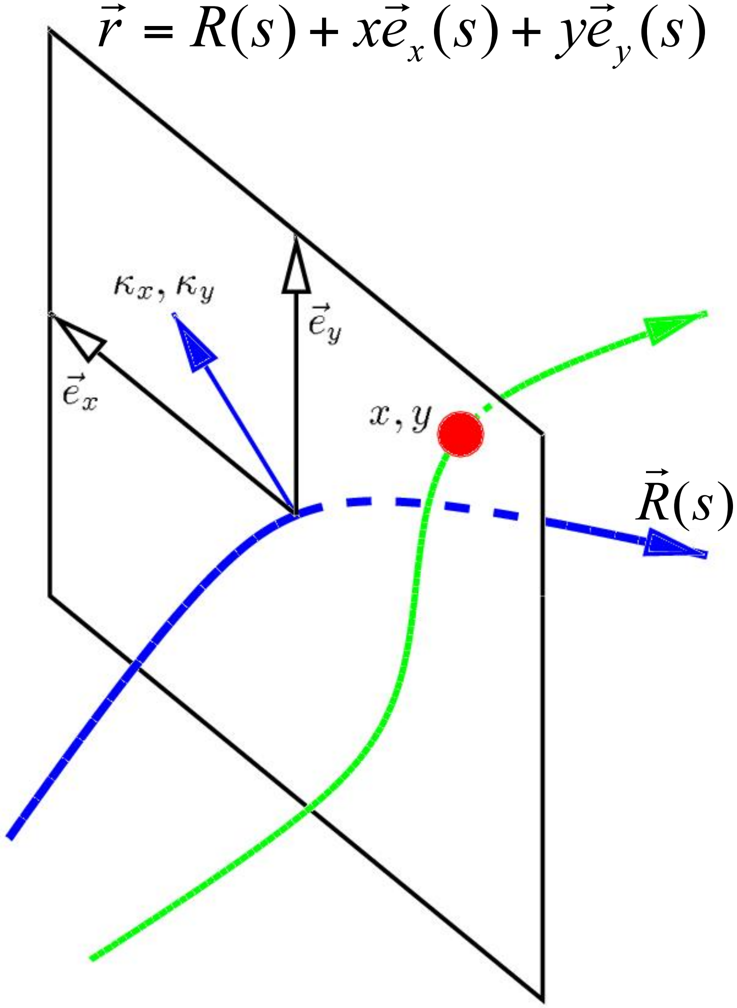
The Frenet Coordinate System

CHESS & LEPP

$$\vec{r} = \vec{R}(s) + x\vec{e}_x(s) + y\vec{e}_y(s)$$



$$\left|d\vec{R}\right| = ds$$

$$\vec{e}_s \equiv \tfrac{d}{ds}\vec{R}(s)$$

$$\vec{e}_\kappa \equiv -\tfrac{d}{ds}\vec{e}_s \Big/ \left|\tfrac{d}{ds}\vec{e}_s\right|$$

$$\vec{e}_b \equiv \vec{e}_s \times \vec{e}_\kappa$$

$$\tfrac{d}{ds}\vec{e}_s = -\kappa\,\vec{e}_\kappa \quad \text{with} \quad \kappa = \frac{1}{\rho}$$

$$0 = \tfrac{d}{ds}\left(\vec{e}_\kappa \cdot \vec{e}_s\right) = \vec{e}_s \cdot \tfrac{d}{ds}\vec{e}_\kappa - \kappa$$

Accumulated torsion angle T

$$\tfrac{d}{ds}\vec{e}_\kappa = \kappa\,\vec{e}_s + T'\vec{e}_b$$

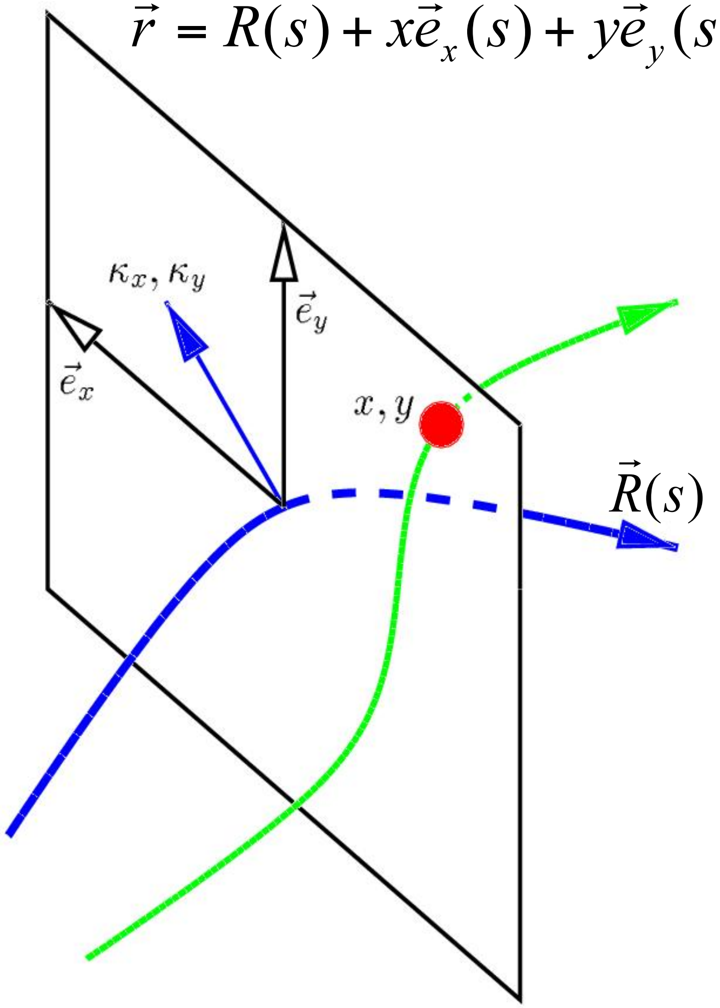
$$0 = \tfrac{d}{ds}\left(\vec{e}_b \cdot \vec{e}_\kappa\right) = \vec{e}_\kappa \cdot \tfrac{d}{ds}\vec{e}_b + T'$$

$$\tfrac{d}{ds}\vec{e}_b = -T'\vec{e}_\kappa$$

$$\vec{r}\,' = (x' - yT')\vec{e}_\kappa + (y' + xT')\vec{e}_b + (1 + x\kappa)\vec{e}_s$$

The Curvi-linear System

$$\vec{r} = \vec{R}(s) + x\vec{e}_x(s) + y\vec{e}_y(s)$$



$$\vec{e}_x \equiv \vec{e}_\kappa \cos(T) - \vec{e}_b \sin(T)$$
$$\vec{e}_y \equiv \vec{e}_\kappa \sin(T) + \vec{e}_b \cos(T)$$

$$\frac{d}{ds}\vec{e}_s = -\kappa_x \vec{e}_x - \kappa_y \vec{e}_y$$

$$\frac{d}{ds}\vec{e}_x = \kappa \cos(T)\vec{e}_s = \kappa_x \vec{e}_s$$

$$\frac{d}{ds}\vec{e}_y = \kappa \sin(T)\vec{e}_s = \kappa_y \vec{e}_s$$

$$\frac{d}{ds}\vec{r} = x'\vec{e}_\kappa + y'\vec{e}_b + (1 + x\kappa_x + y\kappa_y)\vec{e}_s$$

Phase Space ODE

$$\frac{d}{ds}\vec{r} = x'\vec{e}_x + y'\vec{e}_y + \underbrace{(1 + x\kappa_x + y\kappa_y)}_{h}\vec{e}_s$$

$$\frac{d^2}{dt^2}\vec{r} = \vec{F}$$

$$\frac{d}{ds}\vec{r} = \dot{s}^{-1}\frac{d}{dt}\vec{r} = \dot{s}^{-1}\frac{1}{m\gamma}\vec{p} = \frac{h}{p_s}\vec{p}$$

$$\frac{d}{ds}\vec{p} = (p_x' - p_s\kappa_x)\vec{e}_x + (p_y' - p_s\kappa_y)\vec{e}_y + (p_s' + \kappa_x p_x + \kappa_y p_y)\vec{e}_s$$

$$= \dot{s}^{-1}\frac{d}{dt}\vec{p} = \dot{s}^{-1}\vec{F} = \frac{m\gamma h}{p_s}\vec{F}$$

$$\begin{pmatrix} x' \\ y' \\ p_x' \\ p_y' \end{pmatrix} = \begin{pmatrix} \frac{h}{p_s}p_x \\ \frac{h}{p_s}p_y \\ \frac{m\gamma h}{p_s}F_x + p_s\kappa_x \\ \frac{m\gamma h}{p_s}F_y + p_s\kappa_y \end{pmatrix}$$

$$t' = \dot{s}^{-1} = \frac{hm\gamma}{p_s}$$

$$E = \sqrt{(pc)^2 + (mc^2)^2}$$

$$E' = \frac{d}{dp}\sqrt{(pc)^2 + (mc^2)^2}\,\frac{d}{ds}p = c^2\frac{\vec{p}}{E}\frac{d}{ds}\vec{p} = \frac{h}{p_s}\vec{p}\cdot\vec{F}$$

