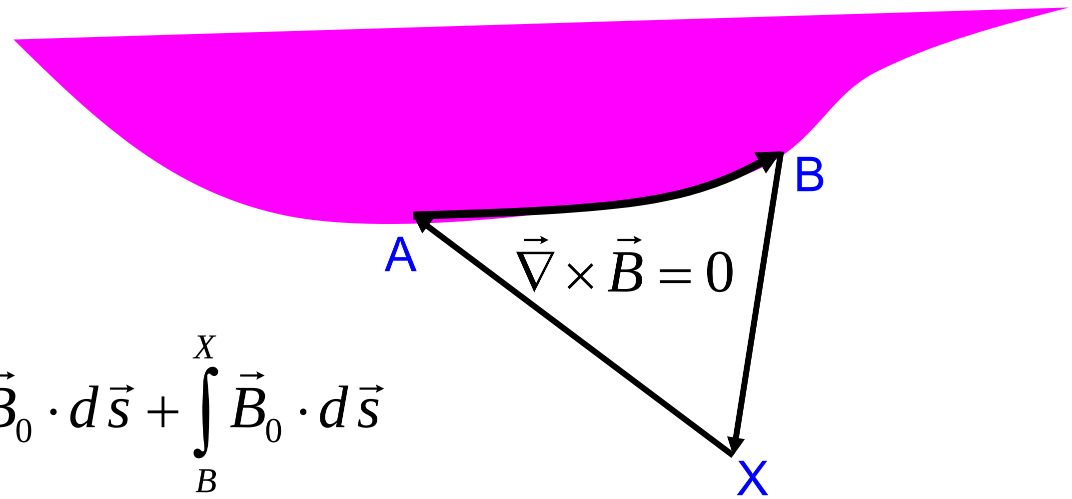
$$\vec{B}_\perp(\text{out}) = \vec{B}_\perp(\text{in})$$

$$\vec{H}_{\text{parallel}}(\text{out}) = \vec{H}_{\text{parallel}}(\text{in})$$

$$\vec{B}_{\text{parallel}}(\text{out}) = \frac{1}{\mu_r}\vec{B}_{\text{parallel}}(\text{in})$$

$$\vec{B}(\vec{r}) = -\vec{\nabla}\Psi(\vec{r})$$

For large permeability, H(out) is perpendicular to the surface.




$$0 = \oint \vec{B}\cdot d\vec{s} = \int_X^A \vec{B}_0\cdot d\vec{s} + \int_A^B \vec{B}_0\cdot d\vec{s} + \int_B^X \vec{B}_0\cdot d\vec{s}$$

$$= \int_X^A \vec{B}_0\cdot d\vec{s} + \frac{1}{\mu_r}\int_A^B \vec{B}_0\cdot d\vec{s} + \int_B^X \vec{B}_0\cdot d\vec{s}$$

$$\approx \int_X^A \vec{B}_0\cdot d\vec{s} + \int_B^X \vec{B}_0\cdot d\vec{s} = \Psi(A) - \Psi(B)$$

For highly permeable materials (like iron) surfaces have a constant potential.

# Green's Theorem



$$\vec{\nabla}^2\psi = 0$$

Green function:

$$\vec{\nabla}_0^2 G(\vec{r},\vec{r}_0) = \delta(\vec{r}-\vec{r}_0)$$

$$\begin{aligned}\psi(\vec{r}) &= \int_V \psi(\vec{r}_0)\delta(\vec{r}-\vec{r}_0)\, d^3\vec{r}_0 \\ &= \int_V \left[\psi(\vec{r}_0)\vec{\nabla}_0^2 G - G\vec{\nabla}_0^2\psi(\vec{r}_0)\right] d^3\vec{r}_0 \\ &= \int_V \vec{\nabla}_0\left[\psi(\vec{r}_0)\vec{\nabla}_0 G - G\vec{\nabla}_0\psi(\vec{r}_0)\right] d^3\vec{r}_0 \\ &= \int_{\partial V} \left[\psi(\vec{r}_0)\vec{\nabla}_0 G - G\vec{\nabla}_0\psi(\vec{r}_0)\right]\cdot d^2\vec{r}_0 \\ &= \int_{\partial V} \left[\psi(\vec{r}_0)\vec{\nabla}_0 G + \vec{B}(\vec{r}_0)G\right]\cdot d^2\vec{r}_0\end{aligned}$$

Knowledge of the field and the scalar magnetic potential on a closed surface inside a magnet determines the magnetic field for the complete volume which is enclosed.

# Potential Expansion



If field data in a plane (for example the midplane of a cyclotron or of a beam line magnet) is known, the complete filed is determined:

$$\psi(x,y,z) = \sum_{n=0}^{\infty} b_n(x,z) y^n \quad \Rightarrow \quad \vec{B}(x,0,z) = -\begin{pmatrix} \partial_x b_0(x,z) \\ b_1(x,z) \\ \partial_z b_0(x,z) \end{pmatrix}$$

$$0 = \vec{\nabla}^2 \psi = \sum_{n=0}^{\infty} (\partial_x^2 + \partial_z^2)\, b_n\, y^n + \sum_{n=2}^{\infty} n(n-1)\, b_n\, y^{n-2}$$

$$= \sum_{n=0}^{\infty} \left[ (\partial_x^2 + \partial_z^2)\, b_n + (n+2)(n+1)\, b_{n+2} \right] y^n$$

$$b_{n+2}(x,z) = -\frac{1}{(n+2)(n+1)} (\partial_x^2 + \partial_y^2)\, b_n(x,z)$$

Data of the magnetic field in the plane y=0 is used to determine $b_0(x,z)$ and $b_1(x,z)$.

# Complex Potentials



$$w = x + iy \quad , \quad \overline{w} = x - iy$$

$$\partial_x = \partial_w + \partial_{\overline{w}} \; , \quad \partial_y = i\partial_w - i\partial_{\overline{w}} = i(\partial_w - \partial_{\overline{w}})$$

$$\vec{\nabla}^2 = \partial_x^2 + \partial_y^2 + \partial_z^2 = (\partial_w + \partial_{\overline{w}})^2 - (\partial_w - \partial_{\overline{w}})^2 + \partial_z^2 = 4\partial_w \partial_{\overline{w}} + \partial_z^2$$

$$\psi = \mathrm{Im}\{ \sum_{\nu,\lambda=0}^{\infty} a_{\nu\lambda}(z) \cdot (w\overline{w})^{\lambda} \overline{w}^{\nu} \}$$

$$\vec{\nabla}^2 \psi = \mathrm{Im}\{ \sum_{\nu=0,\lambda=1}^{\infty} 4 a_{\nu\lambda} (\lambda+\nu)\lambda (w\overline{w})^{\lambda-1} \overline{w}^{\nu} + \sum_{\nu=0,\lambda=0}^{\infty} a_{\nu\lambda}'' (w\overline{w})^{\lambda} \overline{w}^{\nu} \}$$

$$= \mathrm{Im}\{ \sum_{\nu,\lambda=0}^{\infty} [4(\lambda+1+\nu)(\lambda+1) a_{\nu\lambda+1} + a_{\nu\lambda}''] (w\overline{w})^{\lambda} \overline{w}^{\nu} \} = 0$$

Iteration equation: $a_{\nu\lambda+1} = \dfrac{-1}{4(\lambda+1+\nu)(\lambda+1)} a_{\nu\lambda}'' \quad , \quad a_{\nu 0} = \Psi_{\nu}(z)$

The functions $\Psi_\nu(z)$ along a line determine the complete field inside a magnet.

# Multipole Coefficients

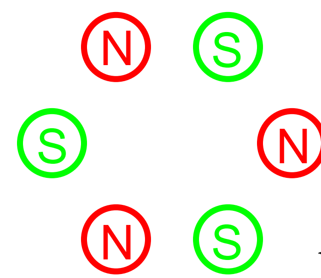


$\Psi_\nu(z)$ are called the z-dependent multipole coefficients

$$\psi(x,y,z) = \mathrm{Im}\{\sum_{\nu,\lambda=0}^{\infty} \frac{(-1)^\lambda \nu!}{(\lambda+\nu)!\lambda!} \left(\frac{w\overline{w}}{4}\right)^\lambda \overline{w}^\nu \Psi_\nu^{[2\lambda]}(z)\}$$

$$\psi(r,\varphi,z) = \sum_{\nu,\lambda=0}^{\infty} \frac{(-1)^\lambda \nu!}{(\lambda+\nu)!\lambda!} \left(\frac{r}{2}\right)^{2\lambda} r^\nu \, \mathrm{Im}\{\Psi_\nu^{[2\lambda]}(z) e^{-i\nu\varphi}\}$$

