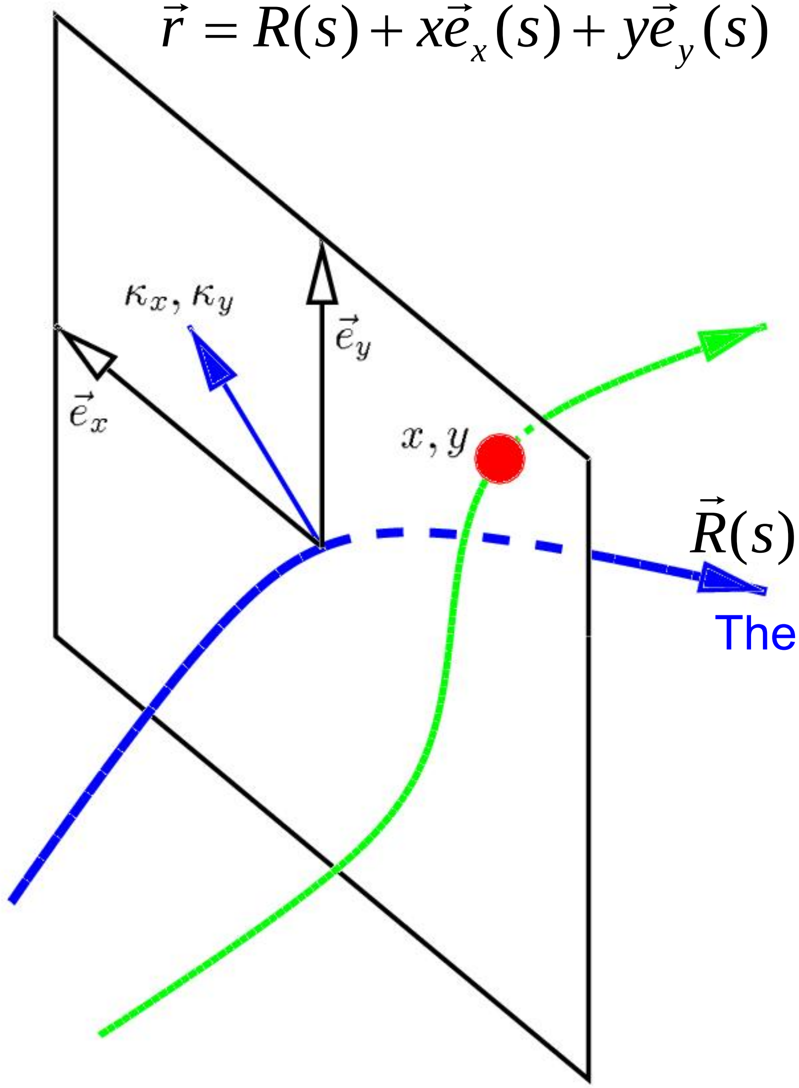
The comoving Coordinate System

$$\vec{r} = \vec{R}(s) + x\vec{e}_x(s) + y\vec{e}_y(s)$$

$$\left|d\vec{R}\right| = ds$$

$$\vec{e}_s \equiv \tfrac{d}{ds}\vec{R}(s)$$



The time dependence of a particle's motion is often not as interesting as the trajectory along the accelerator length "s".

The 4D Equation of Motion

$$\frac{d^2}{dt^2}\vec{r} = \vec{f}_r(\vec{r},\tfrac{d}{dt}\vec{r},t)$$

3 dimensional ODE of 2nd order can be changed to a

6 dimensional ODE of 1st order:

$$\left.\begin{array}{l}\frac{d}{dt}\vec{r} = \frac{1}{m\gamma}\vec{p} = \frac{c}{\sqrt{p^2-(mc)^2}}\vec{p} \\ \frac{d}{dt}\vec{p} = \vec{F}(\vec{r},\vec{p},t)\end{array}\right\} \quad \frac{d}{dt}\vec{Z} = \vec{f}_Z(\vec{Z},t)\,, \quad \vec{Z} = (\vec{r},\vec{p})$$

If the force does not depend on time, as in a typical beam line magnet, the energy is conserved so that one can reduce the dimension to 5. The equation of motion is then autonomous.

