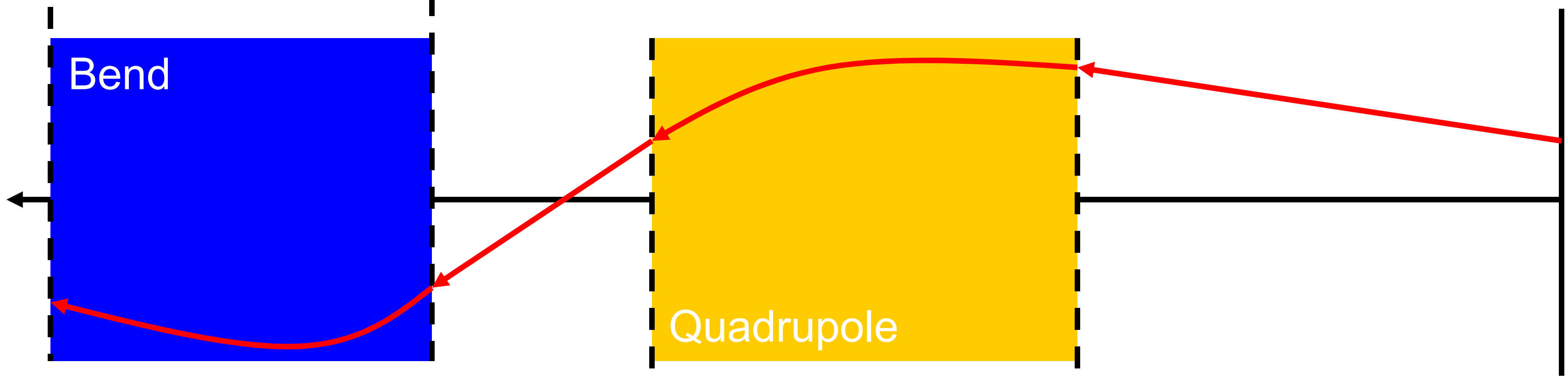
Matrix Solutions

Linear equation of motion: $\vec{z}\,' = \underline{F}(s)\vec{z}$

Matrix solution of the starting condition $\vec{z}(0) = \vec{z}_0$

$$\vec{z} = \underline{M}_{\text{bend}}(L_4)\underline{M}_{\text{drift}}(L_3)\underline{M}_{\text{quad}}(L_2)\underline{M}_{\text{drift}}(L_1)\vec{z}_0$$

$$\vec{z} = \underline{M}_{\text{drift}}(L_3)\underline{M}_{\text{quad}}(L_2)\underline{M}_{\text{drift}}(L_1)\vec{z}_0$$



$$\vec{z} = \underline{M}_{\text{drift}}(L_1)\vec{z}_0 \qquad \vec{z}_0$$

$$\vec{z} = \underline{M}_{\text{quad}}(L_2)\underline{M}_{\text{drift}}(L_1)\vec{z}_0$$

The Drift

$$\begin{pmatrix} x' \\ a' \\ y' \\ b' \\ \tau' \\ \delta' \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ b \\ 0 \\ \frac{1}{\gamma_0^2}\beta_0^{-4}\delta \\ 0 \end{pmatrix}$$

Note that in nonlinear expansion $x' \neq a$ so that the drift does not have a linear transport map even though $x(s) = x_0 + x_0' s$ is completely linear.

$$\frac{1}{\gamma^2} << 1 \quad \Rightarrow \quad \begin{pmatrix} x \\ a \\ y \\ b \\ \tau \\ \delta \end{pmatrix} = \begin{pmatrix} x_0 + s a_0 \\ a \\ y_0 + s b_0 \\ b_0 \\ \tau_0 \\ \delta_0 \end{pmatrix} = \begin{pmatrix} \begin{matrix} 1 & s \\ 0 & 1 \end{matrix} & \underline{0} & \underline{0} \\ \underline{0} & \begin{matrix} 1 & s \\ 0 & 1 \end{matrix} & \underline{0} \\ \underline{0} & \underline{0} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{pmatrix} \vec{z}_0$$

The Dipole Equation of Motion

$$\frac{1}{\gamma^2} << 1 \quad \Rightarrow \quad \begin{aligned} x'' &= -x\,\kappa^2 + \delta\kappa \\ y'' &= 0 \\ \tau' &= -x\,\kappa \end{aligned}$$

Homogeneous solution:

$$x_H'' = -x_H\,\kappa^2 \quad \Rightarrow \quad x_H = A\cos(\kappa\,s) + B\sin(\kappa\,s)$$ (natural ring focusing)

Variation of constants:

$$x = A(s)\cos(\kappa\,s) + B(s)\sin(\kappa\,s)$$

$$x' = -A\kappa\sin(\kappa\,s) + B\kappa\cos(\kappa\,s) + \underbrace{A'\cos(\kappa\,s) + B'\sin(\kappa\,s)}_{\equiv 0}$$

$$x'' = -\kappa^2 x \underbrace{- A'\kappa\sin(\kappa\,s) + B'\kappa\cos(\kappa\,s)}_{=\delta\kappa} = -\kappa^2 x + \delta\kappa$$

$$\begin{pmatrix} \cos(\kappa\,s) & \sin(\kappa\,s) \\ -\sin(\kappa\,s) & \cos(\kappa\,s) \end{pmatrix} \begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} 0 \\ \delta\,\beta_0^{-2} \end{pmatrix}$$