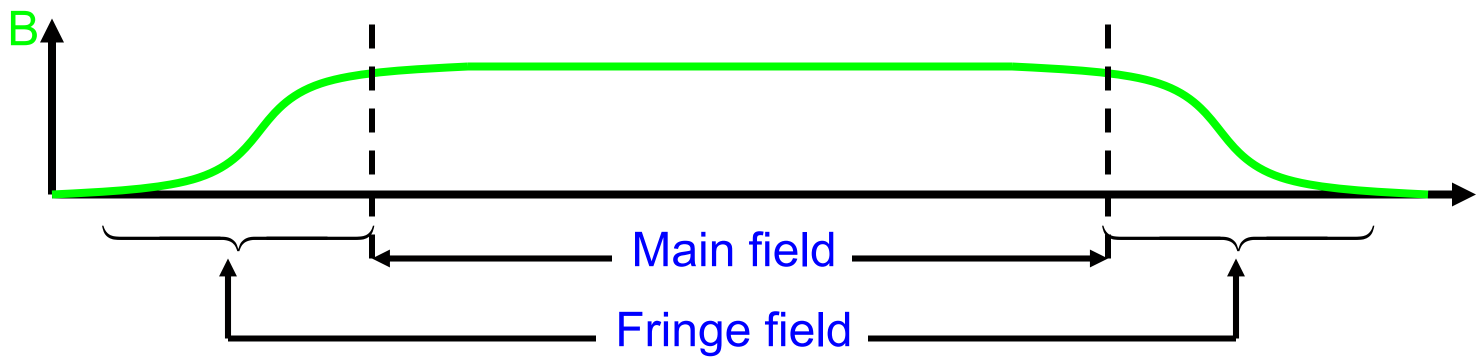
The index ν describes $C_\nu$ Symmetry
around the z-axis $\vec{e}_z$
due to a sign change after $\Delta\varphi = \frac{\pi}{\nu}$




$\nu = 3$

# Fringe Fields and Main Fields






Only the fringe field region has terms with $\lambda \neq 0$ and $\partial_z^2 \psi \neq 0$

Main fields in accelerator physics: $\lambda = 0\,, \quad \partial_z^2 \psi = 0$

$$\Psi_\nu = \begin{cases} e^{i\nu\vartheta_\nu}\,|\Psi_\nu| & \text{for} \quad \nu \neq 0 \\ i\;|\Psi_0| & \text{for} \quad \nu = 0 \end{cases}$$

$$\psi(r,\varphi) = \sum_{\nu=1}^{\infty} r^\nu |\Psi_\nu| \,\mathrm{Im}\{e^{-i\nu(\varphi-\vartheta_\nu)}\} + |\Psi_0|$$

# Main Field Potential



Main field potential: $$\psi = |\Psi_0| - \sum_{\nu=1}^{\infty} r^{\nu} |\Psi_{\nu}| \sin[\nu(\varphi - \vartheta_{\nu})]$$

The isolated multipole: $$\psi = -r^{\nu} |\Psi_{\nu}| \sin(\nu\varphi)$$

Where the rotation $\vartheta_{\nu}$ of the coordinate system is set to 0

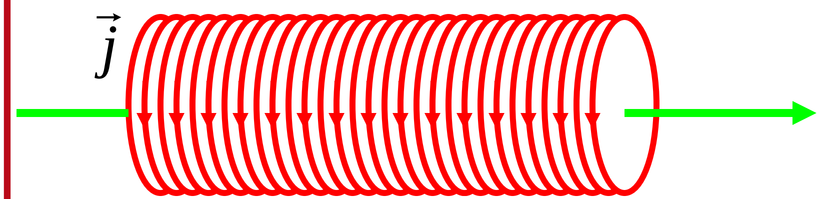
The potentials produced by different multipole components $\Psi_{\nu}$ have

a) Different rotation symmetry $C_{\nu}$

b) Different radial dependence $r^{\nu}$

# Multipoles in Accelerators  ν=0: Solenoids






$$m\gamma \begin{pmatrix}\ddot x\\ \ddot y\\ \ddot z\end{pmatrix} = q\begin{pmatrix}\dot x\\ \dot y\\ \dot z\end{pmatrix}\times\begin{pmatrix}-\frac{x}{2}B_z'\\ -\frac{y}{2}B_z'\\ B_z\end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix}\ddot x\\ \ddot y\end{pmatrix} = \frac{qB_z}{m\gamma}\begin{pmatrix}\dot y\\ -\dot x\end{pmatrix} + \frac{qB_z'\dot z}{2m\gamma}\begin{pmatrix}y\\ -x\end{pmatrix}$$

$$\Downarrow$$

$$\ddot w = -i\frac{qB_z}{m\gamma}\dot w - i\frac{q\dot B_z}{2m\gamma}w$$

$$\psi = \Psi_0(z) - \frac{w\bar w}{4}\Psi_0''(z) \pm \dots$$

$$\vec B = \begin{pmatrix}\frac{x}{2}\Psi_0''\\ \frac{y}{2}\Psi_0''\\ -\Psi_0'\end{pmatrix} \quad\Rightarrow\quad \vec\nabla\cdot\vec B = 0$$

$$g = \frac{qB_z}{2m\gamma}\ , \quad w_0 = w\,e^{i\int_0^t g\,dt}$$

$$\ddot w_0 = (\ddot w + i2g\dot w + i\dot g w - g^2 w)\,e^{i\int_0^t g\,dt}$$
$$= -g^2 w_0$$

$$\ddot x_0 = -g^2 x_0$$
$$\ddot y_0 = -g^2 y_0$$

Focusing in a rotating coordinate system

# Solenoid vs. Strong Focusing



If the solenoids field was perpendicular to the particle's motion,

its bending radius would be $\rho_z = \frac{p}{qB_z}$

$$\ddot{\vec{r}} = -\left(\frac{qB_z}{2m\gamma}\right)^2 \vec{r} = -\frac{qv_z}{m\gamma} B_z \frac{\vec{r}}{4\rho_z}$$

Solenoid focusing is weak compared to the deflections created by a transverse magnetic field.

Transverse fields: $\vec{B} = B_x \, \vec{e}_x + B_y \, \vec{e}_y$

$$m\gamma \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = q \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} \times \begin{pmatrix} B_x \\ B_y \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \frac{qv_z}{m\gamma} \begin{pmatrix} -B_y \\ B_x \end{pmatrix}$$ Strong focusing

Weak focusing < Strong focusing by about $r/\rho$

# Solenoid Focusing

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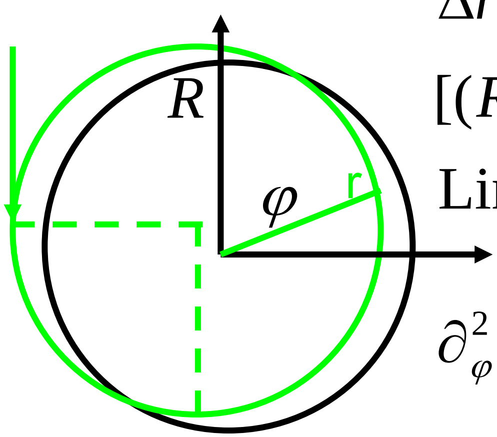
Solenoid magnets are used in detectors for particle identification via $\rho = \frac{p}{qB}$

The solenoid's rotation $\dot{\varphi} = -\frac{qB_z}{2m\gamma}$ of the beam is often compensated by a reversed solenoid called compensator.

Solenoid or Weak Focusing:

Solenoids are also used to focus low γ beams: $\ddot{w} = -\left(\frac{qB_z}{2m\gamma}\right)^2 w$

Weak focusing from natural ring focusing:




$$\Delta r = r - R$$

$$[(R+\Delta r)\cos\varphi - \Delta x_0]^2 + [(R+\Delta r)\sin\varphi - \Delta y_0]^2 = R^2$$

$$\text{Linearization in } \Delta: \quad \Delta r = (\cos\varphi\,\Delta x_0 + \sin\varphi\,\Delta y_0)$$

$$\partial_\varphi^2 \Delta r = -\Delta r \quad \Rightarrow \quad \Delta\ddot{r} = -\dot{\varphi}^2 \Delta r = -\left(\frac{v}{\rho}\right)^2 \Delta r = -\left(\frac{qB}{m\gamma}\right)^2 \Delta r$$

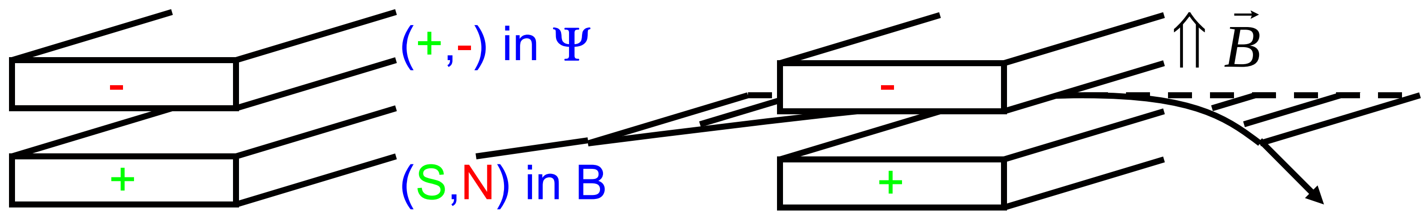
# Multipoles in Accelerators    ν=1: Dipoles