Furthermore, the time dependence is often not as interesting as the trajectory along the accelerator length "s". Using "s" as the independent variable reduces the dimensions to 4. The equation of motion is then no longer autonomous.

$$\frac{d}{ds}\vec{z} = \vec{f}_z(\vec{z},s)\,, \quad \vec{z} = (x,y,p_x,p_y)$$

The 6D Equation of Motion

Usually one prefers to compute the trajectory as a function of "s" along the accelerator even when the energy is not conserved, as when accelerating cavities are in the accelerator.

Then the energy "E" and the time "t" at which a particle arrives at the cavities are important. And the equations become 6 dimensional again:

$$\tfrac{d}{ds}\vec{z} = \vec{f}_z(\vec{z}, s)\,, \quad \vec{z} = (x, y, p_x, p_y, -t, E)$$

But: $\vec{z} = (\vec{r}, \vec{p})$ is an especially suitable variable, since it is a phase space vector so that its equation of motion comes from a Hamiltonian, or by variation principle from a Lagrangian.

$$\delta\int\left[p_x\dot{x} + p_y\dot{y} + p_s\dot{s} - H(\vec{r}, \vec{p}, t)\right]dt = 0 \quad \Rightarrow \quad \text{Hamiltonian motion}$$

$$\delta\int\left[p_x x' + p_y y' - H\,t' + p_s(x, y, p_x, p_y, t, H)\right]ds = 0 \quad \Rightarrow \quad \text{Hamiltonian motion}$$

The new canonical coordinates are: $\vec{z} = (x, y, p_x, p_y, -t, E) \quad \text{with} \quad \mathrm{E} = \mathrm{H}$

The new Hamiltonian is: $K = -p_s(\vec{z}, s)$

Significance of Hamiltonian

The equations of motion can be determined by one function:

$$\tfrac{d}{ds}x = \partial_{p_x} H(\vec{z},s)\,, \quad \tfrac{d}{ds}p_x = -\partial_x H(\vec{z},s)\,, \quad \ldots$$

$$\tfrac{d}{ds}\vec{z} = \underline{J}\,\vec{\partial}H(\vec{z},s) = \vec{F}(\vec{z},s) \quad \text{with} \quad \underline{J} = \mathrm{diag}(\underline{J}_2)\,, \quad \underline{J}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The force has a Hamiltonian Jacobi Matrix:

A linear force: $\vec{F}(\vec{z},s) = \underline{F}(s)\cdot\vec{z}$

The Jacobi Matrix of a linear force: $\underline{F}(s)$

The general Jacobi Matrix : $F_{ij} = \partial_{z_j} F_i$ or $\underline{F} = \left(\vec{\partial}\,\vec{F}^T\right)^T$

Hamiltonian Matrices: $\underline{F}\,\underline{J} + \underline{J}\,\underline{F}^T = 0$

Prove : $F_{ij} = \partial_{z_j} F_i = \partial_{z_j} J_{ik} \partial_{z_k} H = J_{ik}\,\partial_k \partial_j H \quad \Rightarrow \quad \underline{F} = \underline{J}\,\underline{DH}$

$$\underline{F}\,\underline{J} + \underline{J}\,\underline{F}^T = \underline{J}\,\underline{DJH} + \underline{J}\,\underline{D}^T\,\underline{J}^T H = 0$$

H $\rightarrow$ Symplectic Flows

The flow of a Hamiltonian equation of motion has a symplectic Jacobi Matrix

The flow or transport map: $\vec{z}(s) = \vec{M}(s, \vec{z}_0)$

A linear flow: $\vec{z}(s) = \underline{M}(s) \cdot \vec{z}_0$

The Jacobi Matrix of a linear flow: $\underline{M}(s)$

The general Jacobi Matrix : $M_{ij} = \partial_{z_{0j}} M_i$ or $\underline{M} = \left(\vec{\partial}_0 \vec{M}^T\right)^T$

The Symplectic Group SP(2N) : $\underline{M}\,\underline{J}\,\underline{M}^T = \underline{J}$

$$\tfrac{d}{ds}\vec{z} = \tfrac{d}{ds}\vec{M}(s, \vec{z}_0) = \underline{J}\,\vec{\nabla} H = \vec{F}$$

$$\tfrac{d}{ds} M_{ij} = \partial_{z_{0j}} F_i(\vec{z}, s) = \partial_{z_{0j}} M_k \partial_{z_k} F_i(\vec{z}, s)$$

$$\tfrac{d}{ds}\underline{M}(s, \vec{z}_0) = \underline{F}(\vec{z}, s)\underline{M}(s, \vec{z}_0)$$

$$K = \underline{M}\,\underline{J}\,\underline{M}^T$$

$$\tfrac{d}{ds}\underline{K} = \tfrac{d}{ds}\underline{M}\,\underline{J}\,\underline{M}^T + \underline{M}\,\underline{J}\,\tfrac{d}{ds}\underline{M}^T = \underline{F}\,\underline{M}\,\underline{J}\,\underline{M}^T + \underline{M}\,\underline{J}\,\underline{M}^T\underline{F}^T = \underline{F}\,\underline{K} + \underline{K}\,\underline{F}^T$$

$\underline{K} = \underline{J}$ is a solution. Since this is a linear ODE , $\underline{K} = \underline{J}$ is the unique solution.

Symplectic Flows → H

For every symplectic transport map there is a Hamilton function

The flow or transport map: $\vec{z}(s) = \vec{M}(s, \vec{z}_0)$

Force vector: $\vec{h}(\vec{z}, s) = -\underline{J}\left[\frac{d}{ds}\vec{M}(s, \vec{z}_0)\right]_{\vec{z}_0 = \vec{M}^{-1}(\vec{z}, s)}$

Since then: $\frac{d}{ds}\vec{z} = \underline{J}\vec{h}(\vec{z}, s)$

There is a Hamilton function H with: $\vec{h} = \vec{\partial}H$

If and only if: $\partial_{z_j} h_i = \partial_{z_i} h_j \quad \Rightarrow \quad \underline{h} = \underline{h}^T$

$$\underline{M}\,\underline{J}\,\underline{M}^T = \underline{J} \quad \Rightarrow \quad \begin{cases} \frac{d}{ds}\underline{M}\,\underline{J}\,\underline{M}^T = -\underline{M}\,\underline{J}\,\frac{d}{ds}\underline{M}^T \\ \underline{M}^{-1} = -\underline{J}\,\underline{M}^T\,\underline{J} \end{cases}$$

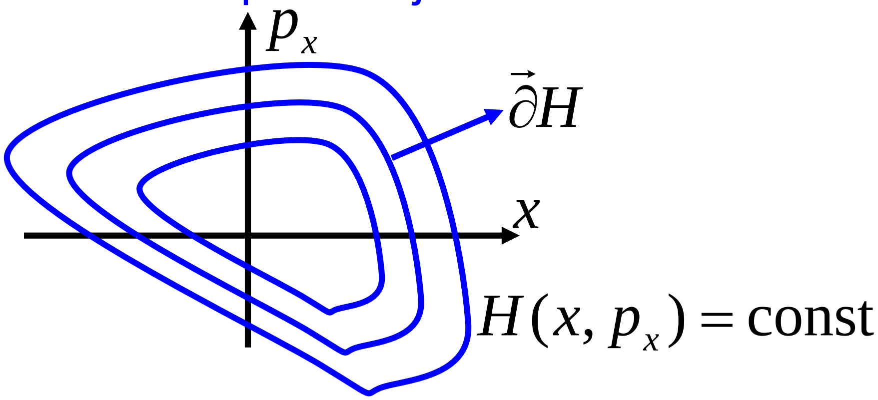
$$\vec{h} \circ \vec{M} = -\underline{J}\,\frac{d}{ds}\vec{M}$$

$$\underline{h}(\vec{M})\underline{M} = -\underline{J}\,\frac{d}{ds}\underline{M}$$

$$\underline{h}(\vec{M}) = -\underline{J}\,\frac{d}{ds}\underline{M}\,\underline{M}^{-1} = \underline{J}\,\frac{d}{ds}\underline{M}\,\underline{J}\,\underline{M}^T\,\underline{J} = -\underline{J}\,\underline{M}\,\underline{J}\,\frac{d}{ds}\underline{M}^T\,\underline{J} = \underline{M}^{-T}\,\frac{d}{ds}\underline{M}^T\,\underline{J} = \underline{h}^T$$

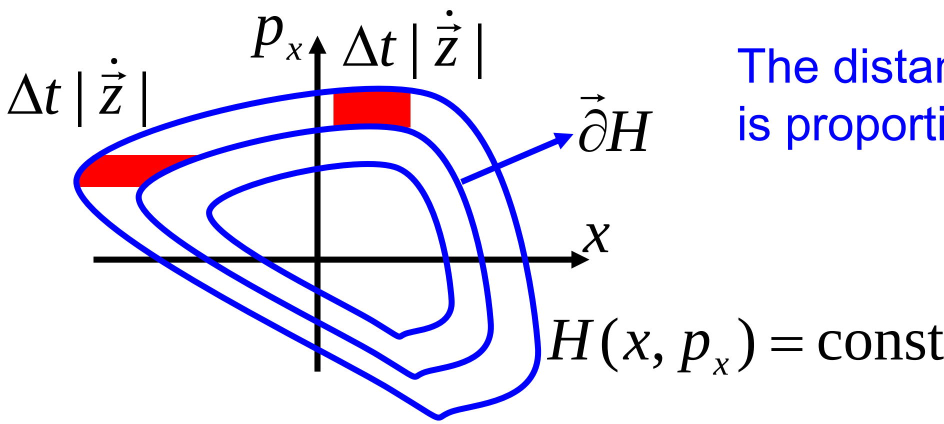
Phase space density in 2D

- Phase space trajectories move on surfaces of constant energy



$$\tfrac{d}{ds}\vec{z} = \underline{J}\,\vec{\partial}H \quad \Rightarrow \quad \tfrac{d}{ds}\vec{z} \perp \vec{\partial}H$$

- A phase space volume does not change when it is transported by Hamiltonian motion.

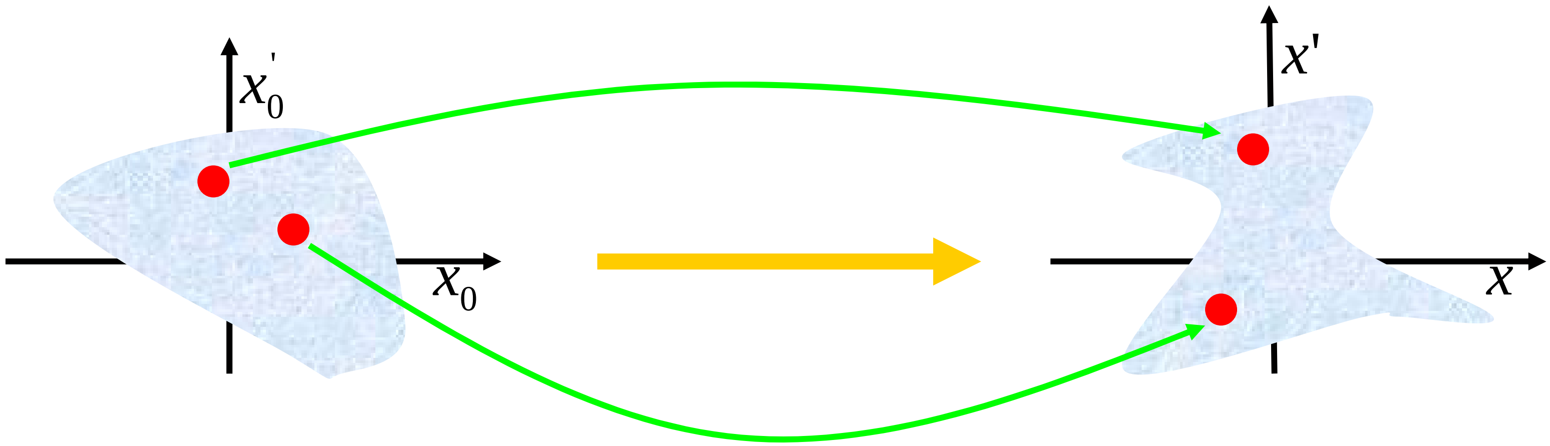


The distance d of lines with equal energy is proportional to $1/|\vec{\partial}H| \propto |\dot{\vec{z}}\,|^{-1}$

$$d * \Delta t\,|\dot{\vec{z}}\,| = \text{const}$$

Lioville's Theorem

- A phase space volume does not change when it is transported by Hamiltonian motion. $\vec{z}(s) = \underline{M}(s)\cdot\vec{z}_0$ with $\det[\underline{M}(s)] = +1$



Volume = $V = \iint\limits_{V} d^n\vec{z} = \iint\limits_{V_0} \left|\frac{\partial\vec{z}}{\partial\vec{z}_0}\right| d^n\vec{z}_0 = \iint\limits_{V_0} |\underline{M}|\, d^n\vec{z}_0 = \iint\limits_{V_0} d^n\vec{z}_0 = V_0$

Hamiltonian Motion → $V = V_0$

But Hamiltonian requires symplecticity, which is much more than just $\det[\underline{M}(s)] = +1$