The Dipole

$$\begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} \cos(\kappa s) & -\sin(\kappa s) \\ \sin(\kappa s) & \cos(\kappa s) \end{pmatrix} \begin{pmatrix} 0 \\ \delta \end{pmatrix}$$

$$\begin{pmatrix} A \\ B \end{pmatrix} = \delta\, \kappa^{-1} \begin{pmatrix} \cos(\kappa s) \\ \sin(\kappa s) \end{pmatrix} + \begin{pmatrix} A_H \\ B_H \end{pmatrix} \quad \text{with} \quad x = A\cos(\kappa s) + B\sin(\kappa s)$$

$$\tau' = -x\kappa$$

$$\underline{M} = \begin{pmatrix} \begin{matrix} \cos(\kappa s) & \frac{1}{\kappa}\sin(\kappa s) \\ -\kappa\sin(\kappa s) & \cos(\kappa s) \end{matrix} & \underline{0} & \begin{matrix} 0 & \kappa^{-1}[1-\cos(\kappa s)] \\ 0 & \sin(\kappa s) \end{matrix} \\ \underline{0} & \begin{matrix} 1 & s \\ 0 & 1 \end{matrix} & \underline{0} \\ \begin{matrix} -\sin(\kappa s) & \kappa^{-1}[\cos(\kappa s)-1] \\ 0 & 0 \end{matrix} & \underline{0} & \begin{matrix} 1 & \kappa^{-1}[\sin(\kappa s)-s\kappa] \\ 0 & 1 \end{matrix} \end{pmatrix}$$

Time of Flight from Symplecticity

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$$\underline{M} = \begin{pmatrix} \underline{M}_4 & \vec{0} & \vec{D} \\ \vec{T}^T & 1 & M_{56} \\ \vec{0}^T & 0 & 1 \end{pmatrix}$$ is in SU(6) and therefore $\underline{M}\,\underline{J}\,\underline{M}^T = \underline{J}$

$$\begin{pmatrix} \underline{M}_4 \underline{J}_4 & -\vec{D} & \vec{0} \\ \vec{T}^T \underline{J}_4 & -M_{56} & 1 \\ \vec{0}^T & -1 & 0 \end{pmatrix} \begin{pmatrix} \underline{M}_4^T & \vec{T} & \vec{0} \\ \vec{0}^T & 1 & 0 \\ \vec{D}^T & M_{56} & 1 \end{pmatrix} = \begin{pmatrix} \underline{J}_4 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \underline{M}_4 \underline{J}_4 \underline{M}_4^T & \underline{M}_4 \underline{J}_4 \vec{T} - \vec{D} & \vec{0} \\ \vec{T}^T \underline{J}_4 \underline{M}_4^T + \vec{D}^T & 0 & 1 \\ \vec{0}^T & -1 & 0 \end{pmatrix} = \begin{pmatrix} \underline{J}_4 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\boxed{\vec{T} = -\underline{J}_4 \underline{M}_4^{-1} \vec{D}}$$

It is sufficient to compute the 4D map $\underline{M}_4$, the Dispersion $\vec{D}$ and the time of flight term M_{56}

The Quadrupole

$$x'' = -x\,k$$

$$y'' = \quad y\,k$$

$$\underline{M}_4 = \begin{pmatrix} \cos(\sqrt{k}\,s) & \frac{1}{\sqrt{k}}\sin(\sqrt{k}\,s) & \multicolumn{2}{c}{\underline{0}} \\ -\sqrt{k}\sin(\sqrt{k}\,s) & \cos(\sqrt{k}\,s) & & \\ & & \cosh(\sqrt{k}\,s) & \frac{1}{\sqrt{k}}\sinh(\sqrt{k}\,s) \\ \multicolumn{2}{c}{\underline{0}} & \sqrt{k}\sinh(\sqrt{k}\,s) & \cosh(\sqrt{k}\,s) \end{pmatrix}$$

As for a drift:

$$\vec{D} = \vec{0} \quad \Rightarrow \quad \vec{T} = \vec{0}$$

$$M_{56} = 0$$

For k<0 one has to take into account that

$$\cos(\sqrt{k}\,s) = \cosh(\sqrt{|k|}\,s)\,, \quad \sin(\sqrt{k}\,s) = i\sinh(\sqrt{|k|}\,s)$$

$$\cosh(\sqrt{k}\,s) = \cos(\sqrt{|k|}\,s)\,, \quad \sinh(\sqrt{k}\,s) = i\sin(\sqrt{|k|}\,s)$$

