Simplified Equation of Motion

Only magnetic fields: $\vec{E} = 0$

Mid-plane symmetry: $B_x(x,y,s) = -B_x(x,-y,s)\,,\quad B_y(x,y,s) = B_y(x,-y,s)$

Linearization in : $B_x(x,y,s) = \frac{p_0}{q} k y\,,\quad B_y(x,y,s) = \frac{p_0}{q}\frac{1}{\rho} + \frac{p_0}{q} k x$

Highly relativistic : $\frac{p-p_0}{p_0} \to \frac{E-E_0}{E_0}\quad,\quad \frac{v-v_0}{v_0} \to 0$

$$\underline{x'} = \frac{p_x}{p_z} = \frac{p_x}{\sqrt{(p_0+dp)^2 - p_x^2 - p_y^2}} =_1 \frac{p_x}{p_0} \underline{= a} \qquad \Rightarrow \qquad \underline{y' = b}$$

$$\underline{a'} = \frac{(\vec{p}\,' + p_s \kappa \vec{e}_x)_x}{p_0} = \frac{h}{p_0 v_s}\frac{d}{dt} p_x + \frac{p_s \kappa}{p_0} = -\frac{1+x\kappa}{p_0 v_s} q v_s B_y + \frac{p_s \kappa}{p_0}$$

$$=_1 -(1+x\kappa)(\kappa + kx) + (1+\delta)\kappa \underline{= -x(\kappa^2 + k) + \delta\kappa} \qquad \Rightarrow \qquad \underline{b' = -ky}$$

$$\tau' = \frac{d(t-t_0)}{ds}\frac{E_0}{p_0} = \left(\frac{1}{v_0} - \frac{h}{v_s}\right)\frac{E_0}{p_0} =_1 -x\kappa \qquad , \qquad \underline{\delta' = 0}$$

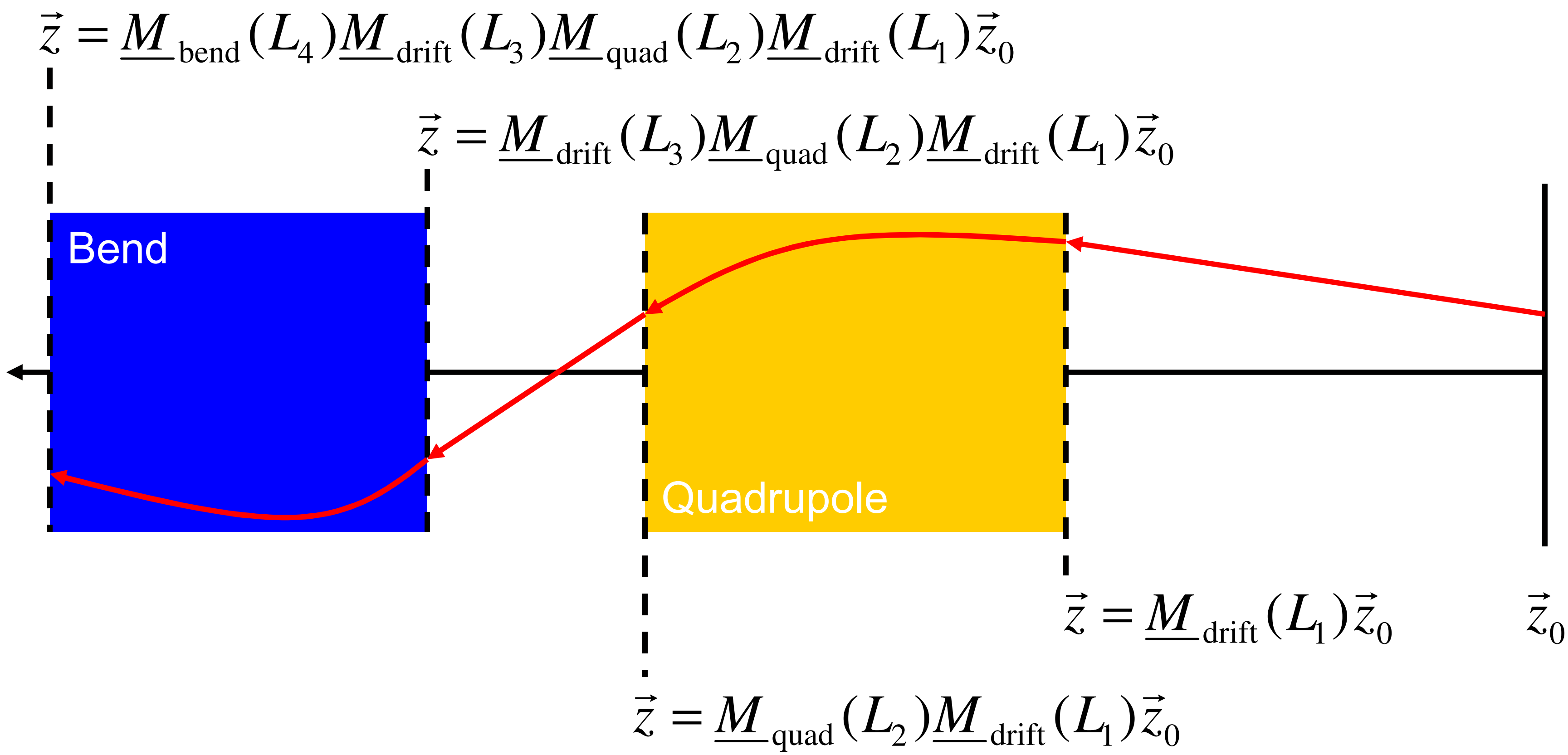
Hamiltonian:

$$H = \tfrac{1}{2}a^2 + \tfrac{1}{2}b^2 + \tfrac{1}{2}k(x^2 - y^2) + \tfrac{1}{2}\kappa^2 x^2 - \kappa x \delta$$

Matrix Solutions

Linear equation of motion: $\vec{z}\,' = \underline{F}(s)\vec{z}$

Matrix solution of the starting condition $\vec{z}(0) = \vec{z}_0$



The Drift

$$\begin{pmatrix} x' \\ a' \\ y' \\ b' \\ \tau' \\ \delta' \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ b \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Note that in nonlinear expansion $x' \neq a$ so that the drift does not have a linear transport map even though $x(s) = x_0 + x_0' s$ is completely linear.

$$\begin{pmatrix} x \\ a \\ y \\ b \\ \tau \\ \delta \end{pmatrix} = \begin{pmatrix} x_0 + s a_0 \\ a \\ y_0 + s b_0 \\ b_0 \\ \tau_0 \\ \delta_0 \end{pmatrix} = \begin{pmatrix} \begin{matrix} 1 & s \\ 0 & 1 \end{matrix} & \underline{0} & \underline{0} \\ \underline{0} & \begin{matrix} 1 & s \\ 0 & 1 \end{matrix} & \underline{0} \\ \underline{0} & \underline{0} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{pmatrix} \vec{z}_0$$

The Dipole Equation of Motion

$$x'' = -x\,\kappa^2 + \delta\kappa$$

$$y'' = 0$$

$$\tau' = -x\,\kappa$$

Homogeneous solution:

$$x_H'' = -x_H\,\kappa^2 \quad \Rightarrow \quad x_H = A\cos(\kappa s) + B\sin(\kappa s)$$ (natural ring focusing)

Variation of constants:

$$x = A(s)\cos(\kappa s) + B(s)\sin(\kappa s)$$

$$x' = -A\kappa\sin(\kappa s) + B\kappa\cos(\kappa s) + \underbrace{A'\cos(\kappa s) + B'\sin(\kappa s)}_{\equiv 0}$$

$$x'' = -\kappa^2 x \underbrace{- A'\kappa\sin(\kappa s) + B'\kappa\cos(\kappa s)}_{=\delta\kappa} = -\kappa^2 x + \delta\kappa$$

$$\begin{pmatrix} \cos(\kappa s) & \sin(\kappa s) \\ -\sin(\kappa s) & \cos(\kappa s) \end{pmatrix} \begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} 0 \\ \delta \end{pmatrix}$$

The Dipole

$$\begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} \cos(\kappa s) & -\sin(\kappa s) \\ \sin(\kappa s) & \cos(\kappa s) \end{pmatrix} \begin{pmatrix} 0 \\ \delta \end{pmatrix}$$

$$\begin{pmatrix} A \\ B \end{pmatrix} = \delta\,\kappa^{-1} \begin{pmatrix} \cos(\kappa s) \\ \sin(\kappa s) \end{pmatrix} + \begin{pmatrix} A_H \\ B_H \end{pmatrix} \quad \text{with} \quad x = A\cos(\kappa s) + B\sin(\kappa s)$$

$$\tau' = -x\kappa$$

$$\underline{M} = \begin{pmatrix} \begin{matrix} \cos(\kappa s) & \frac{1}{\kappa}\sin(\kappa s) \\ -\kappa\sin(\kappa s) & \cos(\kappa s) \end{matrix} & \underline{0} & \begin{matrix} 0 & \kappa^{-1}[1-\cos(\kappa s)] \\ 0 & \sin(\kappa s) \end{matrix} \\ \underline{0} & \begin{matrix} 1 & s \\ 0 & 1 \end{matrix} & \underline{0} \\ \begin{matrix} -\sin(\kappa s) & \kappa^{-1}[\cos(\kappa s)-1] \\ 0 & 0 \end{matrix} & \underline{0} & \begin{matrix} 1 & \kappa^{-1}[\sin(\kappa s)-s\kappa] \\ 0 & 1 \end{matrix} \end{pmatrix}$$