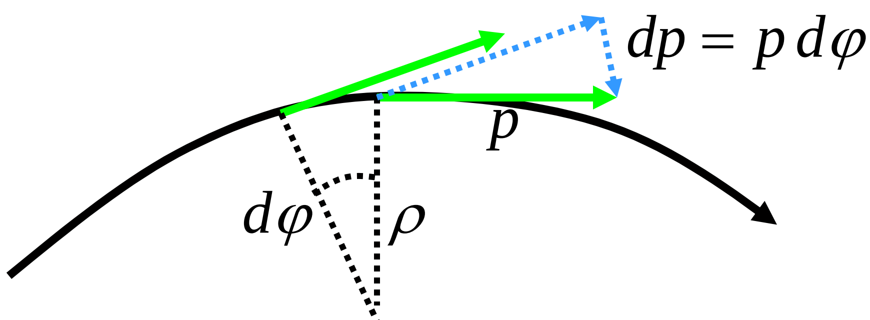
$$\psi = \Psi_1 \,\mathrm{Im}\{x - iy\} = -\Psi_1 \cdot y \quad \Rightarrow \quad \vec{B} = -\vec{\nabla}\psi = \Psi_1 \,\vec{e}_y$$

Equipotential
$y = \text{const.}$

$C_1$ Symmetry




Dipole magnets are used for steering the beams direction



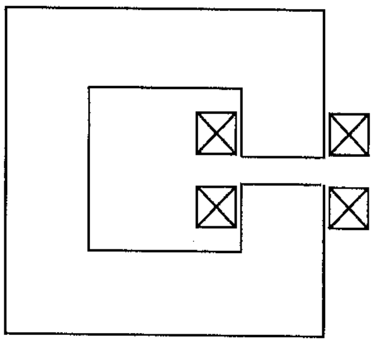

$$\frac{d\vec{p}}{dt} = q\vec{v}\times\vec{B} \quad \Rightarrow \quad \frac{dp}{dt} = qvB_\perp \quad \Rightarrow \quad \rho = \frac{dl}{d\varphi} = \frac{vdt}{dp/p} = \frac{p}{qB_\perp}$$

Bending radius: $\rho = \dfrac{p}{qB}$

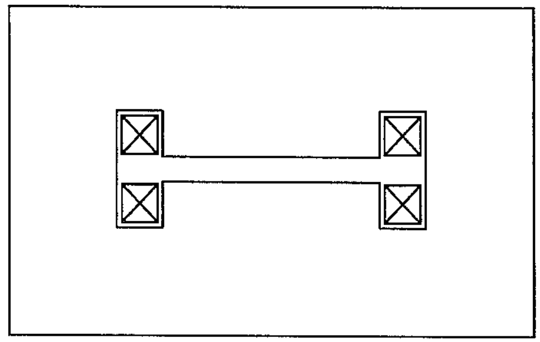
# Different Dipoles



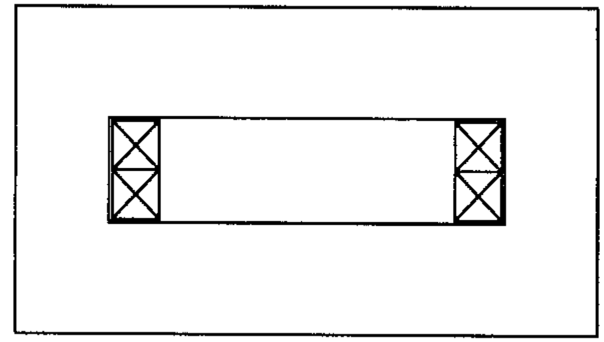
C-shape magnet:

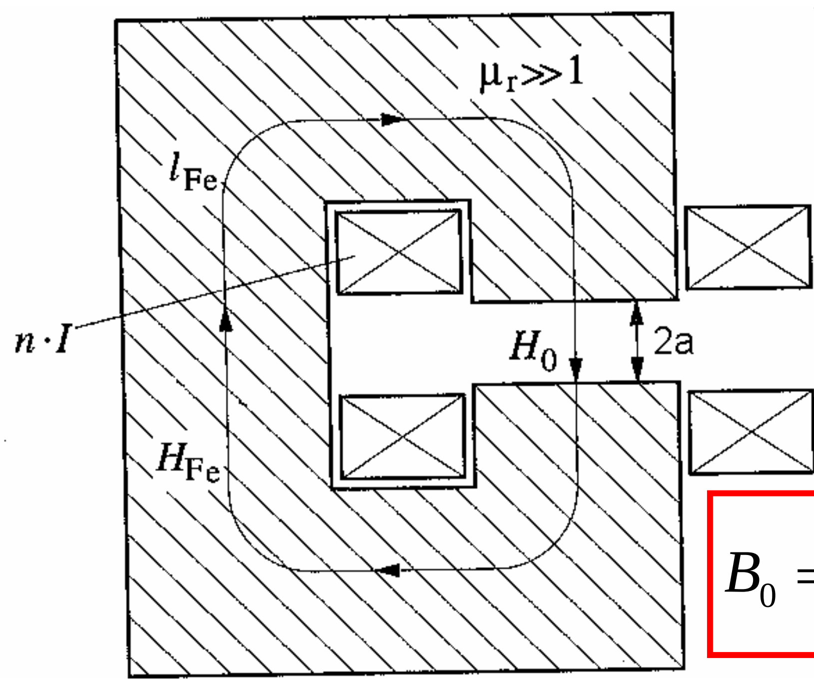


H-shape magnet:



Window frame magnet:






$$\vec{B}_\perp(\text{out}) = \vec{B}_\perp(\text{in})$$

$$\vec{H}_\perp(\text{out}) = \mu_r \vec{H}_\perp(\text{in})$$

$$2nI = \oint \vec{H}\cdot d\vec{s} = H_{Fe} l_{Fe} + H_0 2a$$

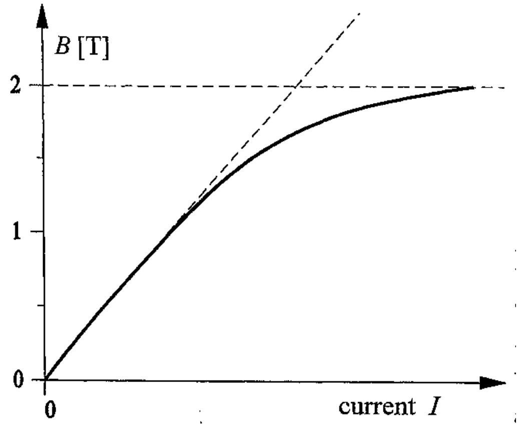
$$= \frac{1}{\mu_r} H_0 l_{Fe} + H_0 2a \approx H_0 2a$$

$$B_0 = \mu_0 \frac{nI}{a}$$

Dipole strength: $\frac{1}{\rho} = \frac{q\mu_0}{p}\frac{nI}{a}$

# Dipole Fields

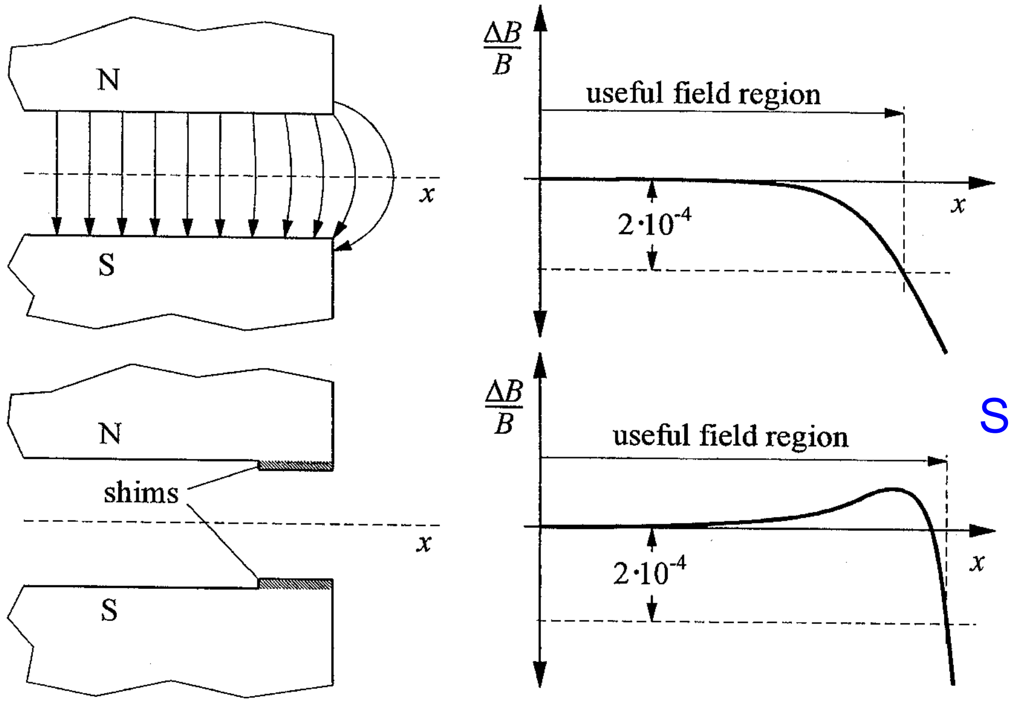





B = 2 T: Typical limit, since the field becomes dominated by the coils, not the iron. Limiting j for Cu is about 100A/mm²

B < 1.5 T: Typically used region

B < 1 T: Region in which $B_0 = \mu_0 \frac{nI}{a}$




Shims reduce the space that is open to the beam, but they also reduce the fringe field region.