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The Combined Function Bend

$$x'' = -x\,(\underbrace{\kappa^2 + k}_{K}) + \delta\kappa$$

$$y'' = y\,k \quad , \quad \tau' = -\kappa\,x$$

$$\underline{M}_\tau = \begin{pmatrix} 1 & M_{56} \\ 0 & 1 \end{pmatrix}$$

$$\underline{M}_6 = \begin{pmatrix} \underline{M}_x & \underline{0} & \vec{0}\,\vec{D} \\ \underline{0} & \underline{M}_y & \underline{0} \\ \underline{0} & \underline{0} & \underline{M}_\tau \end{pmatrix}$$

$$M_{56} = \frac{\kappa^2}{K\sqrt{K}}[\sin(\sqrt{K}s) - \sqrt{K}s]$$

$$\underline{M}_x = \begin{pmatrix} \cos(\sqrt{K}\,s) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}\,s) \\ -\sqrt{K}\sin(\sqrt{K}\,s) & \cos(\sqrt{K}\,s) \end{pmatrix}$$

$$\underline{M}_y = \begin{pmatrix} \cosh(\sqrt{k}\,s) & \frac{1}{\sqrt{k}}\sinh(\sqrt{k}\,s) \\ \sqrt{k}\sinh(\sqrt{k}\,s) & \cosh(\sqrt{k}\,s) \end{pmatrix}$$

$$\vec{D} = \begin{pmatrix} \frac{\kappa}{K}[1-\cos(\sqrt{K}s)] \\ \frac{\kappa}{\sqrt{K}}\sin(\sqrt{K}s) \end{pmatrix}$$

Options:

- For k>0:
  focusing in x, defocusing in y.
- For k<0, K<0:
  defocusing in x, focusing in y.
- For k<0, K>0:
  weak focusing in both planes.

The Thin Lens Approximation

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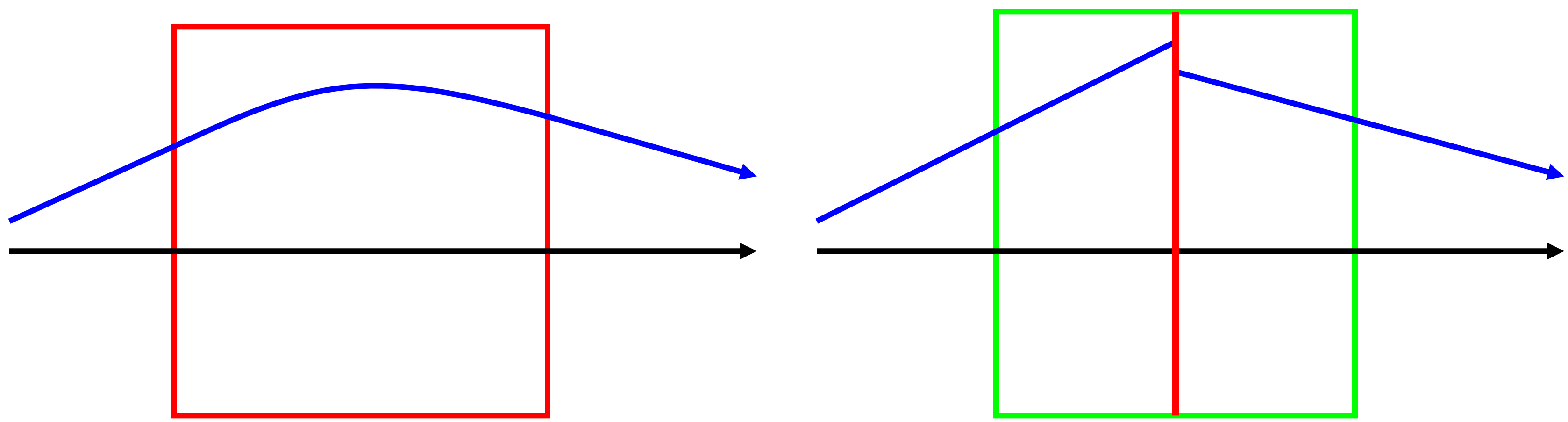
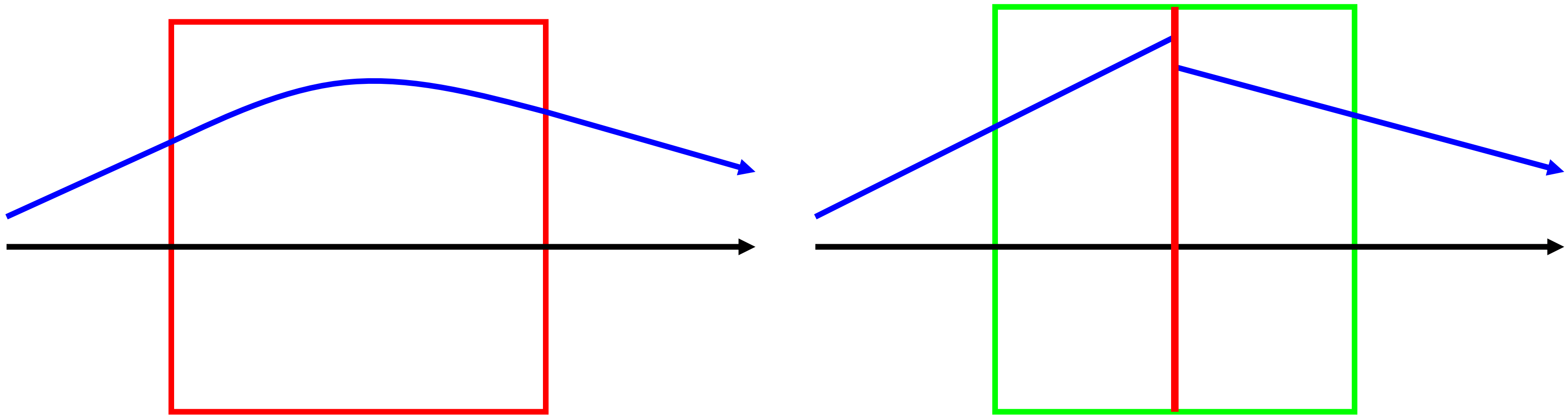


