Symplectic Flows → H

For every symplectic transport map there is a Hamilton function

The flow or transport map: $\vec{z}(s) = \vec{M}(s, \vec{z}_0)$

Force vector: $\vec{h}(\vec{z}, s) = -\underline{J}\left[\frac{d}{ds}\vec{M}(s, \vec{z}_0)\right]_{\vec{z}_0 = \vec{M}^{-1}(\vec{z}, s)}$

Since then: $\frac{d}{ds}\vec{z} = \underline{J}\vec{h}(\vec{z}, s)$

There is a Hamilton function H with: $\vec{h} = \vec{\partial} H$

If and only if: $\partial_{z_j} h_i = \partial_{z_i} h_j \quad \Rightarrow \quad \underline{h} = \underline{h}^T$

$$\underline{M}\,\underline{J}\,\underline{M}^T = \underline{J} \quad \Rightarrow \quad \begin{cases} \frac{d}{ds}\underline{M}\,\underline{J}\,\underline{M}^T = -\underline{M}\,\underline{J}\,\frac{d}{ds}\underline{M}^T \\ \underline{M}^{-1} = -\underline{J}\,\underline{M}^T\,\underline{J} \end{cases}$$

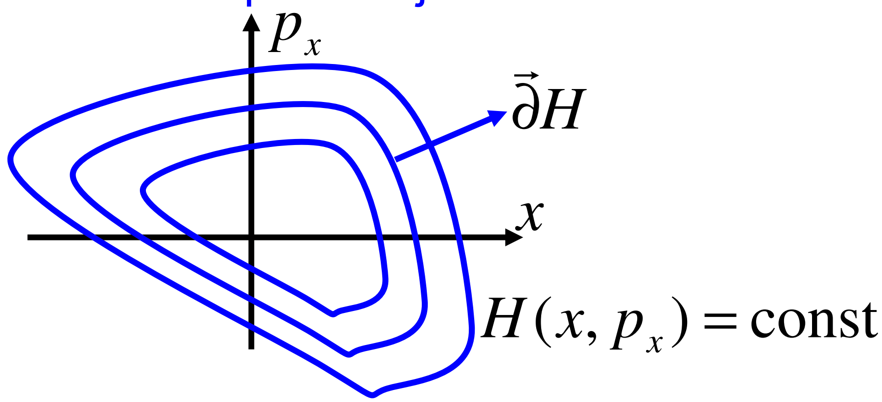
$$\vec{h} \circ \vec{M} = -\underline{J}\,\frac{d}{ds}\vec{M}$$

$$\underline{h}(\vec{M})\underline{M} = -\underline{J}\,\frac{d}{ds}\underline{M}$$

$$\underline{h}(\vec{M}) = -\underline{J}\,\frac{d}{ds}\underline{M}\,\underline{M}^{-1} = \underline{J}\,\frac{d}{ds}\underline{M}\,\underline{J}\,\underline{M}^T\,\underline{J} = -\underline{J}\,\underline{M}\,\underline{J}\,\frac{d}{ds}\underline{M}^T\,\underline{J} = \underline{M}^{-T}\,\frac{d}{ds}\underline{M}^T\,\underline{J} = \underline{h}^T$$

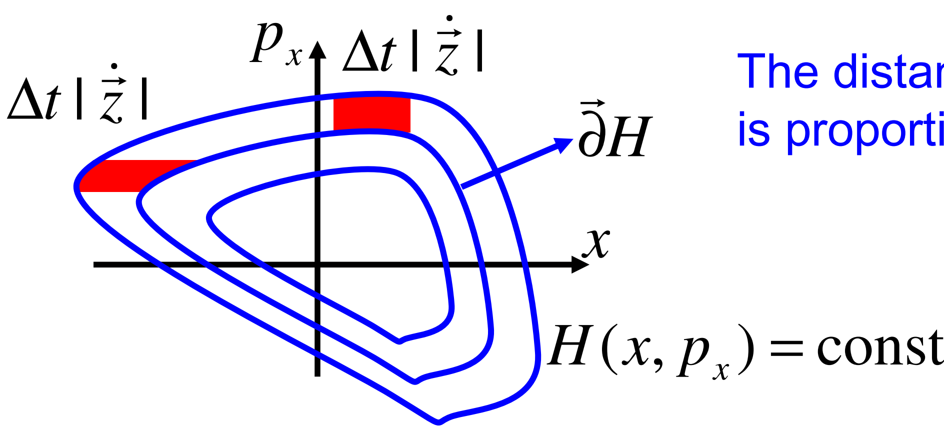
Phase space density in 2D

- Phase space trajectories move on surfaces of constant energy



$$\tfrac{d}{ds}\vec{z} = \underline{J}\,\vec{\partial} H \quad \Rightarrow \quad \tfrac{d}{ds}\vec{z} \perp \vec{\partial} H$$

- A phase space volume does not change when it is transported by Hamiltonian motion.