# Where is the vertical Dipole?

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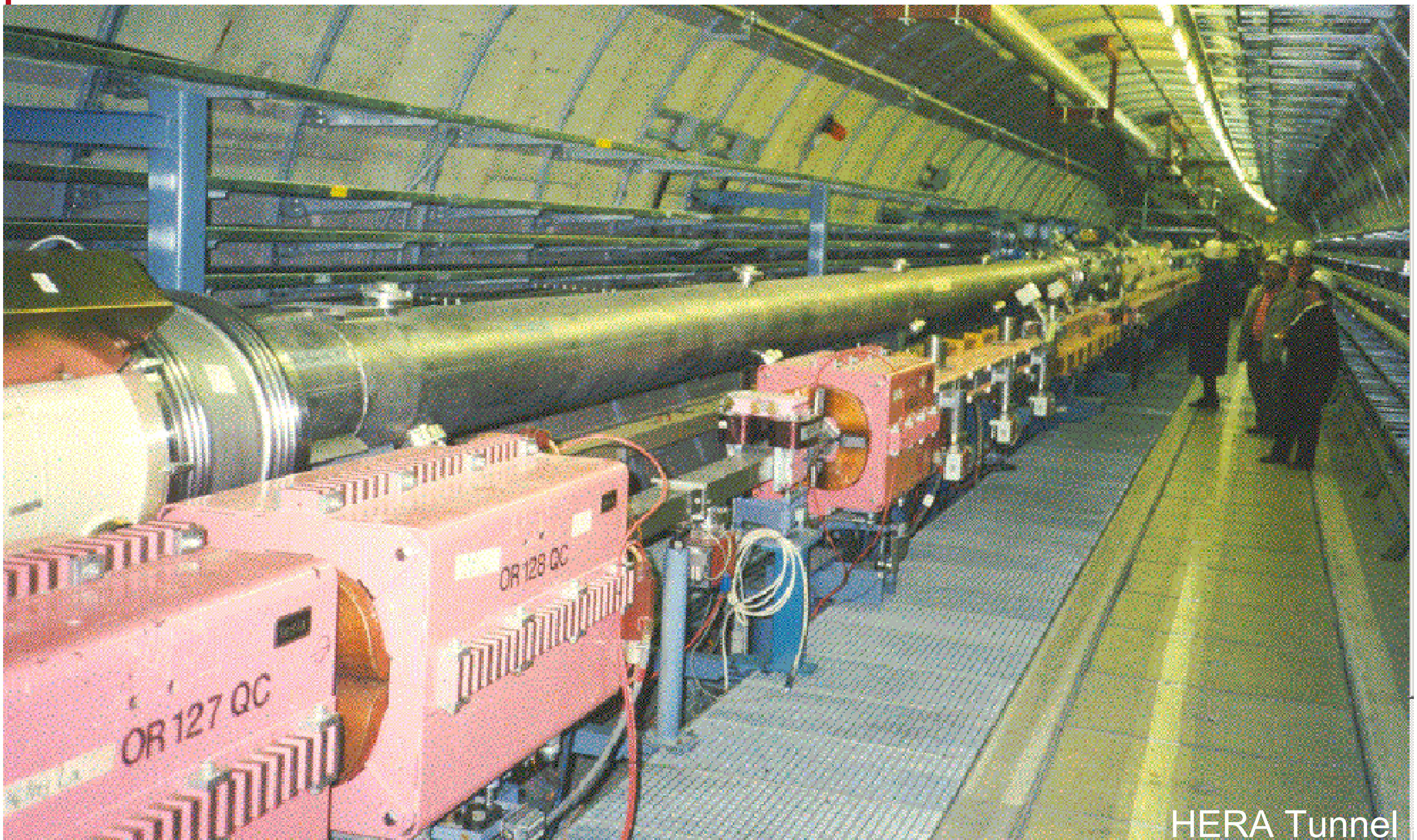
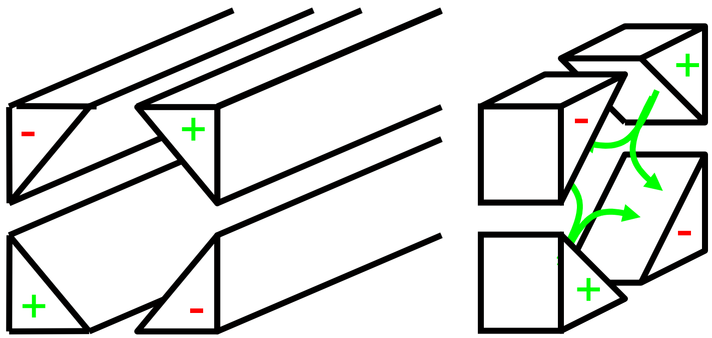


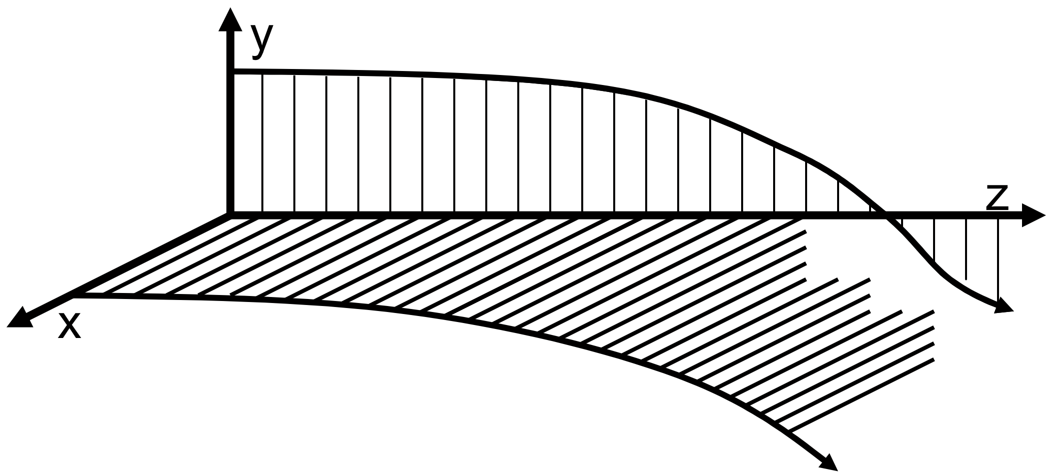
# Multipoles in Accelerators  ν=2: Quadrupoles



$$\psi = \Psi_2 \,\mathrm{Im}\{(x - iy)^2\} = -\Psi_2 \cdot 2xy \quad \Rightarrow \quad \vec{B} = -\vec{\nabla}\psi = \Psi_2 \, 2\begin{pmatrix} y \\ x \end{pmatrix}$$

$C_2$ Symmetry



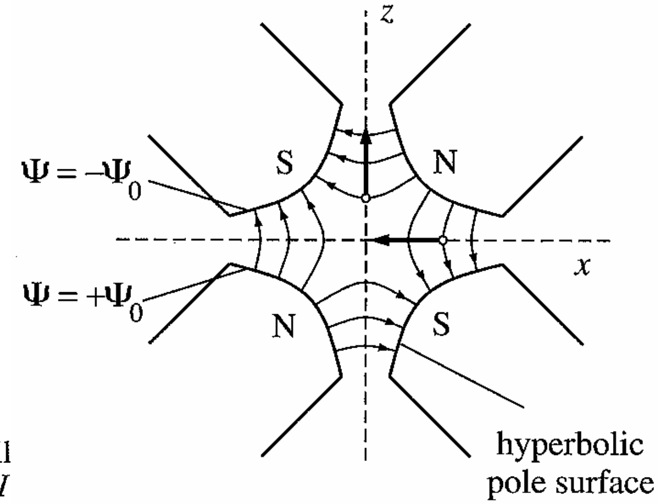



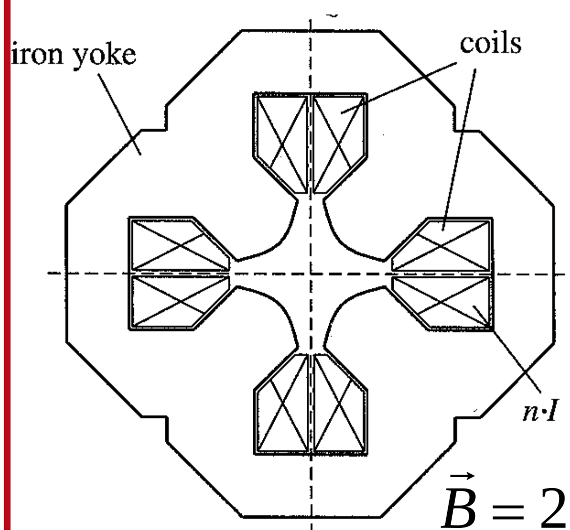

In a quadrupole particles are focused in one plane and defocused in the other plane. Other modes of strong focusing are not possible.