Time of Flight from Symplecticity

$$\underline{M} = \begin{pmatrix} \underline{M}_4 & \vec{0} & \vec{D} \\ \vec{T}^T & 1 & M_{56} \\ \vec{0}^T & 0 & 1 \end{pmatrix}$$

is in SU(6) and therefore $\underline{M}\,\underline{J}\,\underline{M}^T = \underline{J}$

$$\begin{pmatrix} \underline{M}_4 \underline{J}_4 & -\vec{D} & \vec{0} \\ \vec{T}^T \underline{J}_4 & -M_{56} & 1 \\ \vec{0}^T & -1 & 0 \end{pmatrix} \begin{pmatrix} \underline{M}_4^T & \vec{T} & \vec{0} \\ \vec{0}^T & 1 & 0 \\ \vec{D}^T & M_{56} & 1 \end{pmatrix} = \begin{pmatrix} \underline{J}_4 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \underline{M}_4 \underline{J}_4 \underline{M}_4^T & \underline{M}_4 \underline{J}_4 \vec{T} - \vec{D} & \vec{0} \\ \vec{T}^T \underline{J}_4 \underline{M}_4^T + \vec{D}^T & 0 & 1 \\ \vec{0}^T & -1 & 0 \end{pmatrix} = \begin{pmatrix} \underline{J}_4 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\vec{T} = -\underline{J}_4 \underline{M}_4^{-1} \vec{D}$$

It is sufficient to compute the 4D map $\underline{M}_4$, the Dispersion $\vec{D}$ and the time of flight term M_{56}

The Quadrupole (Homework)

$$x'' = -x\,k$$

$$y'' = \quad y\,k$$

$$\underline{M}_4 = \begin{pmatrix} \cos(\sqrt{k}\,s) & \frac{1}{\sqrt{k}}\sin(\sqrt{k}\,s) & \multicolumn{2}{c}{\underline{0}} \\ -\sqrt{k}\sin(\sqrt{k}\,s) & \cos(\sqrt{k}\,s) & & \\ \multicolumn{2}{c}{\underline{0}} & \cosh(\sqrt{k}\,s) & \frac{1}{\sqrt{k}}\sinh(\sqrt{k}\,s) \\ & & \sqrt{k}\sinh(\sqrt{k}\,s) & \cosh(\sqrt{k}\,s) \end{pmatrix}$$

As for a drift:

$$\vec{D} = \vec{0} \quad \Rightarrow \quad \vec{T} = \vec{0}$$

$$M_{56} = 0$$

For k<0 one has to take into account that

$$\cos(\sqrt{k}\,s) = \cosh(\sqrt{|k|}\,s)\,, \quad \sin(\sqrt{k}\,s) = i\sinh(\sqrt{|k|}\,s)$$

$$\cosh(\sqrt{k}\,s) = \cos(\sqrt{|k|}\,s)\,, \quad \sinh(\sqrt{k}\,s) = i\sin(\sqrt{|k|}\,s)$$

The Combined Function Bend (Homework)

$$x'' = -x\,(\underbrace{\kappa^2 + k}_{K}) + \delta\kappa$$

$$y'' = y\,k \quad , \quad \tau' = -\kappa\, x$$

$$\underline{M}_6 = \begin{pmatrix} \underline{M}_x & \underline{0} & \vec{0}\,\vec{D} \\ \underline{0} & \underline{M}_y & \underline{0} \\ \underline{T} & \underline{0} & \underline{M}_\tau \end{pmatrix}$$

$$\underline{M}_\tau = \begin{pmatrix} 1 & M_{56} \\ 0 & 1 \end{pmatrix}$$

$$M_{56} = \frac{\kappa^2}{K\sqrt{K}}[\sin(\sqrt{K}s) - \sqrt{K}s]$$

$\underline{T}$ from symplecticity

$$\underline{M}_x = \begin{pmatrix} \cos(\sqrt{K}\,s) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}\,s) \\ -\sqrt{K}\sin(\sqrt{K}\,s) & \cos(\sqrt{K}\,s) \end{pmatrix}$$

$$\underline{M}_y = \begin{pmatrix} \cosh(\sqrt{k}\,s) & \frac{1}{\sqrt{k}}\sinh(\sqrt{k}\,s) \\ \sqrt{k}\sinh(\sqrt{k}\,s) & \cosh(\sqrt{k}\,s) \end{pmatrix}$$

$$\vec{D} = \begin{pmatrix} \frac{\kappa}{K}[1 - \cos(\sqrt{K}s)] \\ \frac{\kappa}{\sqrt{K}}\sin(\sqrt{K}s) \end{pmatrix}$$

Options:

- For k>0:
  focusing in x, defocusing in y.
- For k<0, K<0:
  defocusing in x, focusing in y.
- For k<0, K>0:
  weak focusing in both planes.