$$\vec{z}(s) = \underline{M}(s)\,\vec{z}_0 = \underline{D}(\tfrac{s}{2})\underline{D}^{-1}(\tfrac{s}{2})\underline{M}(s)\underline{D}^{-1}(\tfrac{s}{2})\underline{D}(\tfrac{s}{2})\vec{z}_0$$

Drift: $\underline{M}^{\text{thin}}_{\text{drift}}(s) = \underline{D}^{-1}(\tfrac{s}{2})\underline{D}(s)\underline{D}^{-1}(\tfrac{s}{2}) = \underline{1}$

The Thin Lens Quadrupole



$$\underline{M}^{\text{thin}}_{\text{quad,x}}(s) = \begin{pmatrix} 1 & -\frac{s}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\sqrt{k}s) & \frac{1}{\sqrt{k}}\sin(\sqrt{k}s) \\ -\sqrt{k}\sin(\sqrt{k}s) & \cos(\sqrt{k}s) \end{pmatrix} \begin{pmatrix} 1 & -\frac{s}{2} \\ 0 & 1 \end{pmatrix}$$

$$\approx \begin{pmatrix} 1 & -\frac{s}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & s \\ -ks & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{s}{2} \\ 0 & 1 \end{pmatrix} \approx \begin{pmatrix} 1 & -\frac{s}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{s}{2} \\ -ks & 1+\frac{ks^2}{2} \end{pmatrix}$$

Weak magnet limit: $\sqrt{k}s << 1$

$$\underline{M}^{\text{thin}}_{\text{quad,x}}(s) \approx \begin{pmatrix} 1 & 0 \\ -ks & 1 \end{pmatrix}$$

The Thin Lens Dipole

$$\underline{M} = \begin{pmatrix} \begin{matrix} \cos(\kappa s) & \frac{1}{\kappa}\sin(\kappa s) \\ -\kappa\sin(\kappa s) & \cos(\kappa s) \end{matrix} & \underline{0} & \begin{matrix} 0 & \kappa^{-1}[1-\cos(\kappa s)] \\ 0 & \sin(\kappa s) \end{matrix} \\ \underline{0} & \begin{matrix} 1 & s \\ 0 & 1 \end{matrix} & \underline{0} \\ \begin{matrix} -\sin(\kappa s) & \kappa^{-1}[\cos(\kappa s)-1] \\ 0 & 0 \end{matrix} & \underline{0} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{pmatrix}$$

Weak magnet limit: $\kappa s << 1$

$$\underline{M}^{\text{thin}}_{\text{bend},x\tau}(s) = \underline{D}(-\tfrac{s}{2})\underline{M}_{\text{bend},x\tau}\underline{D}(-\tfrac{s}{2}) \approx \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\kappa^2 s & 1 & 0 & \kappa s \\ -\kappa s & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Thin Combined Function Bend

$$\underline{M}_6 = \begin{pmatrix} \underline{M}_x & \underline{0} & \vec{0}\,\vec{D} \\ \underline{0} & \underline{M}_y & \underline{0} \\ \underline{0} & \underline{0} & \underline{1} \end{pmatrix}$$

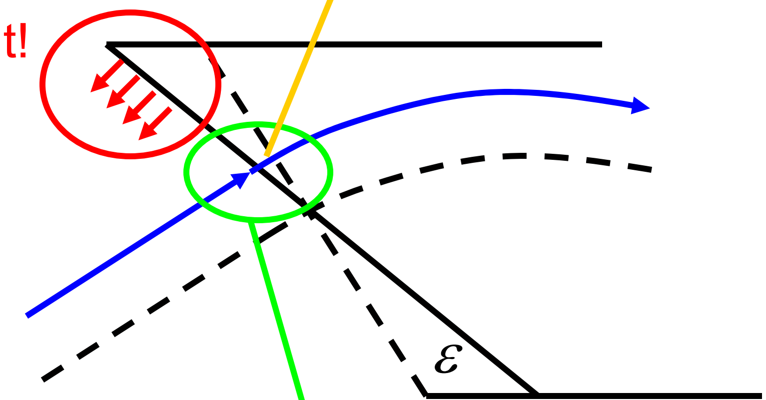
Weak magnet limit: $\kappa s << 1$

$$\underline{M}_x = \begin{pmatrix} \cos(\sqrt{K}\,s) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}\,s) \\ -\sqrt{K}\sin(\sqrt{K}\,s) & \cos(\sqrt{K}\,s) \end{pmatrix} \qquad \underline{M}_x^{\text{thin}} = \begin{pmatrix} 1 & 0 \\ -K\,s & 1 \end{pmatrix}$$