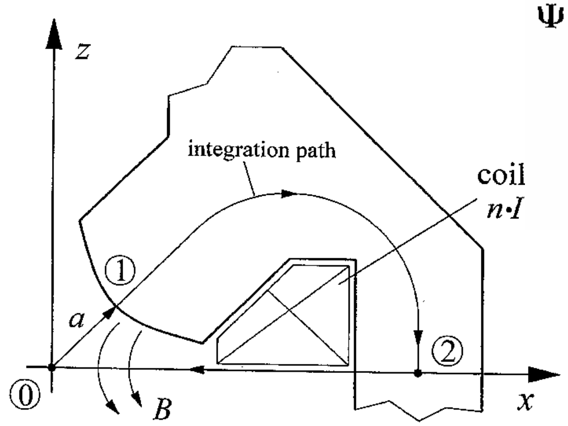
# Quadrupole Fields



$$\psi = -\Psi_2 \cdot 2xy \quad \Rightarrow$$ Equipotential: $$x = \frac{\text{const.}}{y}$$




$$\vec{B} = 2\Psi_2 \begin{pmatrix} y \\ x \end{pmatrix} \quad \Rightarrow \quad \vec{B}(0 \mapsto 1) = 2\Psi_2 r\, \vec{e}_r$$

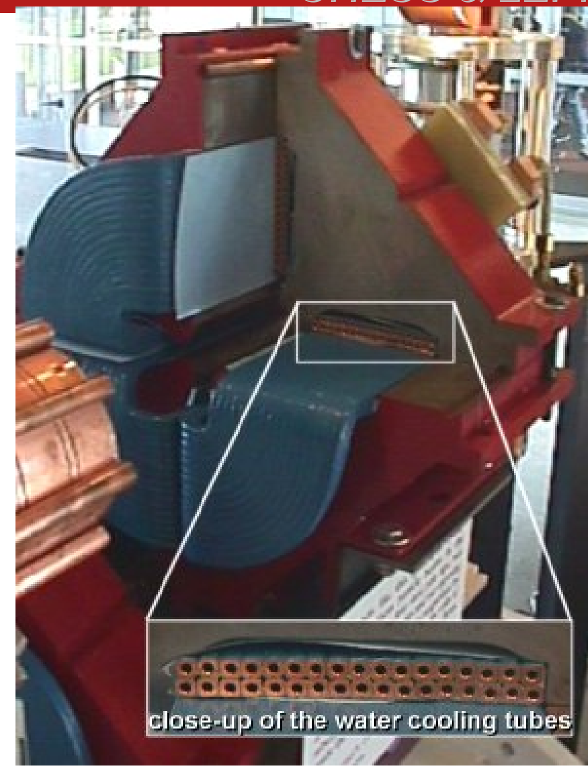
Quadrupole strength:

$$nI = \oint \vec{H} \cdot d\vec{s} \approx \int_0^a H_r dr = \Psi_2 \frac{a^2}{\mu_0}$$

$$k_1 = \frac{q}{p} \partial_x B_y \big|_0 = \frac{q\mu_0}{p} \frac{2nI}{a^2}$$

# Real Quadrupoles





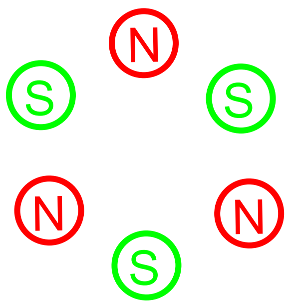
The coils show that this is an upright quadrupole not a rotated or skew quadrupole.

# Multipoles in Accelerators $\nu$=3: Sextupoles



$$\psi = \Psi_3 \, \mathrm{Im}\{(x - iy)^3\} = \Psi_3 \cdot (y^3 - 3x^2 y) \quad \Rightarrow \quad \vec{B} = -\vec{\nabla}\psi = \Psi_3 \, 3 \begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix}$$

$C_3$ Symmetry




i) Sextupole fields hardly influence the particles close to the center, where one can linearize in x and y.

ii) In linear approximation a by $\Delta$x shifted sextupole has a quadrupole field.

iii) When $\Delta$x depends on the energy, one can build an energy dependent quadrupole.

$$\vec{B} = -\vec{\nabla}\psi = \Psi_3 \, 3 \begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix}$$

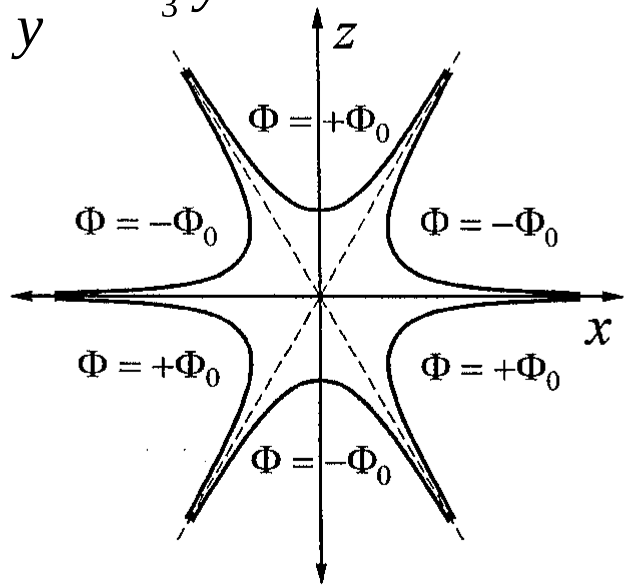
$$x \mapsto \Delta x + x$$

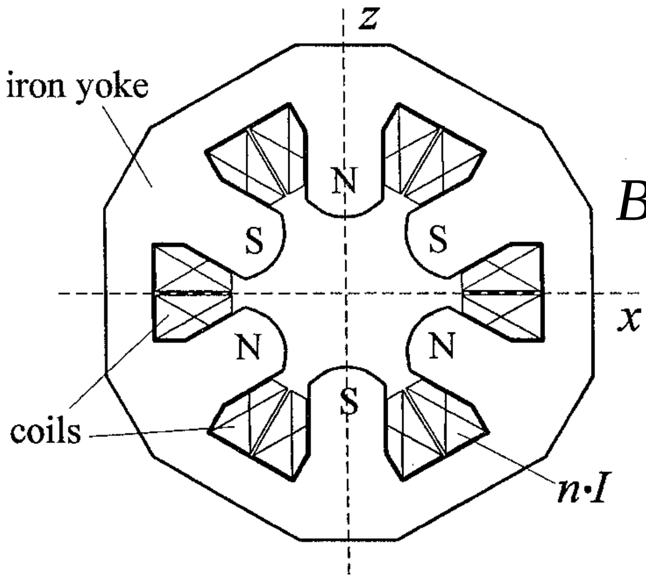
$$\vec{B} \approx \Psi_3 \, 3 \begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix} + 6\Psi_3 \Delta x \begin{pmatrix} y \\ x \end{pmatrix} + O(\Delta x^2)$$

# Sextupole Fields



$$\psi = \Psi_2 \cdot (y^3 - 3x^2 y) \quad \Rightarrow$$ Equipotential: $$x = \sqrt{\frac{\text{const.}}{y} + \tfrac{1}{3} y^2}$$




$$B_y\big|_{x=0} = -\Psi_3 3y^2$$

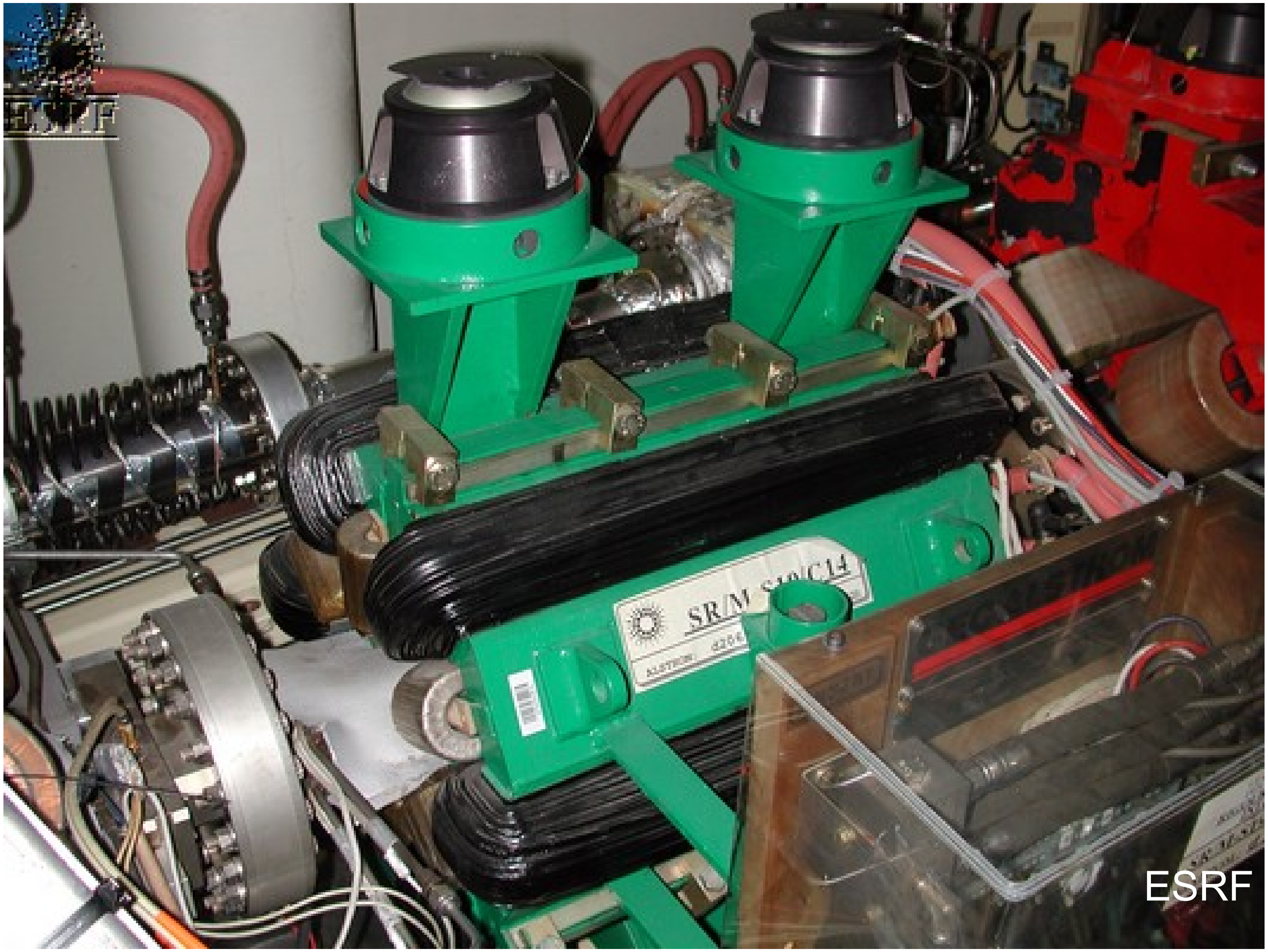
$$nI = \oint \vec{H} \cdot d\vec{s} \approx \int_0^a H_r dr = \Psi_3 \frac{a^3}{\mu_0}$$