6 Dimensional Phase Space

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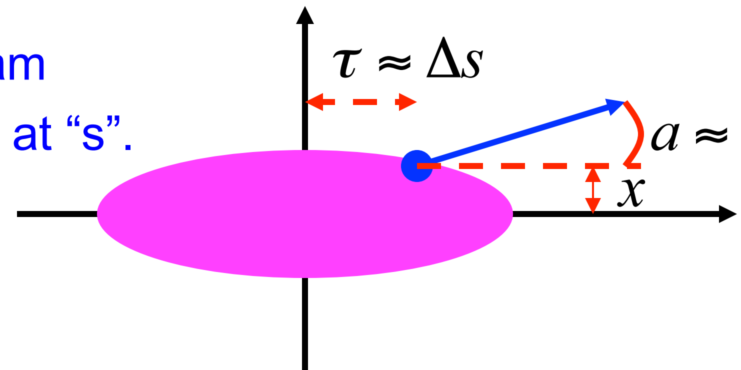
Using a reference momentum p_0 and a reference time t_0:

$$\vec{z} = (x, a, y, b, \tau, \delta)$$

$$a = \frac{p_x}{p_0}, \quad b = \frac{p_y}{p_0}, \quad \delta = \frac{E - E_0}{E_0}, \quad \tau = (t_0 - t)\frac{c^2}{v_0} = (t_0 - t)\frac{E_0}{p_0}$$

Usually p_0 is the design momentum of the beam

And t_0 is the time at which the bunch center is at “s”.



$$\left.\begin{array}{l} x' = \partial_{p_x} K \\ p_x' = -\partial_x K \end{array}\right\} \Rightarrow \left\{\begin{array}{ll} x' = \partial_a K/p_0 , & a' = -\partial_x K/p_0 \\ y' = \partial_b K/p_0 , & b' = -\partial_y K/p_0 \end{array}\right.$$

$$-t' = \partial_E K \quad \Rightarrow \quad \tau' = \tfrac{c^2}{v_0}\partial_\delta K/E_0 = \partial_\delta K/p_0$$

$$E' = -\partial_{-t} K \quad \Rightarrow \quad \delta' = -\tfrac{1}{E_0}\partial_\tau K \tfrac{c^2}{v_0} = -\partial_\tau K/p_0$$

New Hamiltonian:

$$\tilde{H} = K/p_0$$

Simplified Equation of Motion

Only magnetic fields: $\vec{E} = 0$

Mid-plane symmetry: $B_x(x,y,s) = -B_x(x,-y,s)\ ,\quad B_y(x,y,s) = B_y(x,-y,s)$

Linearization in : $B_x(x,y,s) = \frac{p_0}{q} ky\ ,\quad B_y(x,y,s) = \frac{p_0}{q}\frac{1}{\rho} + \frac{p_0}{q} kx$

Highly relativistic : $\frac{p-p_0}{p_0} \to \frac{E-E_0}{E_0}\quad ,\quad \frac{v-v_0}{v_0} \to 0$

$$x' = \frac{p_x}{p_z} = \frac{p_x}{\sqrt{(p_0+dp)^2 - p_x^2 - p_y^2}} =_1 \frac{p_x}{p_0} = a \qquad \Rightarrow \qquad y' = b$$

$$a' = \frac{(\vec{p}' + p_s\kappa\vec{e}_x)_x}{p_0} = \frac{h}{p_0 v_s}\frac{d}{dt}p_x + \frac{p_s\kappa}{p_0} = -\frac{1+x\kappa}{p_0 v_s} q v_s B_y + \frac{p_s\kappa}{p_0}$$

$$=_1 -(1+x\kappa)(\kappa + kx) + (1+\delta)\kappa = -x(\kappa^2 + k) + \delta\kappa \qquad \Rightarrow \qquad b' = ky$$

$$\tau' = \frac{d(t-t_0)}{ds}\frac{E_0}{p_0} = \left(\frac{1}{v_0} - \frac{h}{v_s}\right)\frac{E_0}{p_0} =_1 -x\kappa \qquad , \qquad \delta' = 0$$

Hamiltonian:

$$H = \tfrac{1}{2}a^2 + \tfrac{1}{2}b^2 + \tfrac{1}{2}k(x^2 - y^2) + \tfrac{1}{2}\kappa^2 x^2 - \kappa x\delta$$

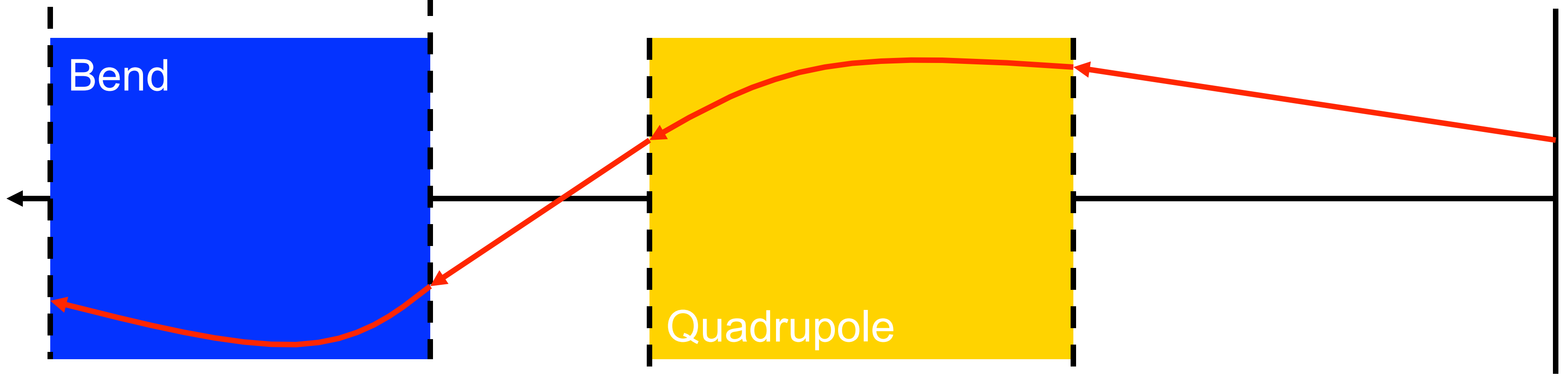
Matrix Solutions

Linear equation of motion: $\vec{z}\,' = \underline{F}(s)\vec{z}$

Matrix solution of the starting condition $\vec{z}(0) = \vec{z}_0$

$$\vec{z} = \underline{M}_{\text{bend}}(L_4)\underline{M}_{\text{drift}}(L_3)\underline{M}_{\text{quad}}(L_2)\underline{M}_{\text{drift}}(L_1)\vec{z}_0$$

$$\vec{z} = \underline{M}_{\text{drift}}(L_3)\underline{M}_{\text{quad}}(L_2)\underline{M}_{\text{drift}}(L_1)\vec{z}_0$$



$$\vec{z} = \underline{M}_{\text{drift}}(L_1)\vec{z}_0 \qquad \vec{z}_0$$

$$\vec{z} = \underline{M}_{\text{quad}}(L_2)\underline{M}_{\text{drift}}(L_1)\vec{z}_0$$

The Drift

$$\begin{pmatrix} x' \\ a' \\ y' \\ b' \\ \tau' \\ \delta' \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ b \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Note that in nonlinear expansion $x' \neq a$ so that the drift does not have a linear transport map even though $x(s) = x_0 + x_0' s$ is completely linear.

$$\begin{pmatrix} x \\ a \\ y \\ b \\ \tau \\ \delta \end{pmatrix} = \begin{pmatrix} x_0 + sa_0 \\ a \\ y_0 + sb_0 \\ b_0 \\ \tau_0 \\ \delta_0 \end{pmatrix} = \begin{pmatrix} 1 & s & \underline{0} & & \underline{0} & \\ 0 & 1 & & & & \\ \underline{0} & & 1 & s & \underline{0} & \\ & & 0 & 1 & & \\ \underline{0} & & \underline{0} & & 1 & 0 \\ & & & & 0 & 1 \end{pmatrix} \vec{z}_0$$

The Dipole Equation of Motion

$$x'' = -x\kappa^2 + \delta\kappa$$

$$y'' = 0$$

$$\tau' = -x\kappa$$

Homogeneous solution:

$$x_H'' = -x_H\kappa^2 \quad \Rightarrow \quad x_H = A\cos(\kappa s) + B\sin(\kappa s)$$ (natural ring focusing)

Variation of constants:

$$x = A(s)\cos(\kappa s) + B(s)\sin(\kappa s)$$

$$x' = -A\kappa\sin(\kappa s) + B\kappa\cos(\kappa s) + \underbrace{A'\cos(\kappa s) + B'\sin(\kappa s)}_{\equiv 0}$$

$$x'' = -\kappa^2 x \underbrace{- A'\kappa\sin(\kappa s) + B'\kappa\cos(\kappa s)}_{=\delta\kappa} = -\kappa^2 x + \delta\kappa$$

$$\begin{pmatrix} \cos(\kappa s) & \sin(\kappa s) \\ -\sin(\kappa s) & \cos(\kappa s) \end{pmatrix} \begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} 0 \\ \delta \end{pmatrix}$$

The Dipole

$$\begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} \cos(\kappa s) & -\sin(\kappa s) \\ \sin(\kappa s) & \cos(\kappa s) \end{pmatrix} \begin{pmatrix} 0 \\ \delta \end{pmatrix}$$

$$\begin{pmatrix} A \\ B \end{pmatrix} = \delta\, \kappa^{-1} \begin{pmatrix} \cos(\kappa s) \\ \sin(\kappa s) \end{pmatrix} + \begin{pmatrix} A_H \\ B_H \end{pmatrix} \quad \text{with} \quad x = A\cos(\kappa s) + B\sin(\kappa s)$$

$$\tau' = -x\kappa$$

$$\underline{M} = \begin{pmatrix} \cos(\kappa s) & \frac{1}{\kappa}\sin(\kappa s) & \underline{0} & & 0 & \kappa^{-1}[1-\cos(\kappa s)] \\ -\kappa\sin(\kappa s) & \cos(\kappa s) & & & 0 & \sin(\kappa s) \\ \underline{0} & & 1 & s & \underline{0} & \\ & & 0 & 1 & & \\ -\sin(\kappa s) & \kappa^{-1}[\cos(\kappa s)-1] & \underline{0} & & 1 & \kappa^{-1}[\sin(\kappa s)-s\kappa] \\ 0 & 0 & & & 0 & 1 \end{pmatrix}$$

Time of Flight from Symplecticity

$$\underline{M} = \begin{pmatrix} \underline{M}_4 & \vec{0} & \vec{D} \\ \vec{T}^T & 1 & M_{56} \\ \vec{0}^T & 0 & 1 \end{pmatrix}$$

is in SU(6) and therefore $\underline{M}\,\underline{J}\,\underline{M}^T = \underline{J}$

$$\begin{pmatrix} \underline{M}_4\underline{J}_4 & -\vec{D} & \vec{0} \\ \vec{T}^T\underline{J}_4 & -M_{56} & 1 \\ \vec{0}^T & -1 & 0 \end{pmatrix}\begin{pmatrix} \underline{M}_4^T & \vec{T} & \vec{0} \\ \vec{0}^T & 1 & 0 \\ \vec{D}^T & M_{56} & 1 \end{pmatrix} = \begin{pmatrix} \underline{J}_4 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \underline{M}_4\underline{J}_4\underline{M}_4^T & \underline{M}_4\underline{J}_4\vec{T} - \vec{D} & \vec{0} \\ \vec{T}^T\underline{J}_4\underline{M}_4^T + \vec{D}^T & 0 & 1 \\ \vec{0}^T & -1 & 0 \end{pmatrix} = \begin{pmatrix} \underline{J}_4 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\vec{T} = -\underline{J}_4\underline{M}_4^{-1}\vec{D}$$

It is sufficient to compute the 4D map $\underline{M}_4$, the Dispersion $\vec{D}$ and the time of flight term M_{56}

The Quadrupole (Homework)

$$x'' = -x\,k$$

$$y'' = \;\; y\,k$$

$$\underline{M}_4 = \begin{pmatrix} \cos(\sqrt{k}\,s) & \frac{1}{\sqrt{k}}\sin(\sqrt{k}\,s) & & \\ -\sqrt{k}\sin(\sqrt{k}\,s) & \cos(\sqrt{k}\,s) & & \underline{0} \\ & \underline{0} & \cosh(\sqrt{k}\,s) & \frac{1}{\sqrt{k}}\sinh(\sqrt{k}\,s) \\ & & \sqrt{k}\sinh(\sqrt{k}\,s) & \cosh(\sqrt{k}\,s) \end{pmatrix}$$

As for a drift:

$$\vec{D} = \vec{0} \quad \Rightarrow \quad \vec{T} = \vec{0}$$

$$M_{56} = 0$$

For k<0 one has to take into account that

$$\cos(\sqrt{k}\,s) = \cosh(\sqrt{|k|}\,s)\,, \quad \sin(\sqrt{k}\,s) = i\sinh(\sqrt{|k|}\,s)$$

$$\cosh(\sqrt{k}\,s) = \cos(\sqrt{|k|}\,s)\,, \quad \sinh(\sqrt{k}\,s) = i\sin(\sqrt{|k|}\,s)$$

The Combined Function Bend (Homework)

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$$x'' = -x\,(\underbrace{\kappa^2 + k}_{K}) + \delta\kappa$$

$$y'' = y\,k \quad , \quad \tau' = -\kappa\, x$$

$$\underline{M}_6 = \begin{pmatrix} \underline{M}_x & \underline{0} & \vec{0}\,\vec{D} \\ \underline{0} & \underline{M}_y & \underline{0} \\ \underline{T} & \underline{0} & \underline{M}_\tau \end{pmatrix}$$

$$\underline{M}_\tau = \begin{pmatrix} 1 & M_{56} \\ 0 & 1 \end{pmatrix}$$

$$M_{56} = \frac{\kappa^2}{K\sqrt{K}}[\sin(\sqrt{K}s) - \sqrt{K}s]$$

$\underline{T}$ from symplecticity

$$\underline{M}_x = \begin{pmatrix} \cos(\sqrt{K}\,s) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}\,s) \\ -\sqrt{K}\sin(\sqrt{K}\,s) & \cos(\sqrt{K}\,s) \end{pmatrix}$$

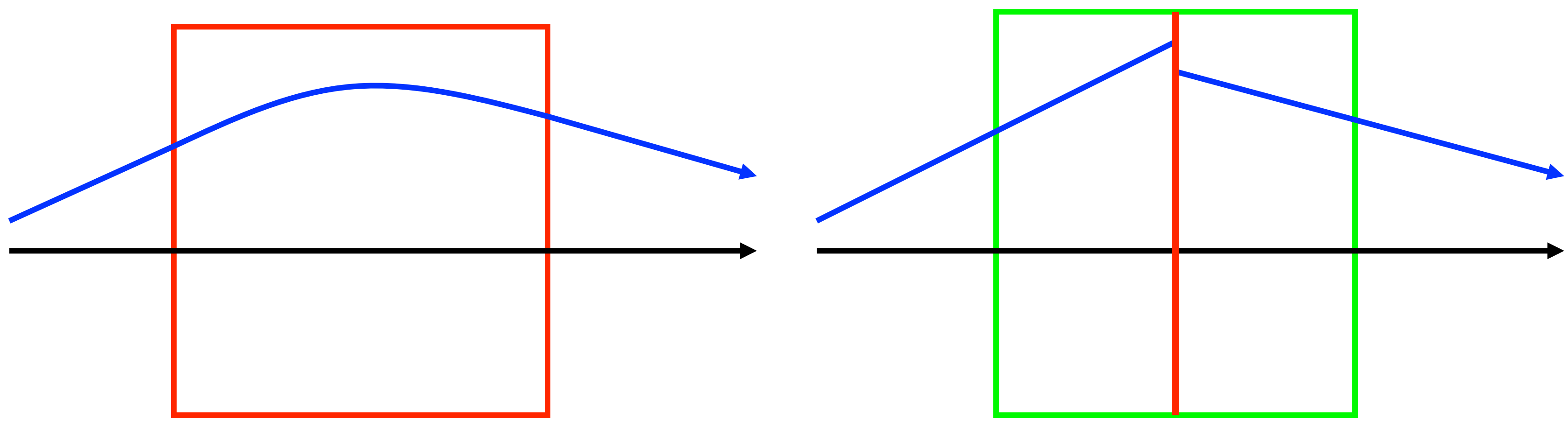
$$\underline{M}_y = \begin{pmatrix} \cosh(\sqrt{k}\,s) & \frac{1}{\sqrt{k}}\sinh(\sqrt{k}\,s) \\ \sqrt{k}\sinh(\sqrt{k}\,s) & \cosh(\sqrt{k}\,s) \end{pmatrix}$$

$$\vec{D} = \begin{pmatrix} \frac{\kappa}{K}[1 - \cos(\sqrt{K}s)] \\ \frac{\kappa}{\sqrt{K}}\sin(\sqrt{K}s) \end{pmatrix}$$

Options:

- For k>0:
  focusing in x, defocusing in y.
- For k<0, K<0:
  defocusing in x, focusing in y.
- For k<0, K>0:
  weak focusing in both planes.

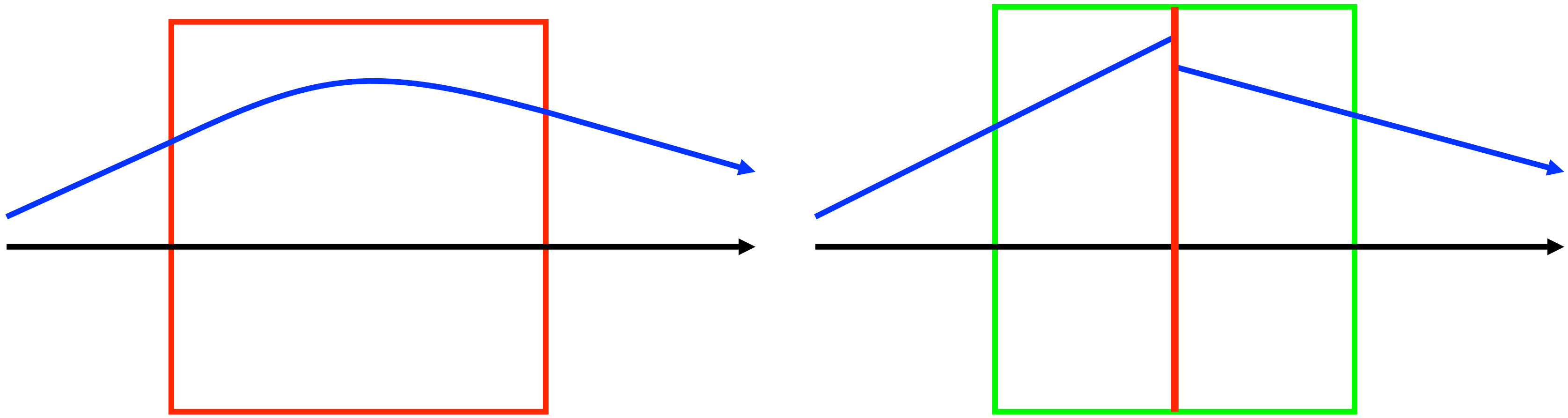
The Thin Lens Approximation



$$\vec{z}(s) = \underline{M}(s)\vec{z}_0 = \underline{D}(\tfrac{s}{2})\underline{D}^{-1}(\tfrac{s}{2})\underline{M}(s)\underline{D}^{-1}(\tfrac{s}{2})\underline{D}(\tfrac{s}{2})\vec{z}_0$$

Drift: $$\underline{M}_{\text{drift}}^{\text{thin}}(s) = \underline{D}^{-1}(\tfrac{s}{2})\underline{D}(s)\underline{D}^{-1}(\tfrac{s}{2}) = \underline{1}$$

The Thin Lens Quadrupole



$$\underline{M}^{\text{thin}}_{\text{quad,x}}(s) = \begin{pmatrix} 1 & -\frac{s}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\sqrt{k}s) & \frac{1}{\sqrt{k}}\sin(\sqrt{k}s) \\ -\sqrt{k}\sin(\sqrt{k}s) & \cos(\sqrt{k}s) \end{pmatrix} \begin{pmatrix} 1 & -\frac{s}{2} \\ 0 & 1 \end{pmatrix}$$

$$\approx \begin{pmatrix} 1 & -\frac{s}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & s \\ -ks & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{s}{2} \\ 0 & 1 \end{pmatrix} \approx \begin{pmatrix} 1 & -\frac{s}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{s}{2} \\ -ks & 1+\frac{ks^2}{2} \end{pmatrix}$$

Weak magnet limit: $\sqrt{k}s << 1$

$$\underline{M}^{\text{thin}}_{\text{quad,x}}(s) \approx \begin{pmatrix} 1 & 0 \\ -ks & 1 \end{pmatrix}$$

The Thin Lens Dipole

$$\underline{M} = \begin{pmatrix} \begin{matrix} \cos(\kappa s) & \frac{1}{\kappa}\sin(\kappa s) \\ -\kappa\sin(\kappa s) & \cos(\kappa s) \end{matrix} & \underline{0} & \begin{matrix} 0 & \kappa^{-1}[1-\cos(\kappa s)] \\ 0 & \sin(\kappa s) \end{matrix} \\ \underline{0} & \begin{matrix} 1 & s \\ 0 & 1 \end{matrix} & \underline{0} \\ \begin{matrix} -\sin(\kappa s) & \kappa^{-1}[\cos(\kappa s)-1] \\ 0 & 0 \end{matrix} & \underline{0} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{pmatrix}$$

Weak magnet limit: $\kappa s << 1$

$$\underline{M}^{\text{thin}}_{\text{bend},x\tau}(s) = \underline{D}(-\tfrac{s}{2})\underline{M}_{\text{bend},x\tau}\underline{D}(-\tfrac{s}{2}) \approx \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\kappa^2 s & 1 & 0 & \kappa s \\ -\kappa s & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Thin Combined Function Bend

$$\underline{M}_6 = \begin{pmatrix} \underline{M}_x & \underline{0} & \vec{0}\,\vec{D} \\ \underline{0} & \underline{M}_y & \underline{0} \\ \underline{T} & \underline{0} & \underline{1} \end{pmatrix}$$

Weak magnet limit: $\kappa s << 1$

$$\underline{M}_x = \begin{pmatrix} \cos(\sqrt{K}\, s) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}\, s) \\ -\sqrt{K}\sin(\sqrt{K}\, s) & \cos(\sqrt{K}\, s) \end{pmatrix}$$

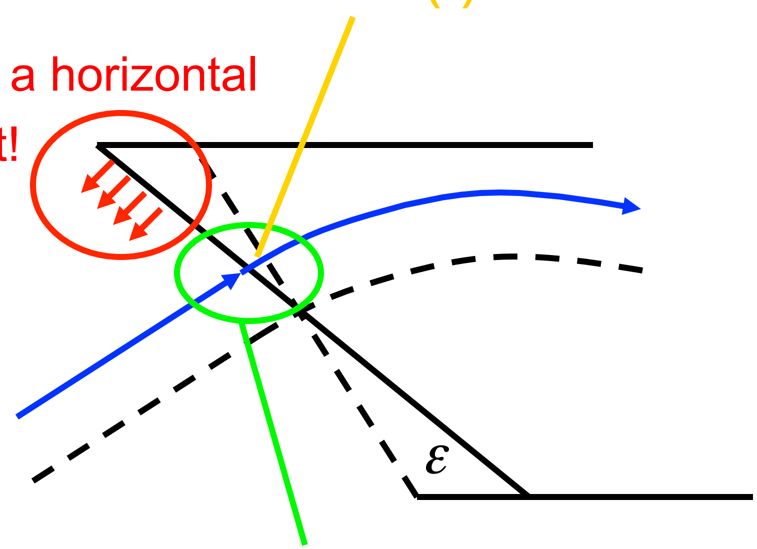
$$\underline{M}_x^{\text{thin}} = \begin{pmatrix} 1 & 0 \\ -K s & 1 \end{pmatrix}$$

$$\underline{M}_y = \begin{pmatrix} \cosh(\sqrt{k}\, s) & \frac{1}{\sqrt{k}}\sinh(\sqrt{k}\, s) \\ \sqrt{k}\sinh(\sqrt{k}\, s) & \cosh(\sqrt{k}\, s) \end{pmatrix} \longrightarrow \underline{M}_y^{\text{thin}} = \begin{pmatrix} 1 & 0 \\ k s & 1 \end{pmatrix}$$

$$\vec{D} = \begin{pmatrix} \frac{\kappa}{K}[1-\cos(\sqrt{K}s)] \\ \frac{\kappa}{\sqrt{K}}\sin(\sqrt{K}s) \end{pmatrix}$$

$$\vec{D} = \begin{pmatrix} 0 \\ \kappa s \end{pmatrix}$$

Edge Focusing



Horizontal focusing with $\Delta x' = -x\,\frac{\tan(\varepsilon)}{\rho}$

$$B_x = \partial_y B_s\Big|_{y=0} y\tan(\varepsilon) = \partial_s B_y\Big|_{y=0} y\tan(\varepsilon)$$

$$y'' = \tfrac{q}{p}\,\partial_s B_y\Big|_{y=0} y\tan(\varepsilon)$$

$$\Delta y' = \int y''\,ds = \tfrac{q}{p} B_y\, y\tan(\varepsilon) = y\,\frac{\tan(\varepsilon)}{\rho}$$

Quadrupole effect with

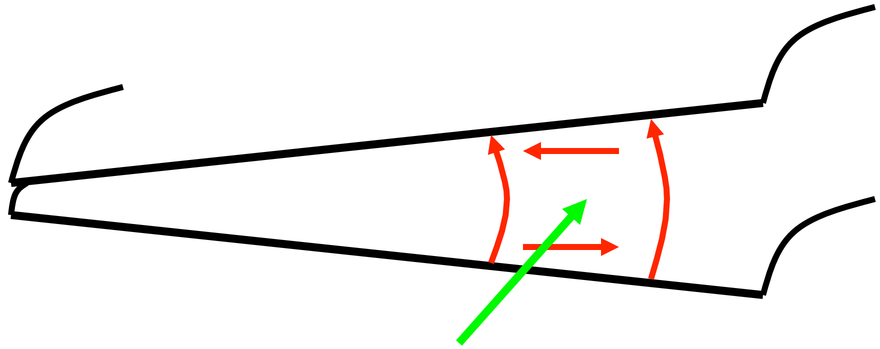
$$kl = \frac{\tan(\varepsilon)}{\rho}$$

$$\vec{z} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{\tan(\varepsilon)}{\rho} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{\tan(\varepsilon)}{\rho} & 1 \end{pmatrix} \vec{z}_0$$

Cyclotrons with edge focusing

- The isocyclotron with constant

$$\omega_z = \frac{q}{m_0\gamma(E)} B_z(r(E))$$



Up to 600MeV but
this vertically defocuses the beam.
Edge focusing is therefore used.

Variation of Constants

$$\vec{z}\,' = \vec{f}(\vec{z},s)$$

$$\vec{z}\,' = \underline{L}(s)\vec{z} + \Delta\vec{f}(\vec{z},s)$$

Field errors, nonlinear fields, etc can lead to $\Delta\vec{f}(\vec{z},s)$

$$\vec{z}\,'_H = \underline{L}(s)\vec{z}_H \quad \Rightarrow \quad \vec{z}_H(s) = \underline{M}(s)\vec{z}_{H0} \quad \text{with} \quad \underline{M}'(s)\vec{a} = \underline{L}(s)\underline{M}(s)\vec{a}$$

$$\vec{z}(s) = \underline{M}(s)\vec{a}(s) \quad \Rightarrow \quad \vec{z}\,'(s) = \underline{M}'(s)\vec{a} + \underline{M}(s)\vec{a}\,'(s) = \underline{L}(s)\vec{z} + \Delta\vec{f}(\vec{z},s)$$

$$\vec{a}(s) = \vec{z}_0 + \int_0^s \underline{M}^{-1}(\hat{s})\Delta\vec{f}(\vec{z}(\hat{s}),\hat{s})\,d\hat{s}$$

$$\vec{z}(s) = \underline{M}(s)\left\{\vec{z}_0 + \int_0^s \underline{M}^{-1}(\hat{s})\Delta\vec{f}(\vec{z}(\hat{s}),\hat{s})\,d\hat{s}\right\}$$

$$= \vec{z}_H(s) + \int_0^s \underline{M}(s-\hat{s})\Delta\vec{f}(\vec{z}(\hat{s}),\hat{s})\,d\hat{s}$$

Perturbations are propagated from s to s’

Aberration Correction

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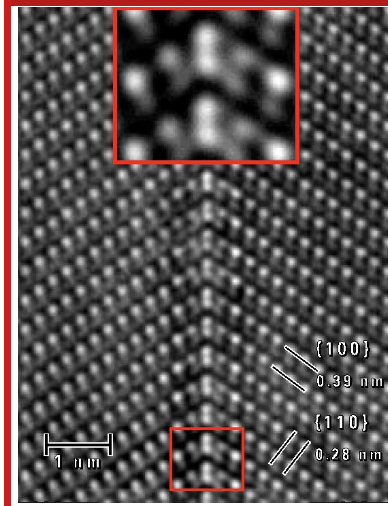
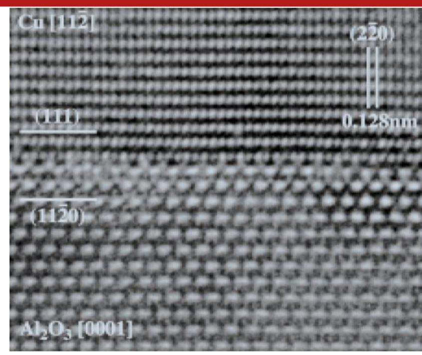




$$w_2(s) = w_H(s) + C(s)w_0^2\overline{w}_0 + \dots$$

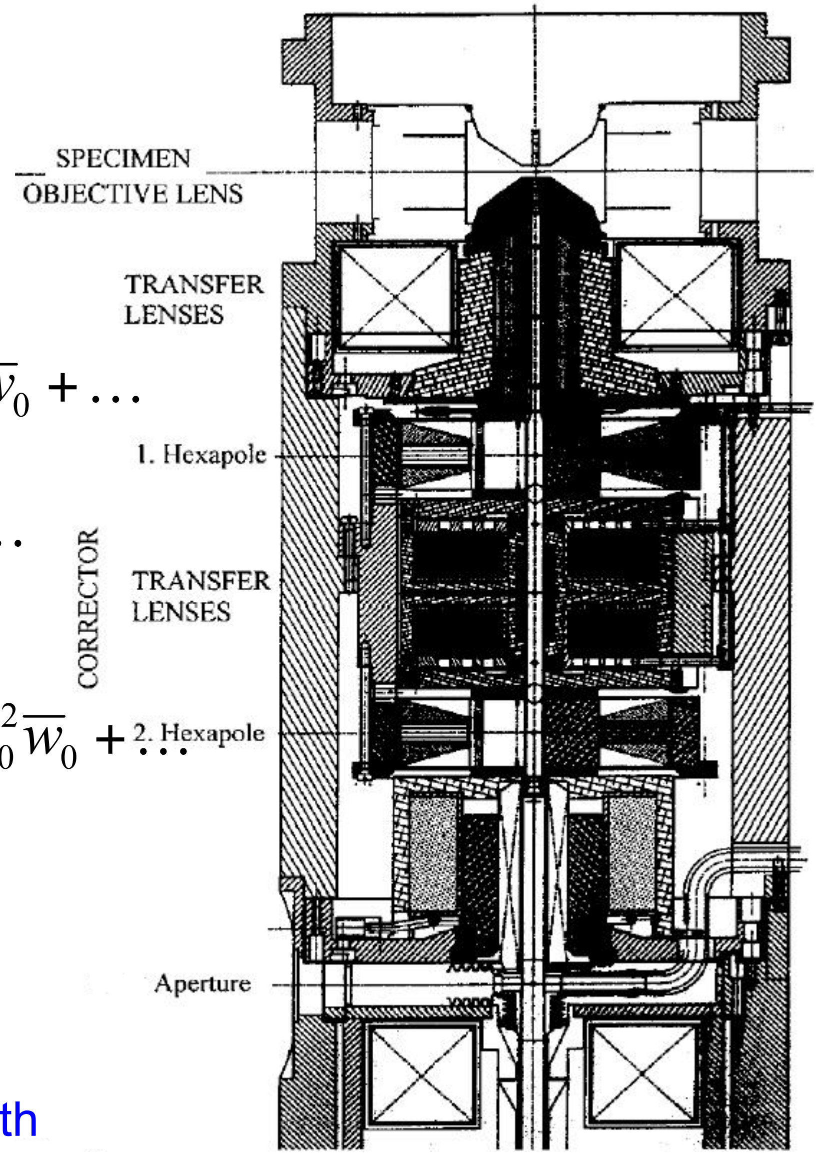
$$w_2(s) = w_H(s) + A(s)\overline{w}_0^2 + B(s)w_0^2\overline{w}_0 + \dots$$

$$w_2(s) = w_H(s) + A(s)\overline{w}_0^2 + 2B(s)w_0^2\overline{w}_0 + \dots$$

2B cancels C !

Quadratic in sextupole strength

Linear in solenoid strength



Beta Function and Betatron Phase

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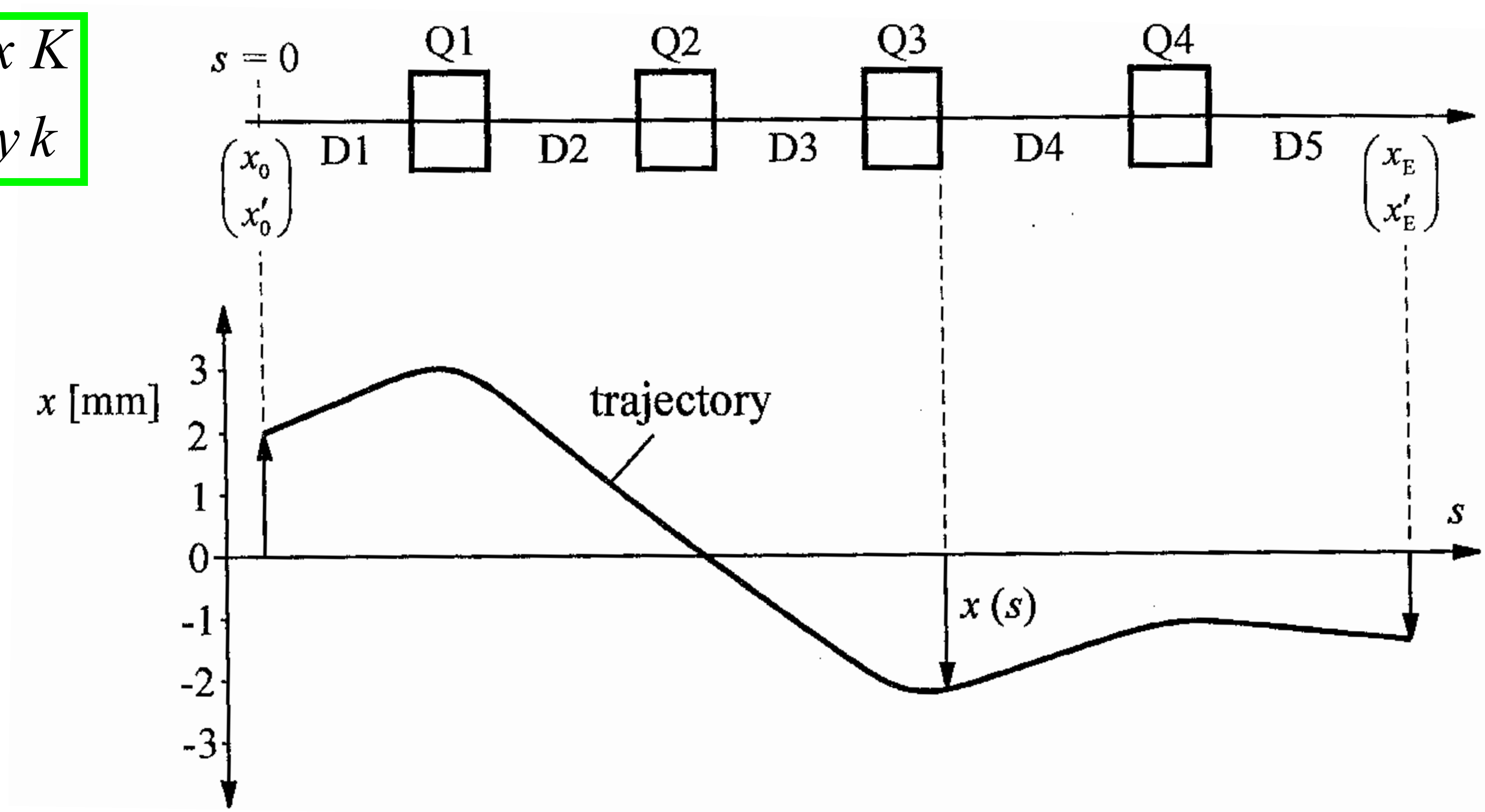
$$x'' = -x\,K$$
$$y'' = \;\; y\,k$$



$$x(s) = M_{11}(s)x_0 + M_{12}(s)x_0'$$
$$x(s) = \sqrt{2J\beta(s)}\,\sin(\psi(s) + \phi_0)$$

Twiss Parameters

$$x'' = -k\,x$$

$$x(s) = \sqrt{2J\beta(s)}\,\sin(\psi(s)+\phi_0)$$

$$x'(s) = \sqrt{\tfrac{2J}{\beta}}[\beta\psi'\cos(\psi(s)+\phi_0) - \alpha\sin(\psi(s)+\phi_0)] \quad \text{with} \quad \alpha = -\tfrac{1}{2}\beta'$$

$$x''(s) = \sqrt{\tfrac{2J}{\beta}}[(\beta\psi'' - 2\alpha\psi')\cos(\psi(s)+\phi_0) - (\alpha' + \tfrac{\alpha^2}{\beta} + \beta\psi'^2)\sin(\psi(s)+\phi_0)]$$

$$= \sqrt{\tfrac{2J}{\beta}}[-k\beta\sin(\psi(s)+\phi_0)]$$

$$\beta\psi'' - 2\alpha\psi' = \beta\psi'' + \beta'\psi' = (\beta\psi')' = 0 \quad \Rightarrow \quad \psi' = \tfrac{I}{\beta}$$

$$\alpha' + \gamma = k\beta \quad with \quad \gamma = \tfrac{I^2+\alpha^2}{\beta}$$

Universal choice: I=1!

$$\beta' = -2\alpha$$

$$\alpha' = k\beta - \gamma$$

$$\psi = \int_0^s \tfrac{I}{\beta(s')}ds'$$

$\alpha, \beta, \gamma, \psi$ are called Twiss parameters.

What are the initial conditions?

Phase Space Ellipse

Particles with a common J and different ϕ all lie on an ellipse in phase space:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \sqrt{2J}\begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{I}{\sqrt{\beta}} \end{pmatrix}\begin{pmatrix} \sin(\psi(s)+\phi_0) \\ \cos(\psi(s)+\phi_0) \end{pmatrix}$$

(Linear transform of a circle)

$$x_{\max} = \sqrt{2J\beta} \text{ at } x' = -\alpha\sqrt{\frac{2J}{\beta}}$$

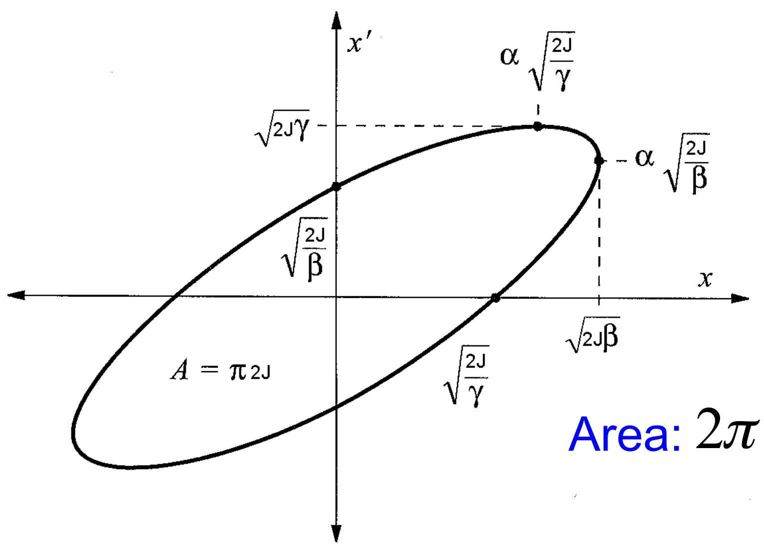
$$(x,x')\begin{pmatrix} \frac{I}{\sqrt{\beta}} & \frac{\alpha}{\sqrt{\beta}} \\ 0 & \sqrt{\beta} \end{pmatrix}\begin{pmatrix} \frac{I}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix}\begin{pmatrix} x \\ x' \end{pmatrix} = (x,x')\begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix}\begin{pmatrix} x \\ x' \end{pmatrix} = 2J$$

(Quadratic form)

$$\beta\gamma - \alpha^2 = I^2$$

Area: $2\pi J / I$

I=1 is therefore a useful choice!



What β is for x, γ is for x'

$$x'_{\max} = \sqrt{2J\gamma} \text{ at } x = -\alpha\sqrt{\frac{2J}{\gamma}}$$

Area: $2\pi J$ → $\int_0^{2\pi}\int_0^{J} dJ\, d\phi = 2\pi J = \int\int dx\, dx'$

The Beam Envelope

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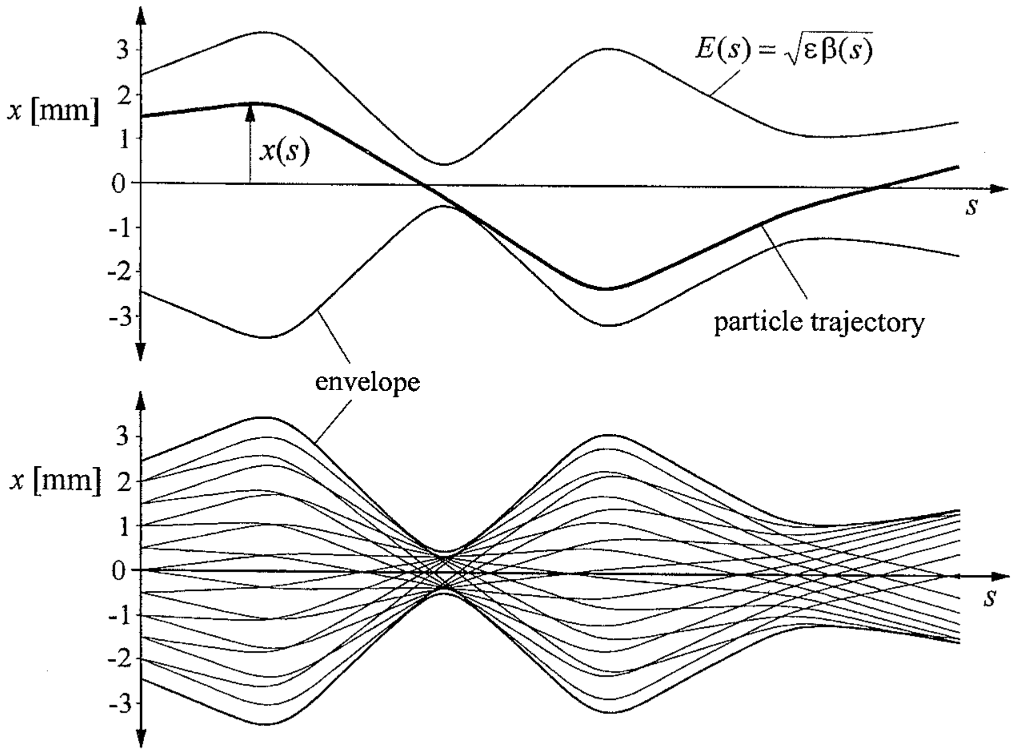


$$x(s) = \sqrt{2J\beta(s)}\,\sin(\psi(s) + \phi_0)$$

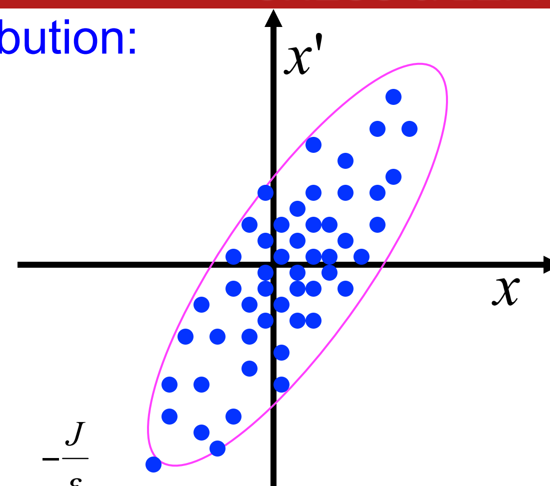
In any beam there is a distribution of initial parameters. If the particles with the largest J are distributed in ϕ over all angles, then the envelope of the beam is described by $\sqrt{2J_{max}\beta(s)}$

The initial conditions of β and α are chosen so that this is approximately the case.

Phase Space Distribution

Often one can fit a Gauss distribution to the particle distribution:

$$\rho(x,x') = \frac{1}{2\pi\varepsilon} e^{-\frac{\gamma x^2 + 2\alpha x x' + \beta x'^2}{2\varepsilon}}$$



The equi-density lines are then ellipses. And one chooses the starting conditions for β and α according to these ellipses!

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \sin\phi_0 \\ \cos\phi_0 \end{pmatrix} \qquad \rho(J,\phi_0) = \frac{1}{2\pi\varepsilon} e^{-\frac{J}{\varepsilon}}$$

$$\langle 1 \rangle = \frac{1}{2\pi\varepsilon} \int_0^{2\pi}\int_0^{\infty} e^{-J/\varepsilon}\, dJ\, d\phi_0 = 1$$

Initial beam distribution ⟶ initial α, β, γ

$$\langle x^2 \rangle = \frac{1}{2\pi\varepsilon} \int\int 2J\beta \sin\phi_0^2 e^{-J/\varepsilon}\, dJ\, d\phi_0 = \varepsilon\beta \longrightarrow \langle x'^2 \rangle = \varepsilon\gamma$$

$$\langle x x' \rangle = -\frac{1}{2\pi\varepsilon} \int\int 2J\alpha \sin\phi_0^2 e^{-J/\varepsilon}\, dJ\, d\phi_0 = \varepsilon\alpha$$

$$\varepsilon = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2}$$ is called the emittance.

Invariant of Motion

$$x(s) = \sqrt{2J\beta(s)}\,\sin(\psi(s)+\phi_0)$$

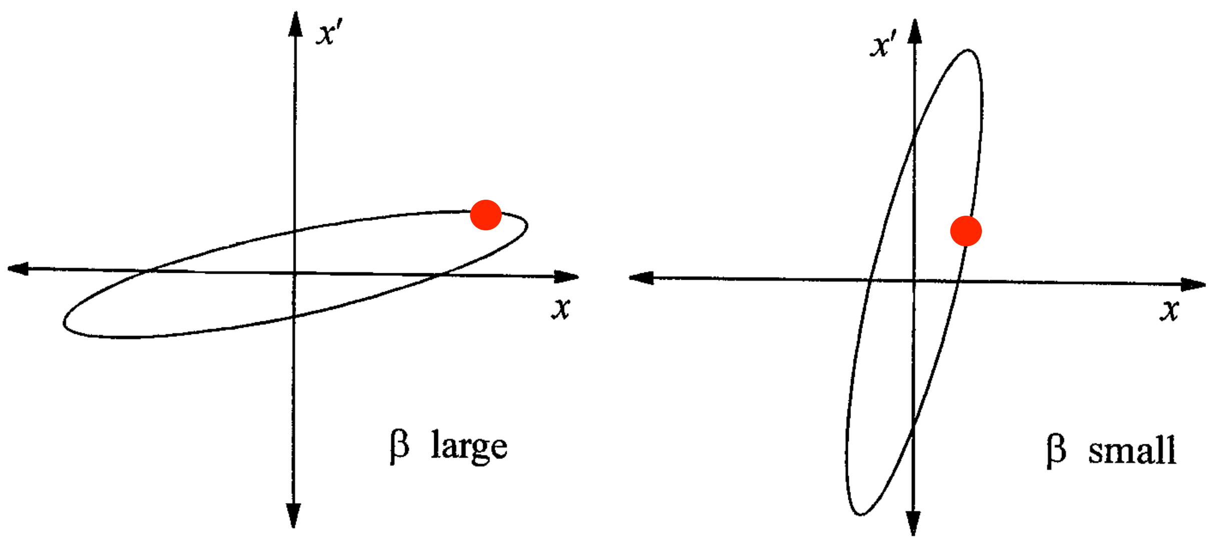
Where J and ϕ are given by the starting conditions x_0 and x'_0.

