F → SP(2N) [for notes]

Generating Functions produce symplectic tranport maps

$$F_1(\vec{q},\vec{q}_0,s) \quad \text{with} \quad \vec{p} = -\vec{\partial}_q F_1(\vec{q},\vec{q}_0,s) \quad , \quad \vec{p}_0 = \vec{\partial}_{q_0} F_1(\vec{q},\vec{q}_0,s)$$

$$\left.\begin{aligned} \vec{z} &= \begin{pmatrix} \vec{q} \\ \vec{p} \end{pmatrix} = \begin{pmatrix} \vec{q} \\ -\vec{\partial}_q F_1(\vec{q},\vec{q}_0,s) \end{pmatrix} = \vec{f}(\vec{Q},s) \\ \vec{z}_0 &= \begin{pmatrix} \vec{q}_0 \\ \vec{p}_0 \end{pmatrix} = \begin{pmatrix} \vec{q}_0 \\ \vec{\partial}_{q_0} F_1(\vec{q},\vec{q}_0,s) \end{pmatrix} = \vec{g}(\vec{Q},s) \end{aligned}\right\}$$

$$\vec{z} = \vec{f}(\vec{g}^{-1}(\vec{z}_0,s),s)$$

$$\vec{M} = \vec{f} \circ \vec{g}^{-1}$$

(function concatenation)

Jacobi matrix of concatenated functions:

$$\vec{C}(\vec{z}_0) = \vec{A} \circ \vec{B}(\vec{z}_0)$$

$$C_{ij} = \partial_j C_i = \sum_k \partial_{z_{0j}} B_k(\vec{z}_0) \left[\partial_{z_k} A_i(\vec{z})\right]_{\vec{z}=\vec{B}(\vec{z}_0)} \quad \Rightarrow \quad \underline{C} = \underline{A}(\vec{B})\underline{B}$$

$$\vec{M} \circ \vec{g} = \vec{f} \quad \Rightarrow \quad \underline{M}(\vec{g}) = \underline{F}\,\underline{G}^{-1}$$

F → SP(2N) [for notes]

$$\vec{f}(\vec{Q},s)=\begin{pmatrix}\vec{q}\\ -\vec{\partial}_q F_1(\vec{q},\vec{q}_0,s)\end{pmatrix} \Rightarrow F=\begin{pmatrix}\underline{1} & \underline{0}\\ -\vec{\partial}_q\vec{\partial}_q^T F_1 & -\vec{\partial}_q\vec{\partial}_{q_0}^T F_1\end{pmatrix}$$

$$\vec{g}(\vec{Q},s)=\begin{pmatrix}\vec{q}_0\\ \vec{\partial}_{q_0} F_1(\vec{q},\vec{q}_0,s)\end{pmatrix} \Rightarrow G=\begin{pmatrix}\underline{0} & \underline{1}\\ \vec{\partial}_{q_0}\vec{\partial}_q^T F_1 & \vec{\partial}_{q_0}\vec{\partial}_{q_0}^T F_1\end{pmatrix}$$

$$F=\begin{pmatrix}1 & 0\\ -F_{11} & -F_{12}\end{pmatrix},\quad G=\begin{pmatrix}0 & 1\\ F_{21} & F_{22}\end{pmatrix} \Rightarrow G^{-1}=\begin{pmatrix}-F_{21}^{-1}F_{22} & F_{21}^{-1}\\ 1 & 0\end{pmatrix}$$

$$\underline{M}(\vec{g})=FG^{-1}=\begin{pmatrix}-F_{21}^{-1}F_{22} & F_{21}^{-1}\\ F_{11}F_{21}^{-1}F_{22}-F_{12} & -F_{11}F_{21}^{-1}\end{pmatrix}$$

$$\underline{M}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\underline{M}^T$$

The map from a generating function is symplectic.

