Transport maps of cavities

(3) Average focusing over one period with relatively little energy change:

$$u'' \approx -u \frac{\Delta^2/4}{p^2}\,, \quad \Delta = \sqrt{\langle p'^2 \rangle}$$

(4) Continuous energy change:

$$p' \approx \Omega\,, \quad \Omega = \langle p' \rangle$$

$$\frac{d^2}{dp^2} u \approx \frac{1}{\Omega^2} u'' \approx -u \frac{(\Delta/\Omega)^2}{4p^2}$$

$$\frac{d^2}{dp^2}(r\sqrt{p}) = \frac{d^2}{dp^2} r\sqrt{p} + \frac{d}{dp} r \frac{1}{\sqrt{p}} - r\frac{1}{4\sqrt{p}^3} \approx -r\frac{(\Delta/\Omega)^2}{4\sqrt{p}^3}$$

$$\frac{d^2}{dp^2} r + \frac{d}{dp} r \frac{1}{p} \approx -r \frac{(\Delta/\Omega)^2 - 1}{4p^2} = -r\frac{\varepsilon^2}{p^2}$$

$$r(p) = \eta(-\ln(p)) \quad \Rightarrow \quad \frac{d^2}{dp^2} r = \frac{1}{p^2}\eta' + \frac{1}{p^2}\eta'' = \frac{1}{p^2}\eta' - \eta\frac{\varepsilon^2}{p^2}$$

Transport maps of traveling wave cavities

$$\eta'' = -\varepsilon^2 \eta\,, \quad \eta(\xi) = A\cos(\varepsilon\xi) - B\sin(\varepsilon\xi)$$

$$\begin{pmatrix} r \\ a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon\frac{p'}{p} \end{pmatrix} \begin{pmatrix} \cos(\varepsilon\ln(p)) & \sin(\varepsilon\ln(p)) \\ -\sin(\varepsilon\ln(p)) & \cos(\varepsilon\ln(p)) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

$$\begin{pmatrix} r \\ a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon\frac{p'}{p} \end{pmatrix} \begin{pmatrix} \cos(\varepsilon\ln(\frac{p}{p_i})) & \sin(\varepsilon\ln(\frac{p}{p_i})) \\ -\sin(\varepsilon\ln(\frac{p}{p_i})) & \cos(\varepsilon\ln(\frac{p}{p_i})) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\varepsilon}\frac{p_i}{p'} \end{pmatrix} \begin{pmatrix} r_i \\ a_i \end{pmatrix}$$

$$E_z = \sum_{n=0}^{\infty} g_n \cos(n\tfrac{2\pi}{L}z + \alpha_n)\cos(\omega t - kz + \varphi_0)$$

$$= \sum_{n=0}^{\infty} g_n \cos(n\tfrac{2\pi}{L}z + \alpha_n)\cos(\varphi_0)\,, \quad \alpha_0 = 0$$

$$\left.\langle p'\rangle = g_0\cos(\varphi_0)\,, \langle p'^2\rangle = \tfrac{1}{2}\sum_{n=0}^{\infty} g_n^2\cos^2(\varphi_0)\right\}\varepsilon = \tfrac{1}{2}\sqrt{\tfrac{1}{2}\sum_{n=0}^{\infty}\left(\tfrac{g_n}{g_0}\right)^2 - 1}$$

Transport maps of standing wave cavities

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$$\begin{pmatrix} r \\ a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \frac{p'}{p} \end{pmatrix} \begin{pmatrix} \cos(\varepsilon \ln(\frac{p}{p_i})) & \sin(\varepsilon \ln(\frac{p}{p_i})) \\ -\sin(\varepsilon \ln(\frac{p}{p_i})) & \cos(\varepsilon \ln(\frac{p}{p_i})) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\varepsilon}\frac{p_i}{p'} \end{pmatrix} \begin{pmatrix} r_i \\ a_i \end{pmatrix}$$

$$E_z = \sum_{n=-\infty}^{\infty} g_n e^{in\frac{2\pi}{L}z} \cos(kz)\cos(\omega t + \varphi_0) \ , \quad \omega t = kz \, , k = m\tfrac{\pi}{L} \, , \quad g_{-n} = g_n^*$$

$$= \tfrac{1}{4} \sum_{n=-\infty}^{\infty} g_n [e^{i(n+m)\frac{2\pi}{L}z+\varphi_0} + e^{i(n-m)\frac{2\pi}{L}z-\varphi_0} + e^{in\frac{2\pi}{L}z+\varphi_0} + e^{in\frac{2\pi}{L}z-\varphi_0}]$$

$$= \tfrac{1}{4} \sum_{n=-\infty}^{\infty} [g_{n-m} e^{i\varphi_0} + g_{n+m} e^{-i\varphi_0} + 2 g_n \cos(\varphi_0)] e^{in\frac{2\pi}{L}z} = \sum_{n=-\infty}^{\infty} f_n e^{in\frac{2\pi}{L}z}$$

$$\left.\begin{aligned} \langle p' \rangle &= f_0 \, , \\ \langle p'^2 \rangle &= \sum_{n=0}^{\infty} |f_n|^2 \end{aligned}\right\} \varepsilon = \tfrac{1}{2} \sqrt{\sum_{n=0}^{\infty} \left|\frac{f_n}{f_0}\right|^2 - 1}$$

Phase space precervation in cavities

Average focusing over one period with relatively little energy change:

$$\begin{pmatrix} r \\ a \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \frac{p'}{p} \end{pmatrix} \begin{pmatrix} \cos(\varepsilon \ln(\frac{p}{p_i})) & \sin(\varepsilon \ln(\frac{p}{p_i})) \\ -\sin(\varepsilon \ln(\frac{p}{p_i})) & \cos(\varepsilon \ln(\frac{p}{p_i})) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\varepsilon} \frac{p_i}{p'} \end{pmatrix}}_{\underline{M}} \begin{pmatrix} r_i \\ a_i \end{pmatrix}$$

$$\det(\underline{M}) = \frac{p_i}{p}$$

Because the determinant is not 1, the phase space volume is no longer conserved but changes by p_0/p.

A new propagation and definition of Twiss parameters is therefore needed:

$$r = \sqrt{2J \frac{1}{\beta_r \gamma_r} \beta} \sin(\psi + \phi_0)$$