Generating Functions

The motion of particles can be represented by Generating Functions

Each flow or transport map: $\vec{z}(s) = \vec{M}(s, \vec{z}_0)$

With a Jacobi Matrix : $M_{ij} = \partial_{z_{0j}} M_i$ or $\underline{M} = \left(\vec{\partial}_0 \vec{M}^T\right)^T$

That is Symplectic: $\underline{M}\,\underline{J}\,\underline{M}^T = \underline{J}$

Can be represented by a Generating Function:

$$F_1(\vec{q}, \vec{q}_0, s) \quad \text{with} \quad \vec{p} = -\vec{\partial}_q F_1 \quad , \quad \vec{p}_0 = \vec{\partial}_{q_0} F_1$$

$$F_2(\vec{p}, \vec{q}_0, s) \quad \text{with} \quad \vec{q} = \vec{\partial}_p F_2 \quad , \quad \vec{p}_0 = \vec{\partial}_{q_0} F_2$$

$$F_3(\vec{q}, \vec{p}_0, s) \quad \text{with} \quad \vec{p} = -\vec{\partial}_q F_3 \quad , \quad \vec{q}_0 = -\vec{\partial}_{p_0} F_3$$

$$F_4(\vec{p}, \vec{p}_0, s) \quad \text{with} \quad \vec{q} = \vec{\partial}_q F_4 \quad , \quad \vec{q}_0 = -\vec{\partial}_{p_0} F_4$$

6-dimensional motion needs only one function ! But to obtain the transport map this has to be inverted.

F → SP(2N) [for notes]

Generating Functions produce symplectic tranport maps

$$F_1(\vec{q},\vec{q}_0,s) \quad \text{with} \quad \vec{p} = -\vec{\partial}_q F_1(\vec{q},\vec{q}_0,s) \quad , \quad \vec{p}_0 = \vec{\partial}_{q_0} F_1(\vec{q},\vec{q}_0,s)$$

$$\vec{z} = \begin{pmatrix} \vec{q} \\ \vec{p} \end{pmatrix} = \begin{pmatrix} \vec{q} \\ -\vec{\partial}_q F_1(\vec{q},\vec{q}_0,s) \end{pmatrix} = \vec{f}(\vec{Q},s)$$

$$\vec{z}_0 = \begin{pmatrix} \vec{q}_0 \\ \vec{p}_0 \end{pmatrix} = \begin{pmatrix} \vec{q}_0 \\ \vec{\partial}_{q_0} F_1(\vec{q},\vec{q}_0,s) \end{pmatrix} = \vec{g}(\vec{Q},s)$$

$$\vec{z} = \vec{f}(\vec{g}^{-1}(\vec{z}_0,s),s)$$

$$\vec{M} = \vec{f} \circ \vec{g}^{-1}$$

(function concatenation)

Jacobi matrix of concatenated functions:

$$\vec{C}(\vec{z}_0) = \vec{A} \circ \vec{B}(\vec{z}_0)$$

$$C_{ij} = \partial_j C_i = \sum_k \partial_{z_{0j}} B_k(\vec{z}_0) \left[\partial_{z_k} A_i(\vec{z})\right]_{\vec{z}=\vec{B}(\vec{z}_0)} \quad \Rightarrow \quad \underline{C} = \underline{A}(\vec{B})\underline{B}$$

$$\vec{M} \circ \vec{g} = \vec{f} \quad \Rightarrow \quad \underline{M}(\vec{g}) = \underline{F}\,\underline{G}^{-1}$$

F → SP(2N) [for notes]

$$\vec{f}(\vec{Q},s)=\begin{pmatrix}\vec{q}\\ -\vec{\partial}_q F_1(\vec{q},\vec{q}_0,s)\end{pmatrix} \Rightarrow F=\begin{pmatrix}\underline{1} & \underline{0}\\ -\vec{\partial}_q\vec{\partial}_q^T F_1 & -\vec{\partial}_q\vec{\partial}_{q_0}^T F_1\end{pmatrix}$$