$$=\begin{pmatrix}-F_{21}^{-1} & -F_{21}^{-1}F_{22}\\ F_{11}F_{21}^{-1} & F_{11}F_{21}^{-1}F_{22}-F_{12}\end{pmatrix}\begin{pmatrix}-F_{22}F_{12}^{-1} & F_{22}F_{12}^{-1}F_{11}-F_{21}\\ F_{12}^{-1} & -F_{12}^{-1}F_{11}\end{pmatrix}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$

SP(2N) → F [for notes]

Symplectic tranport maps have a Generating Functions

$$\vec{z} = \vec{M}(\vec{z}_0)$$

$$\begin{pmatrix} \vec{q} \\ \vec{q}_0 \end{pmatrix} = \begin{pmatrix} \vec{M}_1(\vec{z}_0) \\ \vec{q}_0 \end{pmatrix} = \vec{l}\,(\vec{z}_0)\,, \quad \begin{pmatrix} \vec{p}_0 \\ \vec{p} \end{pmatrix} = \begin{pmatrix} \vec{p}_0 \\ \vec{M}_2(\vec{z}_0) \end{pmatrix} = \vec{h}(\vec{z}_0) = \underline{J}\left[\vec{\partial} F_1(\vec{q}, \vec{q}_0)\right]_{\vec{l}\,(\vec{z}_0)}$$

$$\vec{\partial} F_1 = -\underline{J}\vec{h} \circ \vec{l}^{\,-1} = \vec{F}$$

For F_1 to exist it is necessary and sufficient that $\partial_i F_j = \partial_j F_i \quad \Rightarrow \quad \underline{F} = \underline{F}^T$

$$-\underline{J}\vec{h} = \vec{F} \circ \vec{l} \quad \Rightarrow \quad -\underline{J}\,\underline{h} = \underline{F}(\vec{l}\,)\,\underline{l}$$

Is $\underline{J}\ \underline{h}\ \underline{l}^{-1}$ symmetric ? Yes since:

$$\underline{J}\,\underline{h}\,\underline{l}^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \vec{\partial}^T_{q_0} \vec{M}_2 & \vec{\partial}^T_{p_0} \vec{M}_2 \end{pmatrix} \begin{pmatrix} \vec{\partial}^T_{q_0} \vec{M}_1 & \vec{\partial}^T_{p_0} \vec{M}_1 \\ 1 & 0 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} M_{21} & M_{22} \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ M_{12}^{-1} & -M_{12}^{-1} M_{11} \end{pmatrix} = \begin{pmatrix} M_{22} M_{12}^{-1} & M_{21} - M_{22} M_{12}^{-1} M_{11} \\ M_{12}^{-1} & M_{12}^{-1} M_{11} \end{pmatrix}$$

SP(2N) $\rightarrow$ F [for notes]

$$\underline{J}\,\underline{h}\,\underline{l}^{-1} = \begin{pmatrix} M_{22}M_{12}^{-1} & M_{21} - M_{22}M_{12}^{-1}M_{11} \\ M_{12}^{-1} & M_{12}^{-1}M_{11} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$$\vec{M}(\vec{z}_0) = \begin{pmatrix} \vec{M}_1(\vec{q}_0, \vec{p}_0) \\ \vec{M}_2(\vec{q}_0, \vec{p}_0) \end{pmatrix}, \quad \underline{M} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

$$\underline{M}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\underline{M}^T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \begin{pmatrix} -M_{12} & M_{11} \\ -M_{22} & M_{21} \end{pmatrix}\begin{pmatrix} M_{11}^T & M_{21}^T \\ M_{12}^T & M_{22}^T \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$M_{12}M_{11}^T = M_{11}M_{12}^T \qquad \Rightarrow \quad (M_{12}^{-1}M_{11})^T = [M_{12}^{-1}M_{11}M_{12}^T]M_{12}^{-T} = M_{12}^{-1}M_{11}$$