Quadrupole strength:

$$k_2 = \frac{q}{p} \partial_x^2 B_y\big|_0 = \frac{q\mu_0}{p} \frac{6nI}{a^3}$$

# Real Sextupoles





# The CESR Tunnel

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# Higher order Multipoles



$$\psi = \Psi_n \,\mathrm{Im}\{(x-iy)^n\} = \Psi_n \cdot (\ldots - i\, n\, x^{n-1} y) \quad \Rightarrow \quad \vec{B}(y=0) = \Psi_n \, n \begin{pmatrix} 0 \\ x^{n-1} \end{pmatrix}$$

Multipole strength: $k_n = \frac{q}{p} \partial_x^n B_y \Big|_{x,y=0} = \frac{q}{p} \Psi_{n+1} \, (n+1)!$ units: $\frac{1}{\mathrm{m}^{n+1}}$

p/q is also called Bρ and used to describe the energy of multiply charge ions

Names: dipole, quadrupole, sextupole, octupole, decapole, duodecapole, …

Higher order multipoles come from

- Field errors in magnets
- Magnetized materials
- From multipole magnets that compensate such erroneous fields
- To compensate nonlinear effects of other magnets
- To stabilize the motion of many particle systems
- To stabilize the nonlinear motion of individual particles

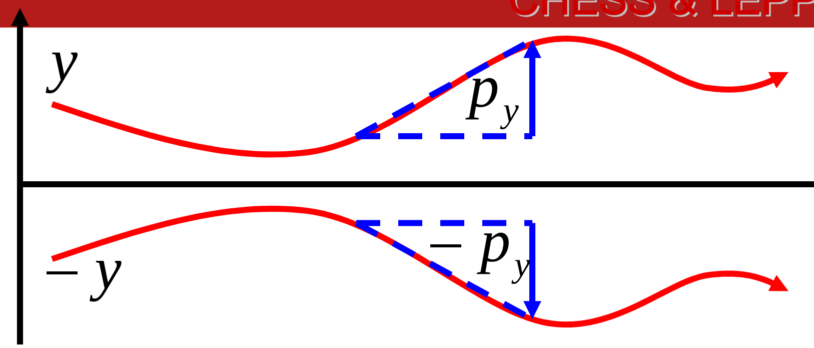
# Midplane Symmetric Motion

CHESS & LEPP

$$\vec{r}^{\oplus} = (x, -y, z)$$

$$\vec{p}^{\oplus} = (p_x, -p_y, p_z)$$

$$\tfrac{d}{dt}\vec{p} = \vec{F}(\vec{r}, \vec{p}) \quad \Rightarrow \quad \tfrac{d}{dt}\vec{p}^{\oplus} = \vec{F}(\vec{r}^{\oplus}, \vec{p}^{\oplus})$$




$$\begin{aligned} v_y B_z - v_z B_y &= -v_y B_z(x,-y,z) - v_z B_y(x,-y,z) & & B_x(x,-y,z) = -B_x(x,y,z) \\ v_z B_x - v_x B_z &= \ \ v_z B_x(x,-y,z) - v_x B_z(x,-y,z) &\Rightarrow\ & B_y(x,-y,z) = \ \ B_y(x,y,z) \\ v_x B_y - v_y B_x &= \ \ v_x B_y(x,-y,z) + v_y B_x(x,-y,z) & & B_z(x,-y,z) = -B_z(x,y,z) \end{aligned}$$

$$\boxed{\psi(x,-y,z) = -\psi(x,y,z)}$$

$$\Psi_n \operatorname{Im}\left\{e^{in\vartheta_n}(x+iy)^n\right\} = -\Psi_n \operatorname{Im}\left\{e^{in\vartheta_n}(x+iy)^n\right\}$$

$$\Rightarrow \quad \Psi_n \operatorname{Im}\left[e^{in\vartheta_n}\, 2\operatorname{Re}\left\{(x+iy)^n\right\}\right] = 0 \quad \Rightarrow \boxed{\vartheta_n = 0}$$

The discussed multipoles

produce midplane symmetric motion. When the field is rotated by $\pi/2$, i.e $\vartheta_n = \pi/2n$, one speaks of a skew multipole.

# Superconducting Magnets



Above 2T the field from the bare coils dominate over the magnetization of the iron.

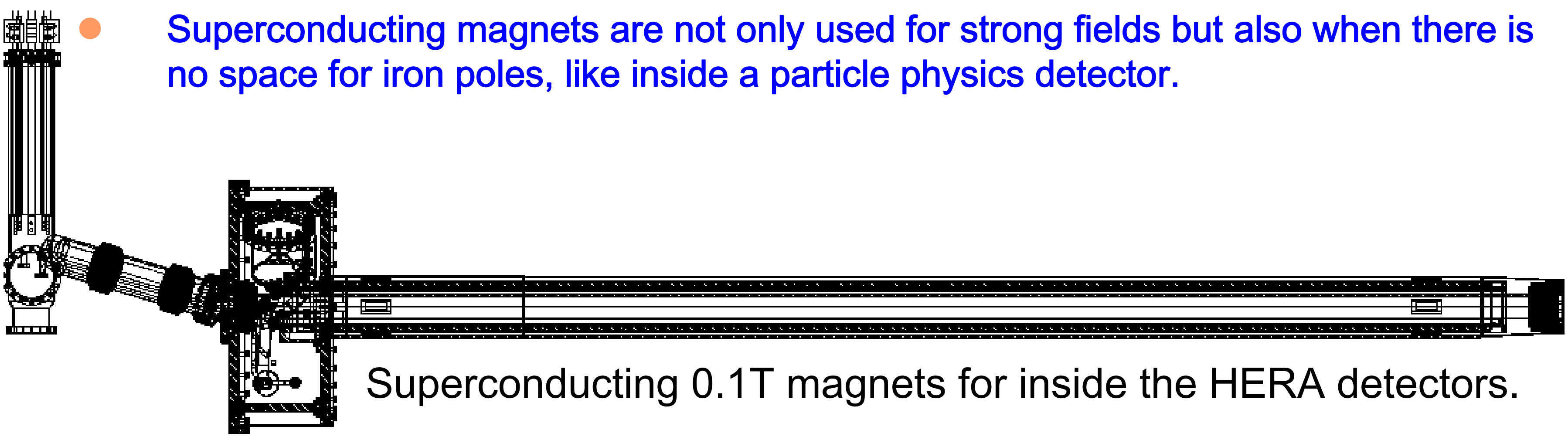
But Cu wires cannot create much filed without iron poles:

5T at 5cm distance from a 3cm wire would require a current density of

$$j = \frac{I}{d^2} = \frac{1}{d^2}\frac{2\pi\, r\, B}{\mu_0} = 1389\,\frac{\mathrm{A}}{\mathrm{mm}^2}$$

Cu can only support about $100\mathrm{A/mm}^2$.

- Superconducting cables routinely allow current densities of $1500\mathrm{A/mm}^2$ at 4.6 K and 6T. Materials used are usually Nb aloys, e.g. NbTi, $Nb_3Ti$ or $Nb_3Sn$.
- Superconducting magnets are not only used for strong fields but also when there is no space for iron poles, like inside a particle physics detector.