$$\vec{g}(\vec{Q},s)=\begin{pmatrix}\vec{q}_0\\ \vec{\partial}_{q_0} F_1(\vec{q},\vec{q}_0,s)\end{pmatrix} \Rightarrow G=\begin{pmatrix}\underline{0} & \underline{1}\\ \vec{\partial}_{q_0}\vec{\partial}_q^T F_1 & \vec{\partial}_{q_0}\vec{\partial}_{q_0}^T F_1\end{pmatrix}$$

$$F=\begin{pmatrix}1 & 0\\ -F_{11} & -F_{12}\end{pmatrix},\quad G=\begin{pmatrix}0 & 1\\ F_{21} & F_{22}\end{pmatrix} \Rightarrow G^{-1}=\begin{pmatrix}-F_{21}^{-1}F_{22} & F_{21}^{-1}\\ 1 & 0\end{pmatrix}$$

$$\underline{M}(\vec{g})=FG^{-1}=\begin{pmatrix}-F_{21}^{-1}F_{22} & F_{21}^{-1}\\ F_{11}F_{21}^{-1}F_{22}-F_{12} & -F_{11}F_{21}^{-1}\end{pmatrix}$$

$$\underline{M}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\underline{M}^T$$

→ The map from a generating function is symplectic.

$$=\begin{pmatrix}-F_{21}^{-1} & -F_{21}^{-1}F_{22}\\ F_{11}F_{21}^{-1} & F_{11}F_{21}^{-1}F_{22}-F_{12}\end{pmatrix}\begin{pmatrix}-F_{22}F_{12}^{-1} & F_{22}F_{12}^{-1}F_{11}-F_{21}\\ F_{12}^{-1} & -F_{12}^{-1}F_{11}\end{pmatrix}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$

SP(2N) → F [for notes]

Symplectic tranport maps have a Generating Functions

$$\vec{z} = \vec{M}(\vec{z}_0)$$

$$\begin{pmatrix} \vec{q} \\ \vec{q}_0 \end{pmatrix} = \begin{pmatrix} \vec{M}_1(\vec{z}_0) \\ \vec{q}_0 \end{pmatrix} = \vec{l}(\vec{z}_0)\,, \quad \begin{pmatrix} \vec{p}_0 \\ \vec{p} \end{pmatrix} = \begin{pmatrix} \vec{p}_0 \\ \vec{M}_2(\vec{z}_0) \end{pmatrix} = \vec{h}(\vec{z}_0) = \underline{J}\left[\vec{\partial} F_1(\vec{q},\vec{q}_0)\right]_{\vec{l}(\vec{z}_0)}$$

$$\vec{\partial} F_1 = -\underline{J}\vec{h} \circ \vec{l}^{\,-1} = \vec{F}$$

For F_1 to exist it is necessary and sufficient that $\partial_i F_j = \partial_j F_i \quad \Rightarrow \quad \underline{F} = \underline{F}^T$

$$-\underline{J}\vec{h} = \vec{F} \circ \vec{l} \quad \Rightarrow \quad -\underline{J}\,\underline{h} = \underline{F}(\vec{l}\,)\,\underline{l}$$

Is $\underline{J}\ \underline{h}\ \underline{l}^{-1}$ symmetric ? Yes since:

$$\underline{J}\,\underline{h}\,\underline{l}^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \vec{\partial}^T_{q_0}\vec{M}_2 & \vec{\partial}^T_{p_0}\vec{M}_2 \end{pmatrix} \begin{pmatrix} \vec{\partial}^T_{q_0}\vec{M}_1 & \vec{\partial}^T_{p_0}\vec{M}_1 \\ 1 & 0 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} M_{21} & M_{22} \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ M_{12}^{-1} & -M_{12}^{-1}M_{11} \end{pmatrix} = \begin{pmatrix} M_{22}M_{12}^{-1} & M_{21} - M_{22}M_{12}^{-1}M_{11} \\ M_{12}^{-1} & M_{12}^{-1}M_{11} \end{pmatrix}$$

SP(2N) → F [for notes]