$$M_{21}M_{22}^T = M_{22}M_{21}^T$$

$$M_{11}M_{22}^T - M_{12}M_{21}^T = 1$$

$$M_{22}M_{11}^T - M_{21}M_{12}^T = 1$$

$$D = D^T$$

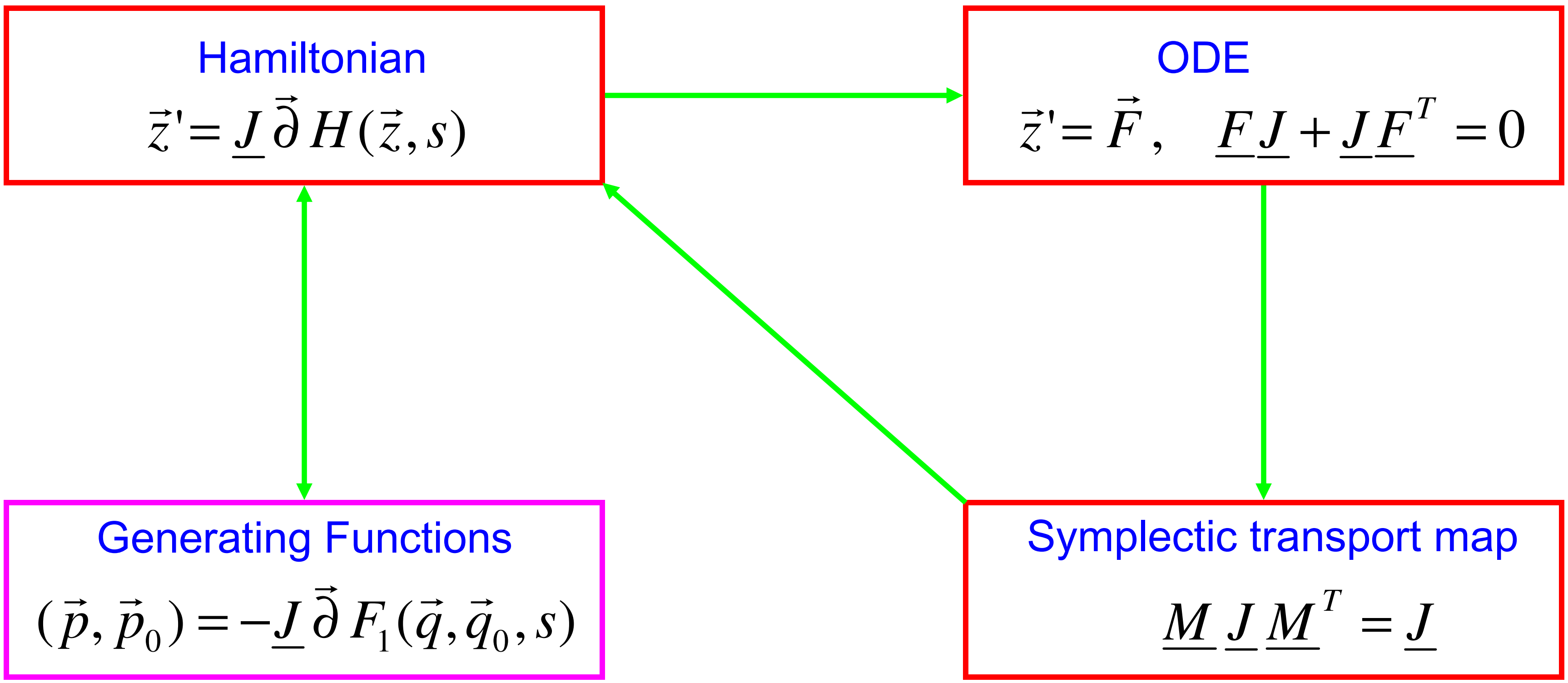
$$A = A^T$$

$$(M_{22}M_{12}^{-1})^T = [M_{22}M_{11}^T M_{12}^{-T} - M_{21}]M_{22}^T = M_{22}[M_{12}^{-1}M_{11}M_{22}^T - M_{21}^T] = M_{22}M_{12}^{-1}$$

$$M_{21} - M_{22}M_{12}^{-1}M_{11} = M_{21} - M_{22}M_{11}M_{12}^{-T} = M_{12}^{-T} \longrightarrow B = C^T$$

Symplectic Representations

CHESS & LEPP



[from notes]