$$\underline{M}_y = \begin{pmatrix} \cosh(\sqrt{k}\,s) & \frac{1}{\sqrt{k}}\sinh(\sqrt{k}\,s) \\ \sqrt{k}\sinh(\sqrt{k}\,s) & \cosh(\sqrt{k}\,s) \end{pmatrix} \longrightarrow \underline{M}_y^{\text{thin}} = \begin{pmatrix} 1 & 0 \\ k\,s & 1 \end{pmatrix}$$

$$\vec{D} = \begin{pmatrix} \frac{\kappa}{K}[1-\cos(\sqrt{K}s)] \\ \frac{\kappa}{\sqrt{K}}\sin(\sqrt{K}s) \end{pmatrix} \qquad \vec{D} = \begin{pmatrix} 0 \\ \kappa\,s \end{pmatrix}$$

Edge Focusing



Horizontal focusing with $\Delta x' = -x\frac{\tan(\varepsilon)}{\rho}$

$$B_x = \partial_y B_s\Big|_{y=0} y\tan(\varepsilon) = \partial_s B_y\Big|_{y=0} y\tan(\varepsilon)$$

$$y'' = \tfrac{q}{p}\partial_s B_y\Big|_{y=0} y\tan(\varepsilon)$$

$$\Delta y' = \int y''ds = \tfrac{q}{p}B_y y\tan(\varepsilon) = y\frac{\tan(\varepsilon)}{\rho}$$

Quadrupole effect with

$$kl = \frac{\tan(\varepsilon)}{\rho}$$

$$\vec{z} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{\tan(\varepsilon)}{\rho} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{\tan(\varepsilon)}{\rho} & 1 \end{pmatrix}\vec{z}_0$$

The Rectangular Bend

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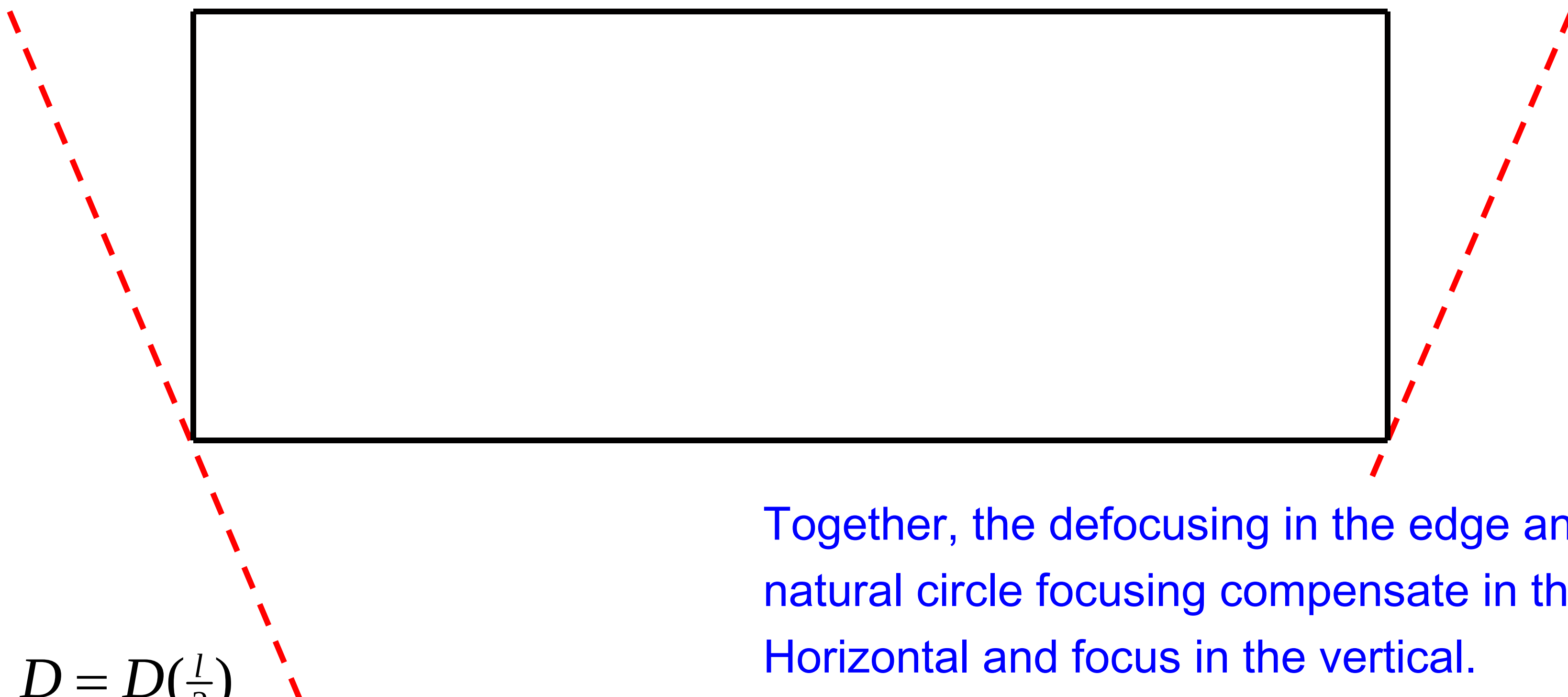
Together, the defocusing in the edge and the natural circle focusing compensate in the Horizontal and focus in the vertical.

$$D = D(\tfrac{l}{2})$$

$$\underline{M}^{\text{thin}}_{\text{rbend,x}} \approx D^{-1}\begin{pmatrix} 1 & 0 \\ \frac{1}{2}\kappa^2 s & 1 \end{pmatrix} D \begin{pmatrix} 1 & 0 \\ -\kappa^2 s & 1 \end{pmatrix} D \begin{pmatrix} 1 & 0 \\ \frac{1}{2}\kappa^2 s & 1 \end{pmatrix} D^{-1}$$

$$= D^{-1}\begin{pmatrix} 1 & 0 \\ \frac{1}{2}\kappa^2 s & 1 \end{pmatrix}\begin{pmatrix} 1 & s \\ -\kappa^2 s & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{1}{2}\kappa^2 s & 1 \end{pmatrix} D^{-1} = D^{-1}\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} D^{-1} = I$$

The Rectangular Bend



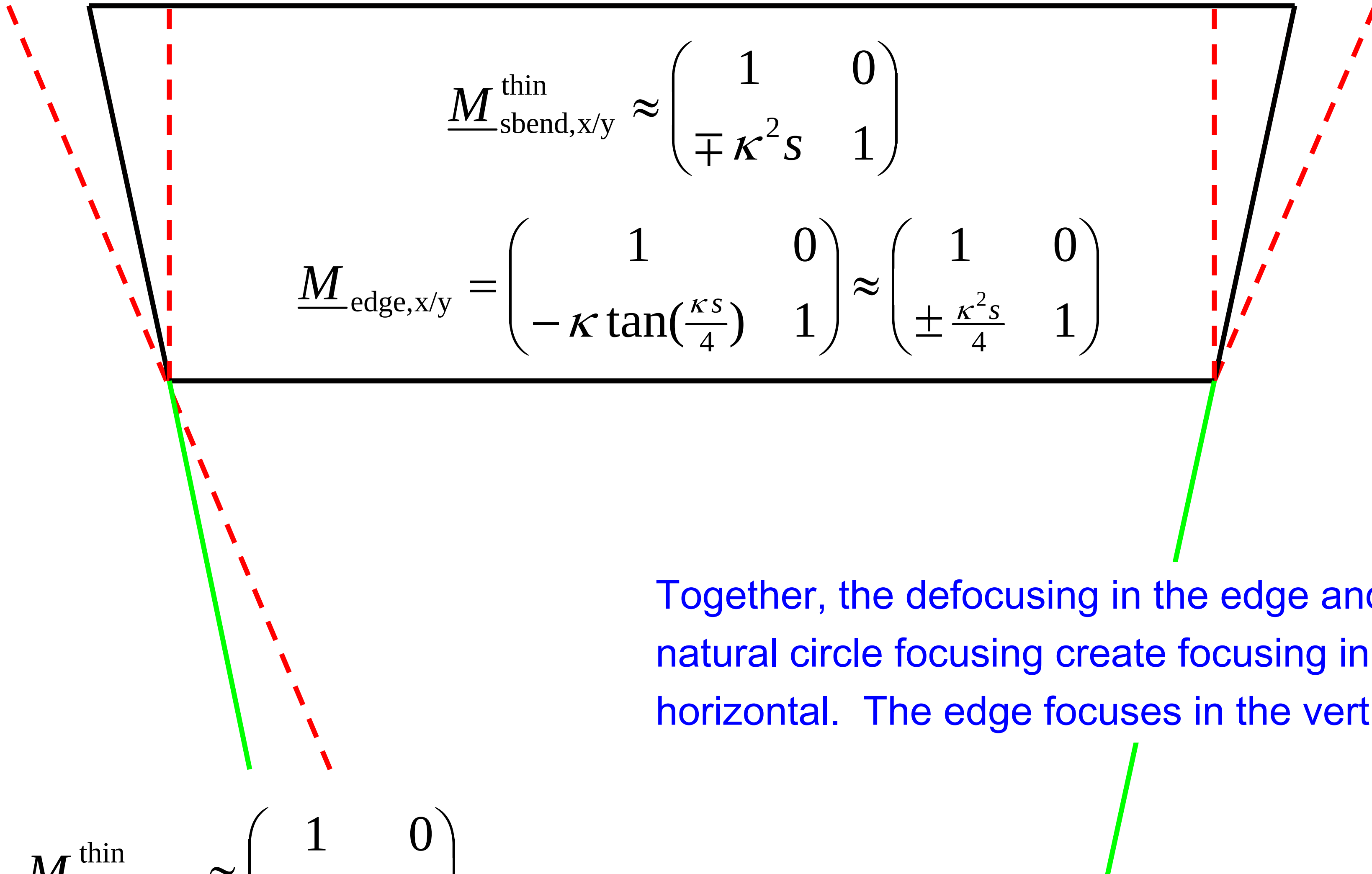
Together, the defocusing in the edge and the natural circle focusing compensate in the Horizontal and focus in the vertical.