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$$\underline{J}\,\underline{h}\,\underline{l}^{-1} = \begin{pmatrix} M_{22}M_{12}^{-1} & M_{21} - M_{22}M_{12}^{-1}M_{11} \\ M_{12}^{-1} & M_{12}^{-1}M_{11} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$$\vec{M}(\vec{z}_0) = \begin{pmatrix} \vec{M}_1(\vec{q}_0,\vec{p}_0) \\ \vec{M}_2(\vec{q}_0,\vec{p}_0) \end{pmatrix}, \quad \underline{M} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

$$\underline{M}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\underline{M}^T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \begin{pmatrix} -M_{12} & M_{11} \\ -M_{22} & M_{21} \end{pmatrix}\begin{pmatrix} M_{11}^T & M_{21}^T \\ M_{12}^T & M_{22}^T \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$M_{12}M_{11}^T = M_{11}M_{12}^T \qquad \Rightarrow \quad \left(M_{12}^{-1}M_{11}\right)^T = [M_{12}^{-1}M_{11}M_{12}^T]M_{12}^{-T} = M_{12}^{-1}M_{11}$$

$$M_{21}M_{22}^T = M_{22}M_{21}^T$$

$$M_{11}M_{22}^T - M_{12}M_{21}^T = 1$$

$$M_{22}M_{11}^T - M_{21}M_{12}^T = 1$$

$$D = D^T$$

$$A = A^T$$

$$\left(M_{22}M_{12}^{-1}\right)^T = [M_{22}M_{11}^T M_{12}^{-T} - M_{21}]M_{22}^T = M_{22}[M_{12}^{-1}M_{11}M_{22}^T - M_{21}^T] = M_{22}M_{12}^{-1}$$

$$M_{21} - M_{22}M_{12}^{-1}M_{11} = M_{21} - M_{22}M_{11}M_{12}^{-T} = M_{12}^{-T} \quad \longrightarrow \quad B = C^T$$

Symplectic Representations

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Hamiltonian

$$\vec{z}' = \underline{J}\,\vec{\partial}\,H(\vec{z},s)$$

ODE

$$\vec{z}' = \vec{F}\,,\quad \underline{F}\,\underline{J} + \underline{J}\,\underline{F}^T = 0$$

Generating Functions

$$(\vec{p},\vec{p}_0) = -\underline{J}\,\vec{\partial}\,F_1(\vec{q},\vec{q}_0,s)$$

[from notes]

Symplectic transport map

$$\underline{M}\,\underline{J}\,\underline{M}^T = \underline{J}$$

Advantages of Symplecticity

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- Determinant of the transfer matrix of linear motion is 1:

$$\vec{z}(s) = \underline{M}(s)\cdot\vec{z}_0 \quad \text{with} \quad \det(\underline{M}(s)) = +1$$

- One function suffices to compute the total nonlinear transfer map:

$$F_1(\vec{q},\vec{q}_0,s) \quad \text{with} \quad \vec{p} = -\vec{\partial}_q F_1(\vec{q},\vec{q}_0,s) \quad , \quad \vec{p}_0 = \vec{\partial}_{q_0} F_1(\vec{q},\vec{q}_0,s)$$

$$\left.\begin{aligned}\vec{z} &= \begin{pmatrix}\vec{q}\\ \vec{p}\end{pmatrix} = \begin{pmatrix}\vec{q}\\ -\vec{\partial}_q F_1(\vec{q},\vec{q}_0,s)\end{pmatrix} = \vec{f}(\vec{Q},s)\\ \vec{z}_0 &= \begin{pmatrix}\vec{q}_0\\ \vec{p}_0\end{pmatrix} = \begin{pmatrix}\vec{q}_0\\ \vec{\partial}_{q_0} F_1(\vec{q},\vec{q}_0,s)\end{pmatrix} = \vec{g}(\vec{Q},s)\end{aligned}\right\} \quad \begin{aligned}\vec{z} &= \vec{f}(\vec{g}^{-1}(\vec{z}_0,s),s)\\ \vec{M} &= \vec{f}\circ\vec{g}^{-1}\end{aligned}$$

- Therefore Taylor Expansion coefficients of the transport map are related.
- Computer codes can numerically approximate $\vec{M}(s,\vec{z}_0)$ with exact symplectic symmetry.
- Liouville's Theorem for phase space densities holds.