Advantages of Symplecticity

- Determinant of the transfer matrix of linear motion is 1:

$$\vec{z}(s) = \underline{M}(s) \cdot \vec{z}_0 \quad \text{with} \quad \det(\underline{M}(s)) = +1$$

- One function suffices to compute the total nonlinear transfer map:

$$F_1(\vec{q}, \vec{q}_0, s) \quad \text{with} \quad \vec{p} = -\vec{\partial}_q F_1(\vec{q}, \vec{q}_0, s) \quad , \quad \vec{p}_0 = \vec{\partial}_{q_0} F_1(\vec{q}, \vec{q}_0, s)$$

$$\left.\begin{aligned} \vec{z} &= \begin{pmatrix} \vec{q} \\ \vec{p} \end{pmatrix} = \begin{pmatrix} \vec{q} \\ -\vec{\partial}_q F_1(\vec{q}, \vec{q}_0, s) \end{pmatrix} = \vec{f}(\vec{Q}, s) \\ \vec{z}_0 &= \begin{pmatrix} \vec{q}_0 \\ \vec{p}_0 \end{pmatrix} = \begin{pmatrix} \vec{q}_0 \\ \vec{\partial}_{q_0} F_1(\vec{q}, \vec{q}_0, s) \end{pmatrix} = \vec{g}(\vec{Q}, s) \end{aligned}\right\} \quad \begin{aligned} \vec{z} &= \vec{f}(\vec{g}^{-1}(\vec{z}_0, s), s) \\ \vec{M} &= \vec{f} \circ \vec{g}^{-1} \end{aligned}$$

- Therefore Taylor Expansion coefficients of the transport map are related.
- Computer codes can numerically approximate $\vec{M}(s, \vec{z}_0)$ with exact symplectic symmetry.
- Liouville's Theorem for phase space densities holds.

Eigenvalues of a Symplectic Matrix

For matrices with real coefficients:

If there is an eigenvector and eigenvalue: $\underline{M}\vec{v}_i = \lambda_i \vec{v}_i$

then the complex conjugates are also eigenvector and eigenvalue: $\underline{M}\vec{v}_i^* = \lambda_i^* \vec{v}_i^*$

For symplectic matrices:

If there are eigenvectors and eigenvalues: $\underline{M}\vec{v}_i = \lambda_i \vec{v}_i$ with $\underline{J} = \underline{M}^T \underline{J}\,\underline{M}$

then $\vec{v}_i^T \underline{J}\vec{v}_j = \vec{v}_i^T \underline{M}^T \underline{J}\,\underline{M}\vec{v}_j = \lambda_i \lambda_j \vec{v}_i^T \underline{J}\vec{v}_j \quad \Rightarrow \quad \vec{v}_i^T \underline{J}\vec{v}_j (\lambda_i \lambda_j - 1) = 0$

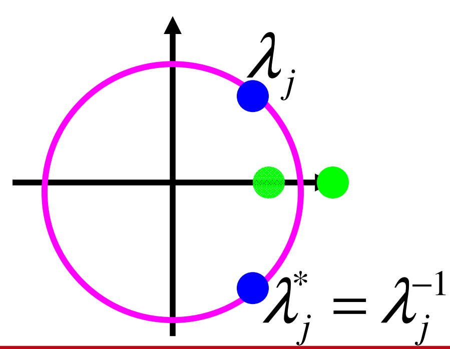
Therefore $\underline{J}\vec{v}_j$ is orthogonal to all eigenvectors with eigenvalues that are not $1/\lambda_j$. Since it cannot be orthogonal to all eigenvectors, there is at least one eigenvector with eigenvalue $1/\lambda_j$

Two dimensions: λ_j is eigenvalue

Then $1/\lambda_j$ and λ_j^* are eigenvalues

$\underline{\lambda_2 = 1/\lambda_1 = \lambda_1^*} \quad \Rightarrow \quad |\lambda_j| = 1$

$\underline{\lambda_2 = 1/\lambda_1 = \lambda_2^*}$



Four dimensions: