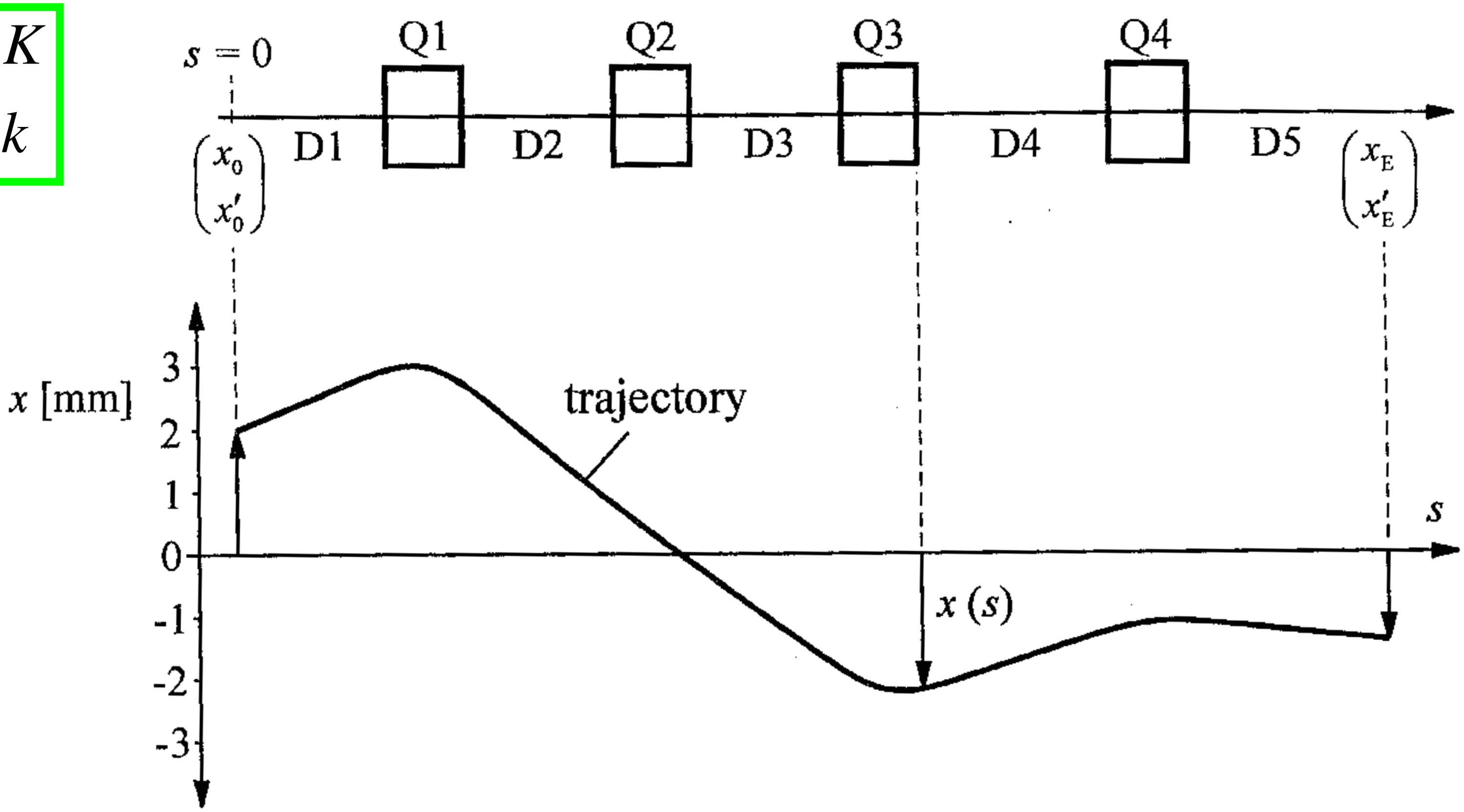
Beta Function and Betatron Phase

$$x'' = -x\,K$$
$$y'' = \quad y\,k$$



$$x(s) = M_{11}(s)x_0 + M_{12}(s)x_0'$$
$$x(s) = \sqrt{2J\beta(s)}\,\sin(\psi(s)+\phi_0)$$

Twiss Parameters

$$x'' = -k\,x$$

$$x(s) = \sqrt{2J\beta(s)}\,\sin(\psi(s)+\phi_0)$$

$$x'(s) = \sqrt{\tfrac{2J}{\beta}}[\beta\psi'\cos(\psi(s)+\phi_0) - \alpha\sin(\psi(s)+\phi_0)] \quad \text{with} \quad \alpha = -\tfrac{1}{2}\beta'$$

$$x''(s) = \sqrt{\tfrac{2J}{\beta}}[(\beta\psi'' - 2\alpha\psi')\cos(\psi(s)+\phi_0) - (\alpha' + \tfrac{\alpha^2}{\beta} + \beta\psi'^2)\sin(\psi(s)+\phi_0)]$$

$$= \sqrt{\tfrac{2J}{\beta}}[-k\beta\sin(\psi(s)+\phi_0)]$$

$$\beta\psi'' - 2\alpha\psi' = \beta\psi'' + \beta'\psi' = (\beta\psi')' = 0 \quad \Rightarrow \quad \psi' = \tfrac{I}{\beta}$$

$$\alpha' + \gamma = k\beta \quad \text{with} \quad \gamma = \tfrac{I^2+\alpha^2}{\beta}$$

Universal choice: I=1!

$$\beta' = -2\alpha$$

$$\alpha' = k\beta - \gamma$$

$$\psi = \int_0^s \frac{I}{\beta(s')}ds'$$

$\alpha, \beta, \gamma, \psi$ are called Twiss parameters.

What are the initial conditions?

Phase Space Ellipse

CHESS & LEPP

Particles with a common J and different ϕ all lie on an ellipse in phase space:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{I}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \sin(\psi(s)+\phi_0) \\ \cos(\psi(s)+\phi_0) \end{pmatrix}$$

(Linear transform of a circle)

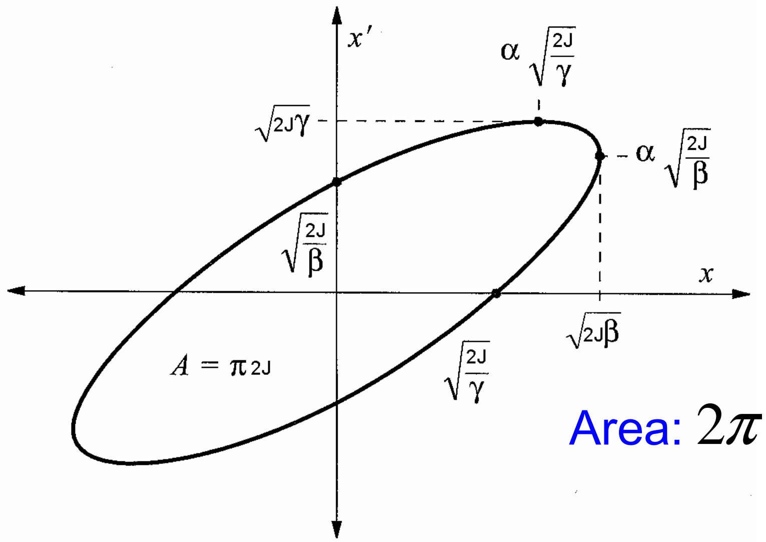
$$x_{\max} = \sqrt{2J\beta} \text{ at } x' = -\alpha\sqrt{\frac{2J}{\beta}}$$

$$(x, x') \begin{pmatrix} \frac{I}{\sqrt{\beta}} & \frac{\alpha}{\sqrt{\beta}} \\ 0 & \sqrt{\beta} \end{pmatrix} \begin{pmatrix} \frac{I}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix} = (x, x') \begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix} = 2J$$

(Quadratic form)

$$\beta\gamma - \alpha^2 = I^2$$

Area: $2\pi J / I$



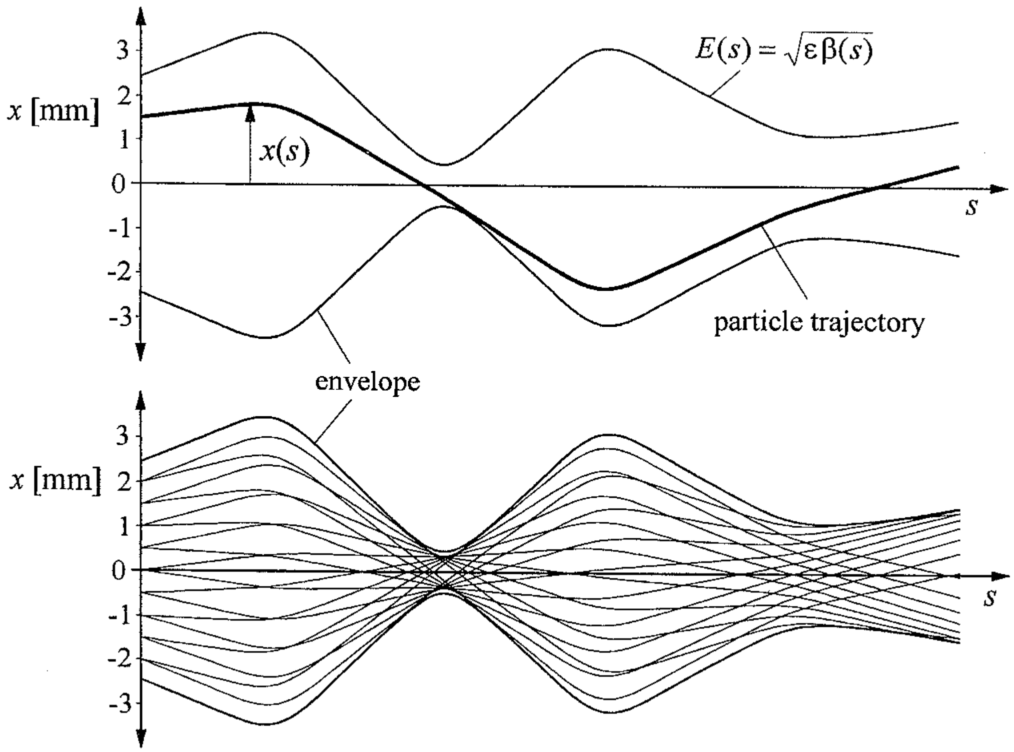
I=1 is therefore a useful choice!

What β is for x, γ is for x'

$$x'_{\max} = \sqrt{2J\gamma} \text{ at } x = -\alpha\sqrt{\frac{2J}{\gamma}}$$

Area: $2\pi J$ → $$\int_0^{2\pi}\int_0^{J} dJ d\phi = 2\pi J = \iint dx dx'$$

The Beam Envelope



$$x(s) = \sqrt{2J\beta(s)}\,\sin(\psi(s) + \phi_0)$$

In any beam there is a distribution of initial parameters. If the particles with the largest J are distributed in ϕ over all angles, then the envelope of the beam is described by $\sqrt{2J_{\max}\beta(s)}$

The initial conditions of β and α are chosen so that this is approximately the case.