$$D = D(\tfrac{l}{2})$$

$$\underline{M}^{\text{thin}}_{\text{rbend,x}} \approx D^{-1}\begin{pmatrix} 1 & 0 \\ \frac{1}{2}\kappa^2 s & 1 \end{pmatrix} D \begin{pmatrix} 1 & 0 \\ -\kappa^2 s & 1 \end{pmatrix} D \begin{pmatrix} 1 & 0 \\ \frac{1}{2}\kappa^2 s & 1 \end{pmatrix} D^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\underline{M}^{\text{thin}}_{\text{rbend,y}} \approx D^{-1}\begin{pmatrix} 1 & 0 \\ -\frac{1}{2}\kappa^2 s & 1 \end{pmatrix} D \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} D \begin{pmatrix} 1 & 0 \\ -\frac{1}{2}\kappa^2 s & 1 \end{pmatrix} D^{-1} = \begin{pmatrix} 1 & 0 \\ -\kappa^2 s & 1 \end{pmatrix}$$

Weak Focusing with Edges



$$\underline{M}_{\text{sbend},x/y}^{\text{thin}} \approx \begin{pmatrix} 1 & 0 \\ \mp\kappa^2 s & 1 \end{pmatrix}$$

$$\underline{M}_{\text{edge},x/y} = \begin{pmatrix} 1 & 0 \\ -\kappa\tan(\frac{\kappa s}{4}) & 1 \end{pmatrix} \approx \begin{pmatrix} 1 & 0 \\ \pm\frac{\kappa^2 s}{4} & 1 \end{pmatrix}$$

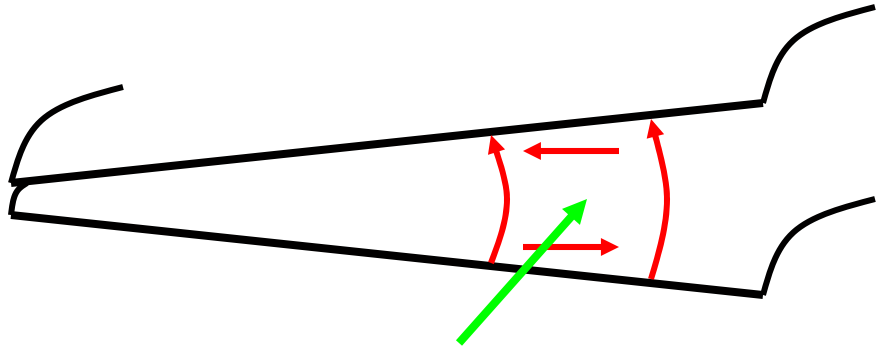
Together, the defocusing in the edge and the natural circle focusing create focusing in the horizontal. The edge focuses in the vertical.

$$\underline{M}_{\text{fbend},x/y}^{\text{thin}} \approx \begin{pmatrix} 1 & 0 \\ -\frac{\kappa^2 s}{2} & 1 \end{pmatrix}$$

Cyclotrons with edge focusing

- The isocyclotron with constant

$$\omega_z = \frac{q}{m_0\gamma(E)} B_z(r(E))$$



Up to 600MeV but
this vertically defocuses the beam.
Edge focusing is therefore used.

Variation of Constants

$$\vec{z}\,' = \vec{f}(\vec{z},s)$$

$$\vec{z}\,' = \underline{L}(s)\vec{z} + \Delta\vec{f}(\vec{z},s)$$

Field errors, nonlinear fields, etc can lead to $\Delta\vec{f}(\vec{z},s)$

$$\vec{z}_H' = \underline{L}(s)\vec{z}_H \quad \Rightarrow \quad \vec{z}_H(s) = \underline{M}(s)\vec{z}_{H0} \quad \text{with} \quad \underline{M}'(s)\vec{a} = \underline{L}(s)\underline{M}(s)\vec{a}$$

$$\vec{z}(s) = \underline{M}(s)\vec{a}(s) \quad \Rightarrow \quad \vec{z}\,'(s) = \underline{M}'(s)\vec{a} + \underline{M}(s)\vec{a}\,'(s) = \underline{L}(s)\vec{z} + \Delta\vec{f}(\vec{z},s)$$

$$\vec{a}(s) = \vec{z}_0 + \int_0^s \underline{M}^{-1}(\hat{s})\Delta\vec{f}(\vec{z}(\hat{s}),\hat{s})\,d\hat{s}$$

$$\vec{z}(s) = \underline{M}(s)\left\{\vec{z}_0 + \int_0^s \underline{M}^{-1}(\hat{s})\Delta\vec{f}(\vec{z}(\hat{s}),\hat{s})\,d\hat{s}\right\}$$

$$= \vec{z}_H(s) + \int_0^s \underline{M}(s-\hat{s})\Delta\vec{f}(\vec{z}(\hat{s}),\hat{s})\,d\hat{s}$$