Eigenvalues of a Symplectic Matrix

For matrices with real coefficients:

If there is an eigenvector and eigenvalue: $\underline{M}\vec{v}_i = \lambda_i \vec{v}_i$

then the complex conjugates are also eigenvector and eigenvalue: $\underline{M}\vec{v}_i^{\,*} = \lambda_i^* \vec{v}_i^{\,*}$

For symplectic matrices:

If there are eigenvectors and eigenvalues: $\underline{M}\vec{v}_i = \lambda_i \vec{v}_i$ with $\underline{J} = \underline{M}^T \underline{J}\,\underline{M}$

then $\vec{v}_i^{\,T} \underline{J}\vec{v}_j = \vec{v}_i^{\,T} \underline{M}^T \underline{J}\,\underline{M}\vec{v}_j = \lambda_i \lambda_j \vec{v}_i^{\,T} \underline{J}\vec{v}_j \quad \Rightarrow \quad \vec{v}_i^{\,T} \underline{J}\vec{v}_j (\lambda_i \lambda_j - 1) = 0$

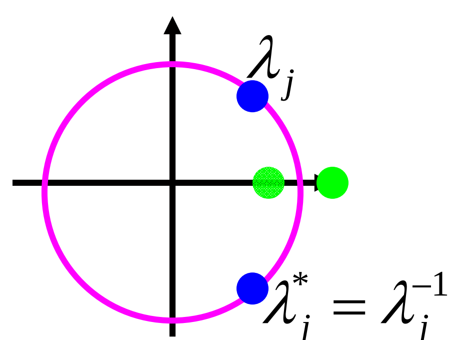
Therefore $\underline{J}\vec{v}_j$ is orthogonal to all eigenvectors with eigenvalues that are not $1/\lambda_j$. Since it cannot be orthogonal to all eigenvectors, there is at least one eigenvector with eigenvalue $1/\lambda_j$

Two dimensions: λ_j is eigenvalue

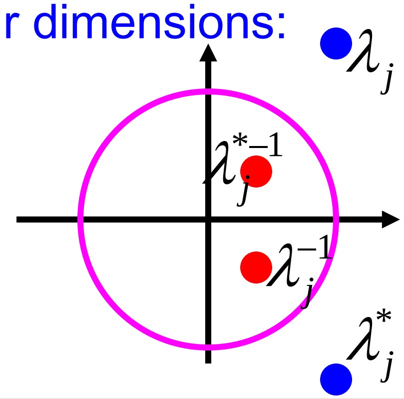
Then $1/\lambda_j$ and λ_j^* are eigenvalues

$\lambda_2 = 1/\lambda_1 = \lambda_1^* \quad \Rightarrow \quad |\lambda_j| = 1$

$\lambda_2 = 1/\lambda_1 = \lambda_2^*$



Four dimensions:



The Frenet Coordinate System

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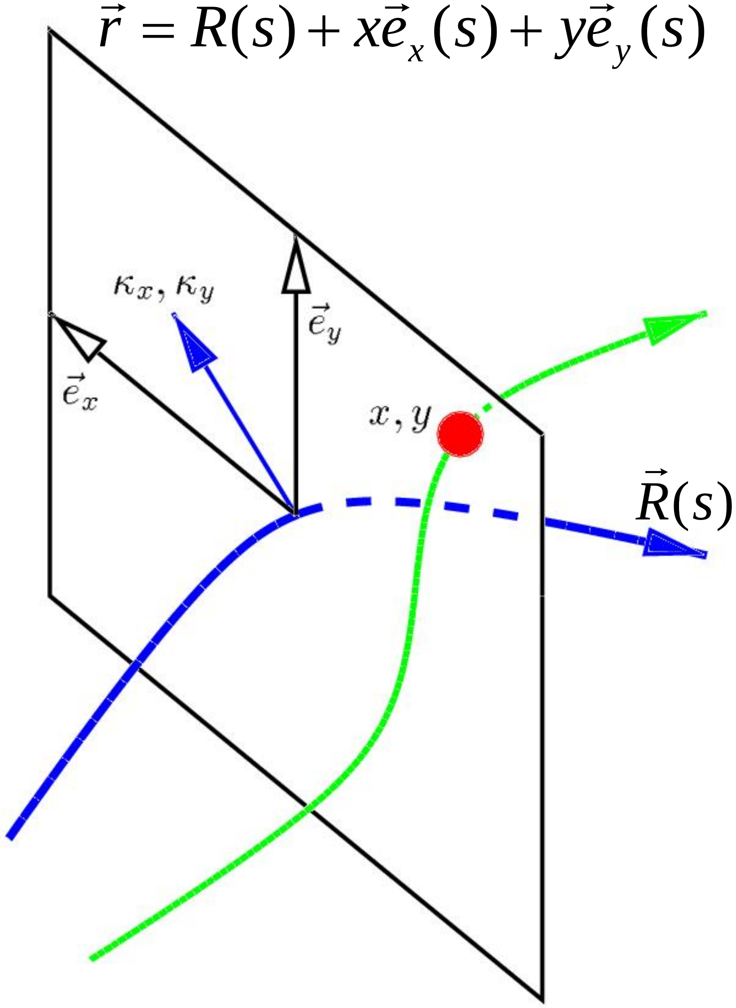
$$\vec{r} = \vec{R}(s) + x\vec{e}_x(s) + y\vec{e}_y(s)$$