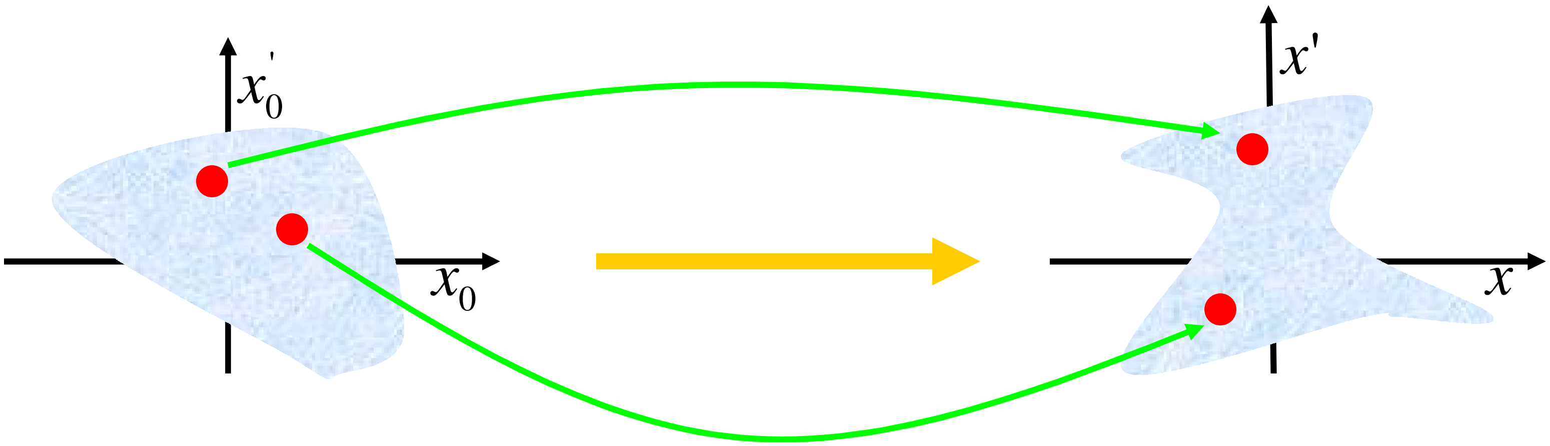


The distance d of lines with equal energy is proportional to $1/|\vec{\partial} H| \propto |\dot{\vec{z}}|^{-1}$

$$d * \Delta t\,|\dot{\vec{z}}| = \text{const}$$

Lioville's Theorem

- A phase space volume does not change when it is transported by Hamiltonian motion. $\vec{z}(s) = \underline{M}(s) \cdot \vec{z}_0$ with $\det[\underline{M}(s)] = +1$



Volume = $V = \iint_V d^n\vec{z} = \iint_{V_0} \left|\frac{\partial\vec{z}}{\partial\vec{z}_0}\right| d^n\vec{z}_0 = \iint_{V_0} |\underline{M}| d^n\vec{z}_0 = \iint_{V_0} d^n\vec{z}_0 = V_0$

Hamiltonian Motion → $V = V_0$

But Hamiltonian requires symplecticity, which is much more than just $\det[\underline{M}(s)] = +1$

Generating Functions

The motion of particles can be represented by Generating Functions

Each flow or transport map: $\vec{z}(s) = \vec{M}(s, \vec{z}_0)$

With a Jacobi Matrix : $M_{ij} = \partial_{z_{0j}} M_i$ or $\underline{M} = (\vec{\partial}_0 \vec{M}^T)^T$

That is Symplectic: $\underline{M}\,\underline{J}\,\underline{M}^T = \underline{J}$

Can be represented by a Generating Function:

$$F_1(\vec{q}, \vec{q}_0, s) \quad \text{with} \quad \vec{p} = -\vec{\partial}_q F_1 \quad , \quad \vec{p}_0 = \vec{\partial}_{q_0} F_1$$

$$F_2(\vec{p}, \vec{q}_0, s) \quad \text{with} \quad \vec{q} = \vec{\partial}_p F_2 \quad , \quad \vec{p}_0 = \vec{\partial}_{q_0} F_2$$

$$F_3(\vec{q}, \vec{p}_0, s) \quad \text{with} \quad \vec{p} = -\vec{\partial}_q F_3 \quad , \quad \vec{q}_0 = -\vec{\partial}_{p_0} F_3$$

$$F_4(\vec{p}, \vec{p}_0, s) \quad \text{with} \quad \vec{q} = \vec{\partial}_q F_4 \quad , \quad \vec{q}_0 = -\vec{\partial}_{p_0} F_4$$

6-dimensional motion needs only one function ! But to obtain the transport map this has to be inverted.