Perturbations are propagated from s to s'

Aberration Correction

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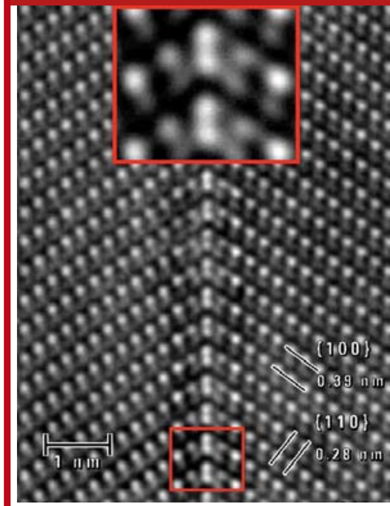
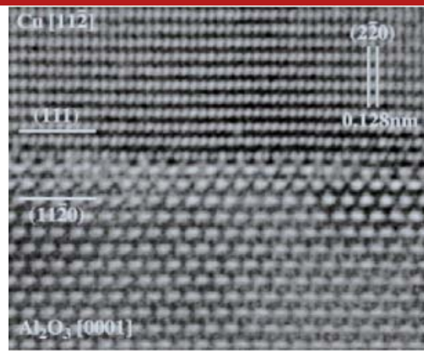
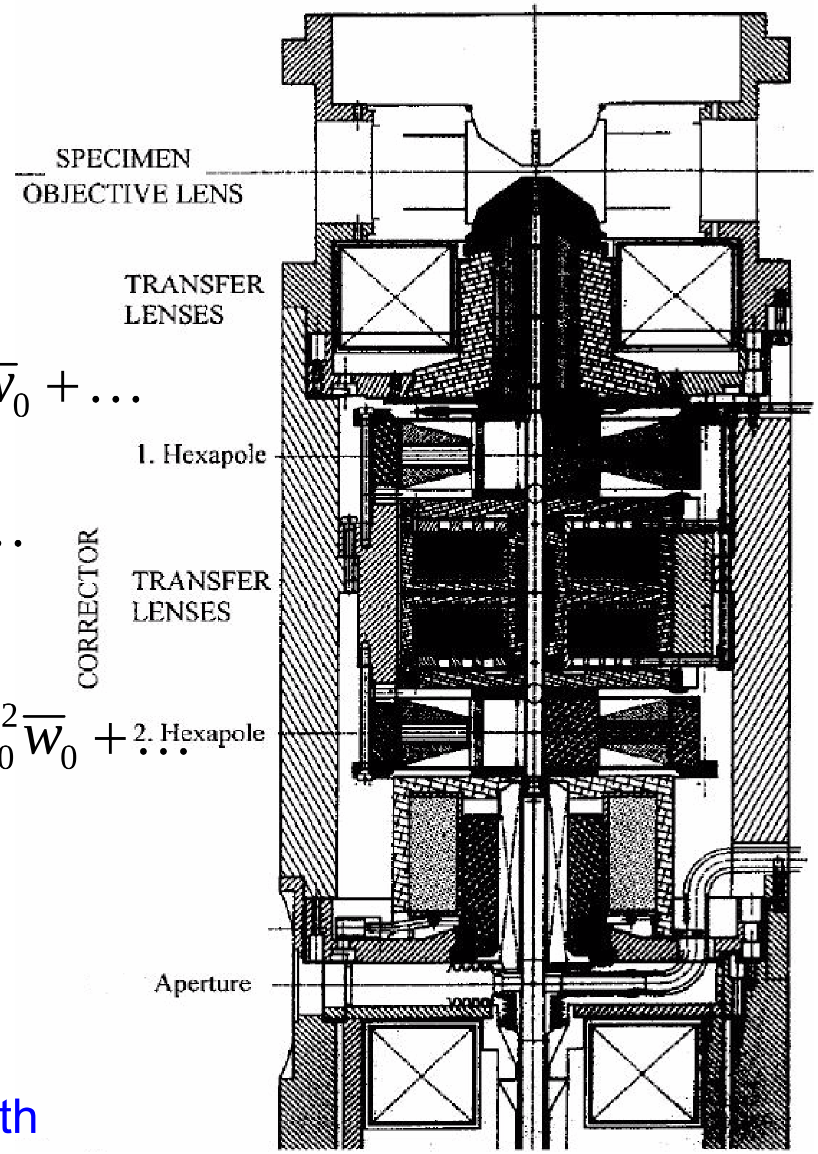




$$w_2(s) = w_H(s) + C(s) w_0^2 \overline{w}_0 + \dots$$

$$w_2(s) = w_H(s) + A(s) \overline{w}_0^2 + B(s) w_0^2 \overline{w}_0 + \dots$$

$$w_2(s) = w_H(s) + A(s) \overline{w}_0^2 + 2B(s) w_0^2 \overline{w}_0 + \dots$$



2B cancels C !

Quadratic in sextupole strength

Linear in solenoid strength