$$|d\vec{R}| = ds$$

$$\vec{e}_s \equiv \tfrac{d}{ds}\vec{R}(s)$$

$$\vec{e}_\kappa \equiv -\tfrac{d}{ds}\vec{e}_s \Big/ \left|\tfrac{d}{ds}\vec{e}_s\right|$$

$$\vec{e}_b \equiv \vec{e}_s \times \vec{e}_\kappa$$

$$\tfrac{d}{ds}\vec{e}_s = -\kappa\,\vec{e}_\kappa \quad \text{with} \quad \kappa = \frac{1}{\rho}$$

$$0 = \tfrac{d}{ds}(\vec{e}_\kappa \cdot \vec{e}_s) = \vec{e}_s \cdot \tfrac{d}{ds}\vec{e}_\kappa - \kappa$$

Accumulated torsion angle T

$$\tfrac{d}{ds}\vec{e}_\kappa = \kappa\,\vec{e}_s + T'\vec{e}_b$$

$$0 = \tfrac{d}{ds}(\vec{e}_b \cdot \vec{e}_\kappa) = \vec{e}_\kappa \cdot \tfrac{d}{ds}\vec{e}_b + T'$$

$$\tfrac{d}{ds}\vec{e}_b = -T'\vec{e}_\kappa$$

$$\vec{r}\,' = (x' - yT')\vec{e}_\kappa + (y' + xT')\vec{e}_b + (1 + x\kappa)\vec{e}_s$$

The Curvi-linerar System

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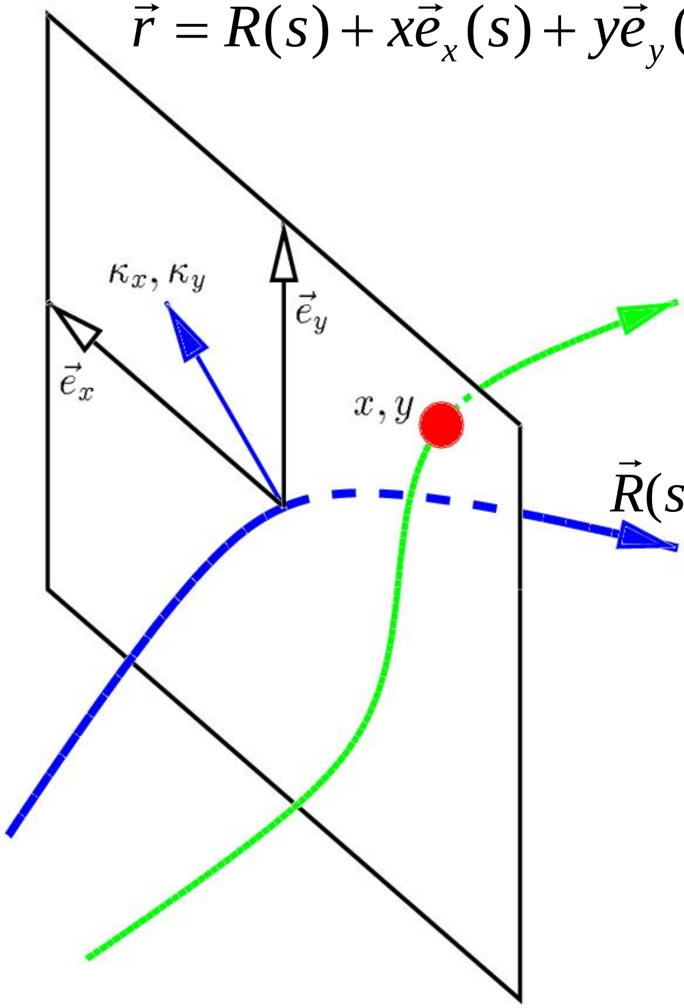
$$\vec{r} = \vec{R}(s) + x\vec{e}_x(s) + y\vec{e}_y(s)$$



$$\vec{e}_x \equiv \vec{e}_\kappa \cos(T) - \vec{e}_b \sin(T)$$
$$\vec{e}_y \equiv \vec{e}_\kappa \sin(T) + \vec{e}_b \cos(T)$$

$$\tfrac{d}{ds}\vec{e}_s = -\kappa_x \vec{e}_x - \kappa_y \vec{e}_y$$

$$\tfrac{d}{ds}\vec{e}_x = \kappa \cos(T)\vec{e}_s = \kappa_x \vec{e}_s$$

$$\tfrac{d}{ds}\vec{e}_y = \kappa \sin(T)\vec{e}_s = \kappa_y \vec{e}_s$$

$$\tfrac{d}{ds}\vec{r} = x'\vec{e}_\kappa + y'\vec{e}_b + (1 + x\kappa_x + y\kappa_y)\vec{e}_s$$

Phase Space ODE

$$\frac{d}{ds}\vec{r} = x'\vec{e}_x + y'\vec{e}_y + \underbrace{(1 + x\kappa_x + y\kappa_y)}_{h}\vec{e}_s$$

$$\frac{d^2}{dt^2}\vec{r} = \vec{F}$$

$$\frac{d}{ds}\vec{r} = \dot{s}^{-1}\frac{d}{dt}\vec{r} = \dot{s}^{-1}\frac{1}{m\gamma}\vec{p} = \frac{h}{p_s}\vec{p}$$

$$\frac{d}{ds}\vec{p} = (p_x' - p_s\kappa_x)\vec{e}_x + (p_y' - p_s\kappa_y)\vec{e}_y + (p_s' + \kappa_x p_x + \kappa_y p_y)\vec{e}_s$$

$$= \dot{s}^{-1}\frac{d}{dt}\vec{p} = \dot{s}^{-1}\vec{F} = \frac{m\gamma h}{p_s}\vec{F}$$

$$\begin{pmatrix} x' \\ y' \\ p_x' \\ p_y' \end{pmatrix} = \begin{pmatrix} \frac{h}{p_s}p_x \\ \frac{h}{p_s}p_y \\ \frac{m\gamma h}{p_s}F_x + p_s\kappa_x \\ \frac{m\gamma h}{p_s}F_y + p_s\kappa_y \end{pmatrix}$$

$$t' = \dot{s}^{-1} = \frac{hm\gamma}{p_s}$$

$$E = \sqrt{(pc)^2 + (mc^2)^2}$$

$$E' = \frac{d}{dp}\sqrt{(pc)^2 + (mc^2)^2}\,\frac{d}{ds}p = c^2\frac{\vec{p}}{E}\frac{d}{ds}\vec{p} = \frac{h}{p_s}\vec{p}\cdot\vec{F}$$

