Accelerator Physics - Homework 2
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Georg H. Hoffstaetter (Cornell University)

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Exercise (Complex potentials)
When the coordinates $w = x + iy$ and $\bar{w} = x - iy$ are used, the Laplace operator has been derived to be $\vec{\nabla}^2 = 4\partial_w\partial_{\bar{w}} + \partial_z^2$.
(a) Check that this is correct.
(b) The static magnetic field in a charge free space is given by $\vec{B} = -\vec{\nabla}\psi$. Writing the magnetic field in x and y direction in complex notation as $B = B_x + iB_y$, derive a formula that expresses B and B_z in terms of $\Psi(w, \bar{w}, z)$ and only ∂_w, $\partial_{\bar{w}}$, and ∂_z.
(c) Given the vector potential in complex notation as $A = A_x + iA_y$ and A_z, derive a formula that expresses B and B_z given by $\vec{B} = \nabla\times\vec{A}$, again only using ∂_w, $\partial_{\bar{w}}$, and ∂_z and A, A_z.

Exercise (Rotational field symmetries)
(a) The field in a bending magnet has usually two symmetries: Midplane symmetry since the upper and lower part of the magnet are built identically, and a mirror symmetry with respect to the vertical plane, since each pole is build with right/left symmetry when viewed along the beam pipe. Which multipoles, in addition to the main dipole component, satisfy this symmetry and can therefore be associated with such a bending magnet.
(b) Similarly, a focusing magnet has C_2 and midplane symmetry. Which multipoles, in addition to the main quadrupole term, satisfy this symmetry and can therefore appear when such a magnet is built.
(c) Generalize your observation to a magnet which is built with exact C_n symmetry and midplane symmetry. Which multipole terms can the field have?

Exercise (Solenoid)
Consider a box shape solenoid field. On the central axis, the solenoid field is given by

$$\vec{B}(z) = \begin{cases} B_0\vec{e}_z & \text{for } z \in [0, L] \\ 0 & \text{else} \end{cases} \quad (1)$$

(a) A particle flies into the solenoid parallel to the central axis with a horizontal distance x_0. Describe its trajectory after the solenoid.
(b) If it touches the central axis somewhere after the solenoid, where would that be? How does the focal length depend on the field B_0 and the length L?

(c) Show that the magnetic field $\vec{B} = \{\frac{x}{2}\Psi_0'' , \ \frac{y}{2}\Psi_0'' , \ -\Psi_0'\}$ can be derived from the vector potential $\vec{A} = \{\frac{y}{2}\Psi_0' , \ -\frac{x}{2}\Psi_0' , \ 0\}$, where Ψ_0 is a function of z.
(d) Show that during this motion, the particle's angular momentum around the z-axis is not conserved. Also show that the z component of its canonical angular momentum $L_z = \left\{\vec{r} \times (\vec{p} + e\vec{A})\right\}_z$ is conserved. To do this, you can show that $\frac{dL_z}{dt} = 0$.
(e) Given a proton beam of $E_k = 5$keV, how many turns of a 100A current is approximately needed for a 10cm coil to have a 1 meter focal length.
(f) Show that within the solenoid the particles perform helical motion of radius $\frac{x_0}{2}$ around the axis $x = \frac{x_0}{2}$.

Exercise (Multipoles)
(a) Describe the magnetic field and the magnetic scalar potential in a duodecapole ?
(b) How strong is a duodecapole for which the distance from the central axis to the iron pole is given by a and around each pole is a winding of n wires each having a current I ?
(c) Show what fields are created when a n pole is shifted by a distance Δ in the transverse direction. For example, show that a shifted sextupole has a quadrupole field.

Exercise (Neutron beam optics)
Dipole magnets are used to guiding charged particles in a beam line or circular accelerator. Neutrons cannot be guided by homogeneous magnetic fields since they have no charge. However, due to their magnetic dipole moment the Stern-Gerlach Force could be used to guide them.
(a) Show that the force, produced by a quadrupole on a particle with horizontal spin, corresponds to the force on a charged particle in a dipole magnet.
(b) Show that a skewed sextupole magnet can be used for focusing neutral particles with horizontal spin.
(c) What would the multipole coefficient need to be to produce an instantaneous bending radius of 10m for a neutron with an energy of 1MeV?

Exercise (Air coil magnet)
(a) Suppose an air-coil magnet has four wires parallel to a beam pipe with the (x, y) coordinates $(a, 0)$, $(0, a)$, $(-a, 0)$, and $(0, -a)$. The fist and the third wire have the current is I, the second and fourth have $-I$. What multipole components Ψ_ν will be created in the center of the beam pipe?
(b) Given an electron beam with 2GeV energy and $a = 5$cm. How much current would one need to create a quadrupole component of $k_1 = 0.01m^{-2}$?

Exercise (Midplane symmetry)
A magnetic field has midplane symmetry. The particle motion through this field is described by a transport map $\vec{M}$ with $\vec{z}(s) = \vec{M}(s, \vec{z}_0)$ for the phase space vector $\vec{z} = (x, p_x, y, p_y)$.
(a) Explain why midplane symmetry requires the condition

$$\underline{S} = \begin{pmatrix} \underline{1} & \underline{0} \\ \underline{0} & -\underline{1} \end{pmatrix} , \quad \vec{M}(s, \vec{z}_0) = \underline{S}\vec{M}(s, \underline{S}\vec{z}_0) \ . \tag{2}$$

The matrices $\underline{0}$ and $\underline{1}$ are the 2×2 dimensional zero and unit matrix.

(b) What conditions do the Taylor coefficients $M_i^{\vec{k}}$ of the transport map satisfy when it is expanded as

$$M_i(\vec{z}_0) = \sum_{j=1}^{4} \sum_{k_j=1}^{\infty} M_i^{\vec{k}} x_0^{k_1} p_{x0}^{k_2} y_0^{k_3} p_{y0}^{k_4} \ . \tag{3}$$