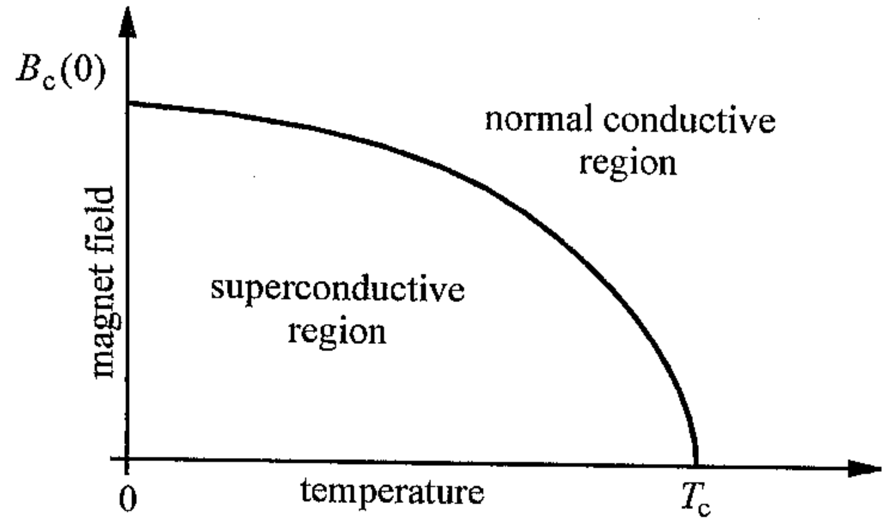
Superconducting 0.1T magnets for inside the HERA detectors.

# Superconducting Magnets



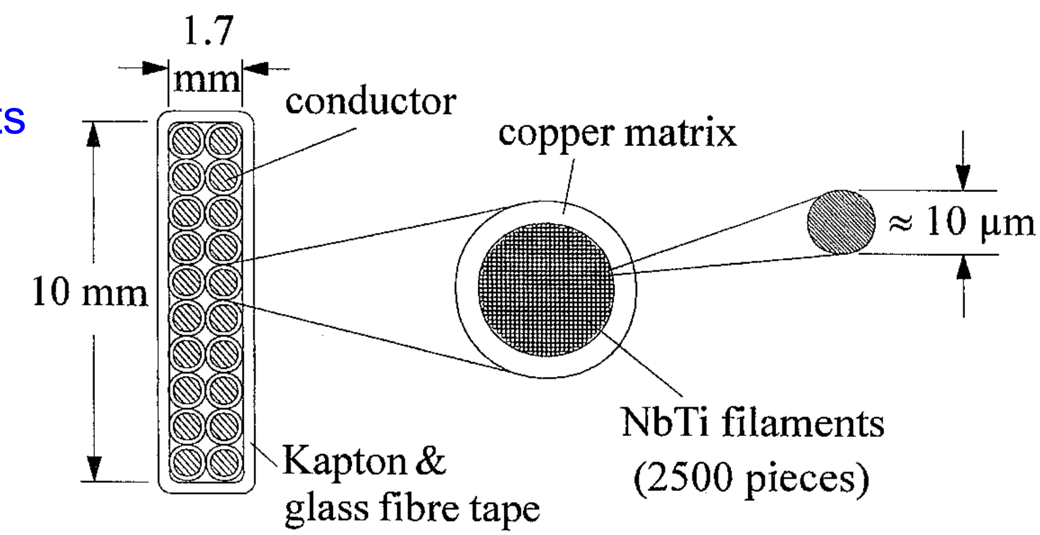
Problems:

- Superconductivity brakes down for too large fields
- Due to the Meissner-Ochsenfeld effect superconductivity current only flows on a thin surface layer.




Remedy:

- Superconducting cable consists of many very thin filaments (about 10μm).



# Complex Potential of a Wire



Straight wire at the origin: $\vec{\nabla}\times\vec{B}=\mu_0\,\vec{j} \quad\Rightarrow\quad \vec{B}(r)=\frac{\mu_0\,I}{2\pi\,r}\vec{e}_\varphi=\frac{\mu_0\,I}{2\pi\,r}\begin{pmatrix}-y\\ x\end{pmatrix}$

Wire at $\vec{a}$ :

$$\vec{B}(x,y)=\frac{\mu_0\,I}{2\pi\,(\vec{r}-\vec{a})^2}\begin{pmatrix}-[y-a_y]\\ x-a_x\end{pmatrix}$$

This can be represented by complex multipole coefficients $\Psi_\nu$

$$\vec{B}(x,y)=-\vec{\nabla}\Psi \quad\Rightarrow\quad B_x+iB_y=-\left(\partial_x+i\partial_y\right)\psi=-2\partial_{\overline{w}}\psi$$

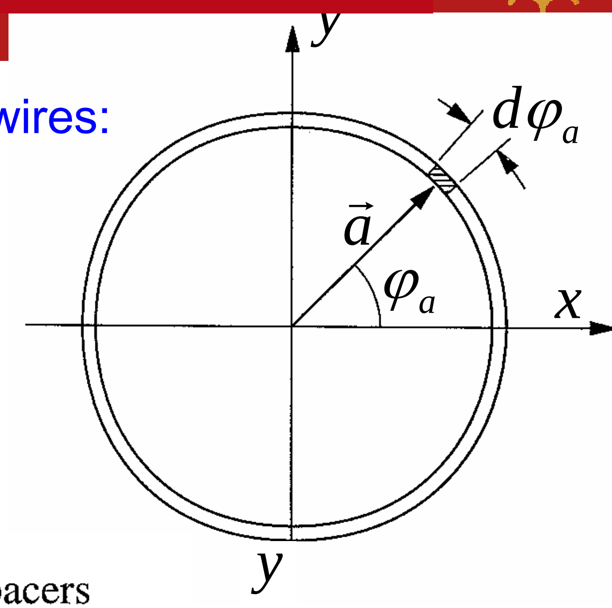
$$B_x+iB_y=\tfrac{\mu_0 I}{2\pi}\frac{-i(w_a-w)}{(w_a-w)(\overline{w}_a-\overline{w})}=i\,\tfrac{\mu_0 I}{2\pi}\frac{-\frac{w_a}{a^2}}{1-\frac{\overline{w}w_a}{a^2}}$$

$$=i\,\tfrac{\mu_0 I}{2\pi}\partial_{\overline{w}}\ln(1-\tfrac{\overline{w}w_a}{a^2})=-2\partial_{\overline{w}}\,\mathrm{Im}\left\{\tfrac{\mu_0 I}{2\pi}\ln(1-\tfrac{\overline{w}w_a}{a^2})\right\}$$

$$\psi=\mathrm{Im}\left\{\tfrac{\mu_0 I}{2\pi}\ln(1-\tfrac{\overline{w}w_a}{a^2})\right\}=-\mathrm{Im}\left\{\tfrac{\mu_0 I}{2\pi}\sum_{\nu=1}^{\infty}\tfrac{1}{\nu}\left(\tfrac{w_a}{a^2}\right)^\nu\overline{w}^\nu\right\} \quad\Rightarrow\quad \Psi_\nu=\tfrac{\mu_0 I}{2\pi}\tfrac{1}{\nu}\tfrac{1}{a^\nu}e^{i\nu\varphi_a}$$

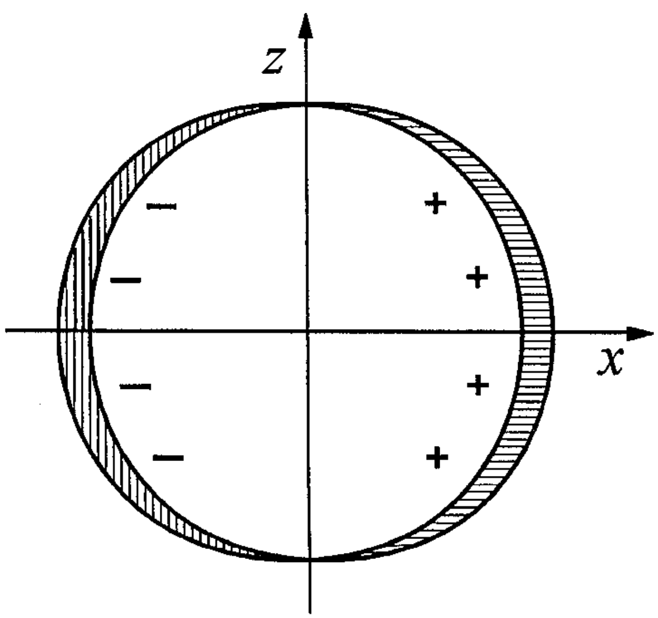
# Air-coil Multipoles

Creating a multipole be created by an arrangement of wires:

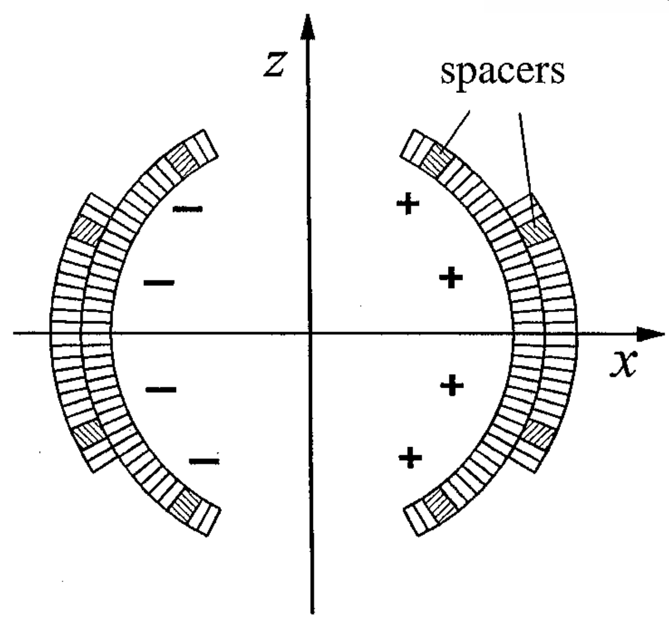



$$\Psi_\nu = \int_0^{2\pi} \frac{\mu_0}{2\pi} \frac{1}{\nu} \frac{1}{a^\nu} e^{i\nu\varphi_a} \frac{dI}{d\varphi_a} d\varphi_a$$

$$\Psi_\nu = \delta_{\nu n} \frac{\mu_0}{2} \frac{1}{n} \frac{1}{a^n} \hat{I} \quad \text{if } I(\varphi_a) = \hat{I} \cos n\varphi_a$$




Ideal multipole

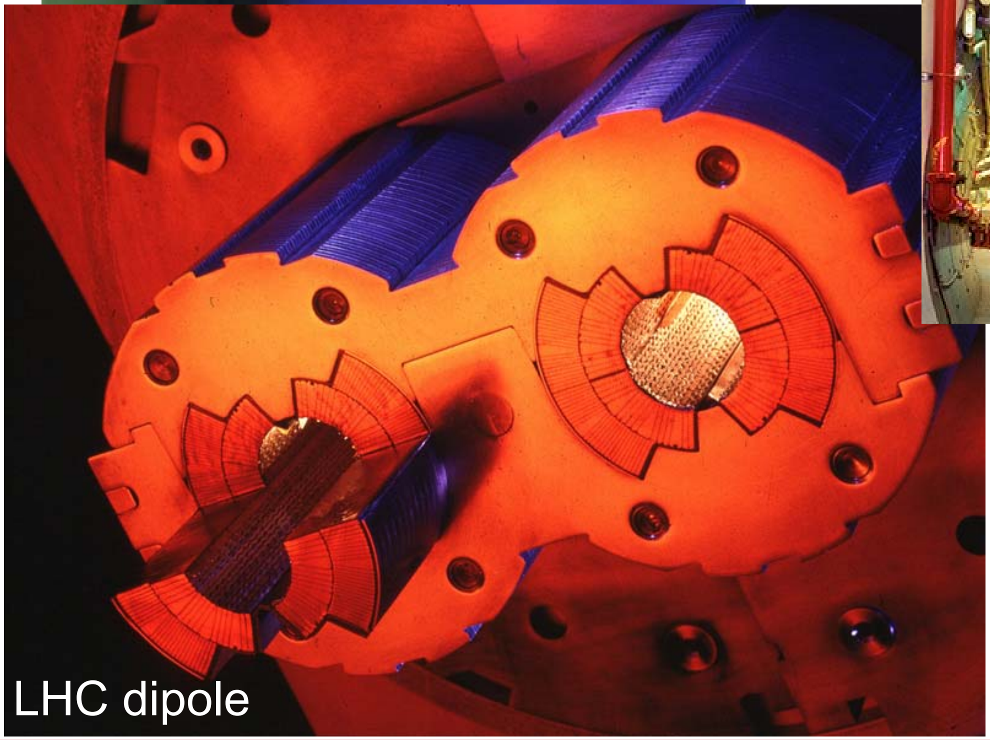



Approximate multipole

# Real Air-coil Multipoles

CHESS & LEPP

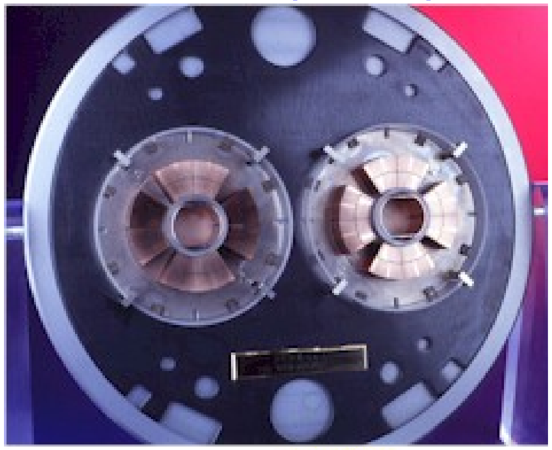
Quadrupole corrector



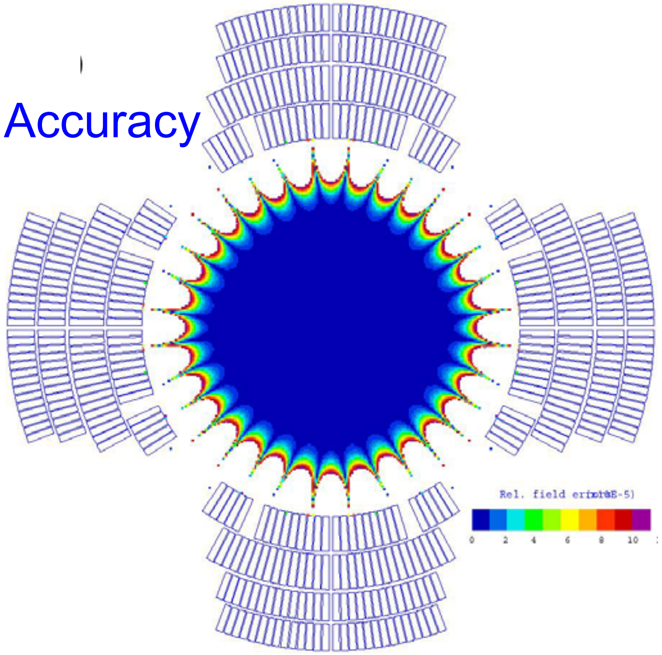

# Special SC Air-coil Magnets

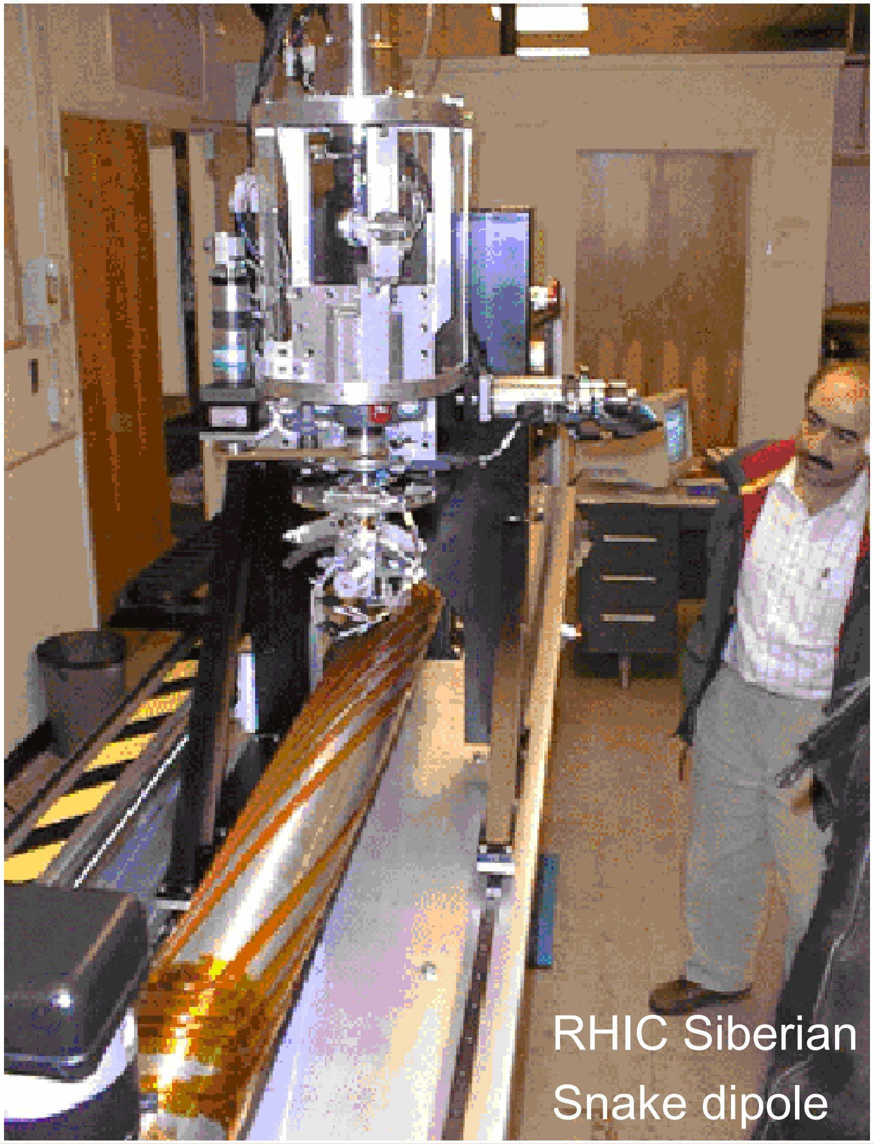
CHESS & LEPP

LHC double quadrupole



Accuracy



RHIC Siberian Snake dipole