6 Dimensional Phase Space

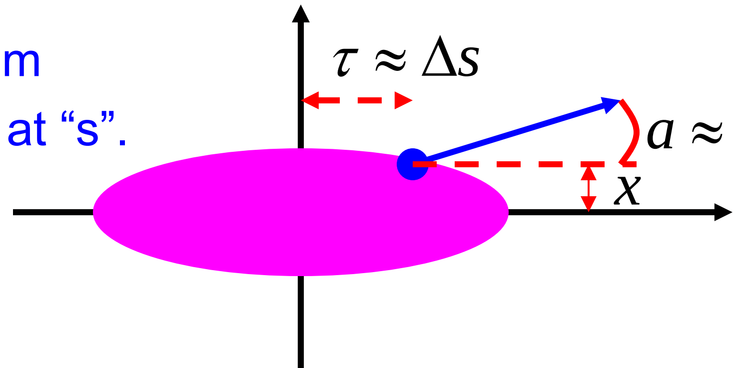
Using a reference momentum p_0 and a reference time t_0:

$$\vec{z} = (x, a, y, b, \tau, \delta)$$

$$a = \frac{p_x}{p_0}, \quad b = \frac{p_y}{p_0}, \quad \delta = \frac{E - E_0}{E_0}, \quad \tau = (t_0 - t)\frac{c^2}{v_0} = (t_0 - t)\frac{E_0}{p_0}$$

Usually p_0 is the design momentum of the beam

And t_0 is the time at which the bunch center is at “s”.



$$\left.\begin{aligned} x' &= \partial_{p_x} K \\ p_x' &= -\partial_x K \end{aligned}\right\} \Rightarrow \begin{cases} x' = \partial_a K/p_0 \,, & a' = -\partial_x K/p_0 \\ y' = \partial_b K/p_0 \,, & b' = -\partial_y K/p_0 \end{cases}$$

$$-t' = \partial_E K \quad \Rightarrow \quad \tau' = \tfrac{c^2}{v_0}\partial_\delta K/E_0 = \partial_\delta K/p_0$$

$$E' = -\partial_{-t} K \quad \Rightarrow \quad \delta' = -\tfrac{1}{E_0}\partial_\tau K \tfrac{c^2}{v_0} = -\partial_\tau K/p_0$$

New Hamiltonian:

$$\tilde{H} = K/p_0$$

The Equation of Motion [for notes]

$$\begin{pmatrix} x' \\ y' \\ p_x' \\ p_y' \end{pmatrix} = \begin{pmatrix} \frac{h}{p_s} p_x \\ \frac{h}{p_s} p_y \\ \frac{m\gamma h}{p_s} F_x + p_s \kappa_x \\ \frac{m\gamma h}{p_s} F_y + p_s \kappa_y \end{pmatrix}$$

$$t' = \dot{s}^{-1} = \frac{hm\gamma}{p_s}$$

$$E' = \frac{h}{p_s} \vec{p} \cdot \vec{F}$$

$$a = \frac{p_x}{p_0}, \quad b = \frac{p_y}{p_0}, \quad \delta = \frac{E - E_0}{E_0}, \quad \tau = (t_0 - t)\frac{E_0}{p_0}$$

$$\begin{pmatrix} x' \\ a' \\ y' \\ b' \\ \tau' \\ \delta' \end{pmatrix} = \begin{pmatrix} h\frac{p_0}{p_s} a \\ \frac{m\gamma h}{p_s p_0} F_x + \frac{p_s}{p_0}\kappa_x \\ h\frac{p_0}{p_s} b \\ \frac{m\gamma h}{p_s p_0} F_y + \frac{p_s}{p_0}\kappa_x \\ \frac{E_0}{p_0}(\frac{m\gamma_0}{p_0} - h\frac{m\gamma}{p_s}) \\ \frac{h}{E_0 p_s} \vec{p} \cdot \vec{F} \end{pmatrix} = \begin{pmatrix} h\frac{p_0}{p_s} a \\ \frac{h}{p_s p_0} q(m\gamma E_x + p_y B_s - p_s B_y) + \frac{p_s}{p_0}\kappa_x \\ h\frac{p_0}{p_s} b \\ \frac{h}{p_s p_0} q(m\gamma E_y + p_s B_x - p_x B_s) + \frac{p_s}{p_0}\kappa_y \\ \frac{c^2}{v_0^2} - h\frac{c^2}{v_0 v_s} \\ \frac{h}{E_0 p_s} q(p_x E_x + p_y E_y + p_s E_s) \end{pmatrix}$$

The 0th Order Equation of Motion [for notes]

One expands around the reference trajectory:

Condition: The reference or design trajectory can be the path of a particle.
The particle transport is then origin preserving.

$$\vec{z}' = \vec{F}(\vec{z}, s) \quad \text{with} \quad \vec{F}(\vec{0}, s) = \vec{0} \quad \Rightarrow \quad \vec{M}(\vec{0}, s) = \vec{0}$$

0th order: $\vec{E} = \vec{E}_0 + \vec{E}_1 + \ldots$

$$\kappa_x = \frac{q}{p_0} B_{y0} - \frac{q}{p_0 v_0} E_{x0}$$

Note: q/p_0 called magnetic rigidity
$q/(p_0 v_0)$ called electric rigidity

$$\kappa_y = -\frac{q}{p_0} B_{x0} - \frac{q}{p_0 v_0} E_{y0}$$

$E_{s0} = 0$ (No acceleration on the design trajectory)

If the energy E changes on the reference trajectory then
$\delta = E - E_0$ does not stay 0. One then works with p_x, p_y, and E rather than with a, b, and δ.

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The Linear Equation of Motion [for notes]

$$p = \frac{1}{c}\sqrt{E^2 - (mc^2)^2} \quad \Rightarrow \quad \frac{dp}{dE} = \frac{E}{pc^2} = \frac{1}{v}$$

$$p_s = \sqrt{p^2 - p_x^2 - p_y^2} = p_0[1 + \beta_0^{-2}\delta] + O^2$$

$$v_s = \frac{v}{p} p_s = \frac{c^2}{E} p_s = v_0[1 + \beta_0^{-2}\delta] + O^2 =$$

1st order: $x' = h\frac{p_0}{p_s} a \quad =_1 a\,, \quad y' = h\frac{p_0}{p_s} b \quad =_1 b$

$$\tau' = \frac{c^2}{v_0^2} - h\frac{c^2}{v_0 v_s} =_1 -\beta_0^{-2}(x\kappa_x + y\kappa_y) + \frac{1}{\gamma_0^2}\beta_0^{-4}\delta$$

$$a' = \frac{h}{p_s p_0} q(m\gamma E_x + p_y B_s - p_s B_y) + \frac{p_s}{p_0}\kappa_x$$

$$= -(x\kappa_x + y\kappa_y)\kappa_x + \frac{q}{p_0}(\frac{1}{v_0}E_{x1} + bB_{s0} - B_{y1}) + \delta\beta_0^{-2}[\kappa_x - \gamma_0^{-1}\beta_0^{-2}\frac{qE_{x0}}{mc^2}]$$

$$b' = \ldots$$

$$\delta' = \frac{h}{E_0 p_s} q(p_x E_x + p_y E_y + p_s E_s) = \frac{1}{E_0} q(aE_{x0} + bE_{y0} + E_{s1})$$

Simplified Equation of Motion

Only bend in the horizontal plane: $\kappa_y = 0\,, \quad \kappa_x = \kappa = 1/\rho$

Only magnetic fields: $\vec{E} = 0$

Mid-plane symmetry: $B_x(x,y,s) = -B_x(x,-y,s)\,, \quad B_y(x,y,s) = B_y(x,-y,s)$

$$a' = -x\kappa^2 - \tfrac{q}{p_0}\partial_x B_y x + \delta\beta_0^{-2}\kappa \quad \Rightarrow \quad x'' = -x\,(\kappa^2 + k) + \delta\beta_0^{-2}\kappa$$

$$b' = \tfrac{q}{p_0}\partial_y B_x y \quad \Rightarrow \quad y'' = k\,y$$

$$\tau' = -x\,\beta_0^{-2}\kappa + \tfrac{1}{\gamma_0^2}\beta_0^{-4}\delta$$

$$\delta' = 0$$

Hamiltonian:

$$H = \tfrac{1}{2}a^2 + \tfrac{1}{2}b^2 + \tfrac{1}{2}k(x^2 - y^2) + \tfrac{1}{2}\kappa^2 x^2 - \beta_0^{-2}\kappa\,x\delta + \tfrac{1}{2}\tfrac{1}{\gamma_0^2}\beta_0^{-4}\delta^2$$