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = 2J$$

Leads to the invariant of motion:

$$f(x,x',s) = \gamma(s)x^2 + 2\alpha(s)xx' + \beta(s)x'^2 \quad \Rightarrow \quad \tfrac{d}{ds} f = 0$$

It is called the Courant-Snyder invariant.



Propagation of Twiss Parameters

$$(x_0, x_0')\begin{pmatrix}\gamma_0 & \alpha_0 \\ \alpha_0 & \beta_0\end{pmatrix}\begin{pmatrix}x_0 \\ x_0'\end{pmatrix} = 2J$$

$$(x, x')\begin{pmatrix}\gamma & \alpha \\ \alpha & \beta\end{pmatrix}\begin{pmatrix}x \\ x'\end{pmatrix} = 2J = (x_0, x_0')\underline{M}^T\begin{pmatrix}\gamma & \alpha \\ \alpha & \beta\end{pmatrix}\underline{M}\begin{pmatrix}x_0 \\ x_0'\end{pmatrix}$$

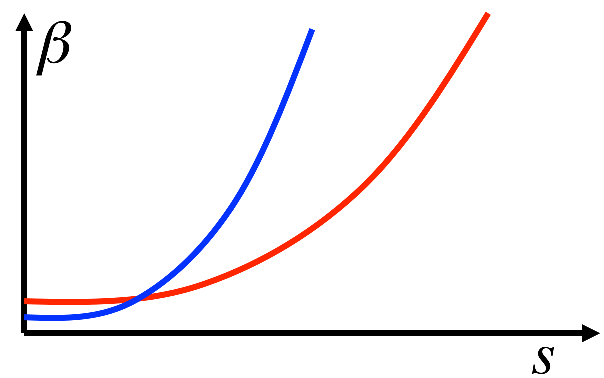
$$\begin{pmatrix}\gamma & \alpha \\ \alpha & \beta\end{pmatrix} = \underline{M}^{-T}\begin{pmatrix}\gamma_0 & \alpha_0 \\ \alpha_0 & \beta_0\end{pmatrix}\underline{M}^{-1}$$

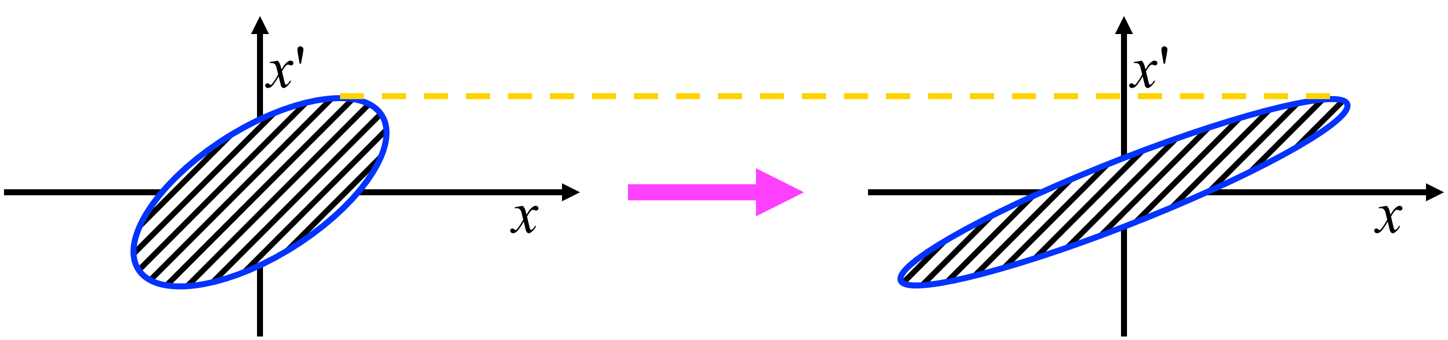
$$\boxed{\begin{pmatrix}\beta & -\alpha \\ -\alpha & \gamma\end{pmatrix} = \underline{M}\begin{pmatrix}\beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0\end{pmatrix}\underline{M}^T}$$

Twiss Parameters in a Drift

$$\begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} = \begin{pmatrix} \beta_0 - 2\alpha_0 s + \gamma_0 s^2 & \gamma_0 s - \alpha_0 \\ \gamma_0 s - \alpha_0 & \gamma_0 \end{pmatrix}$$

$$\beta = \beta_0^*[1 + (\tfrac{s}{\beta_0^*})^2] \quad \text{for} \quad \alpha_0^* = 0$$





Twiss Parameters after a thin Quadrupole

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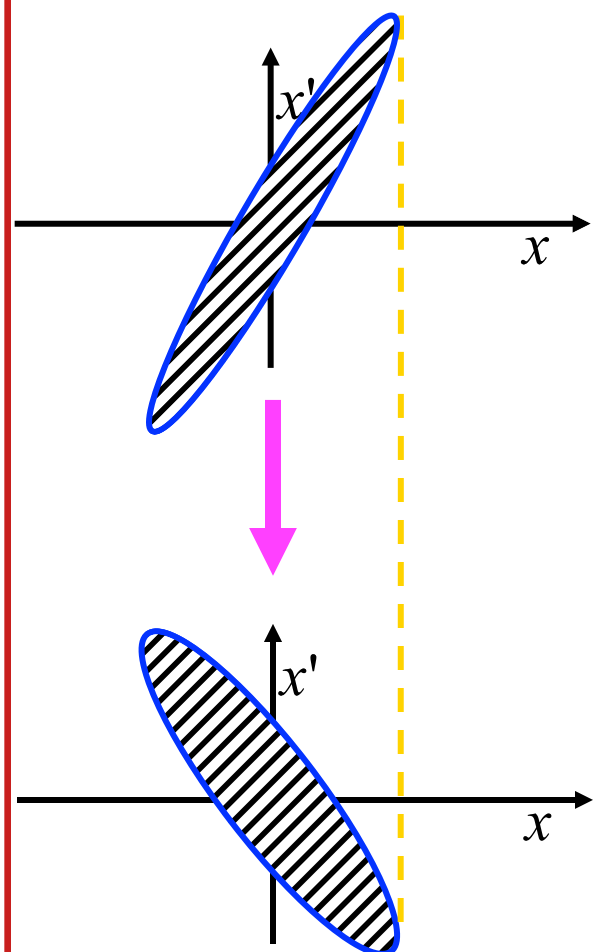
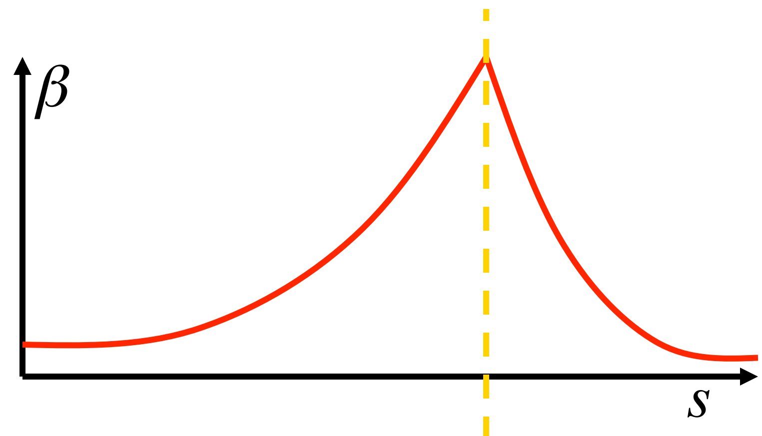


$$\begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix} \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix} \begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix}$$

$$\alpha = \alpha_0 + k\beta_0$$



From Twiss to Transport Matrix

$$\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sqrt{\beta_0} & 0 \\ -\frac{\alpha_0}{\sqrt{\beta_0}} & \frac{1}{\sqrt{\beta_0}} \end{pmatrix} \begin{pmatrix} \sin(\phi_0) \\ \cos(\phi_0) \end{pmatrix}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \sin(\psi(s)+\phi_0) \\ \cos(\psi(s)+\phi_0) \end{pmatrix}$$

$$= \sqrt{2J} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \cos\psi(s) & \sin\psi(s) \\ -\sin\psi(s) & \cos\psi(s) \end{pmatrix} \begin{pmatrix} \sin\phi_0 \\ \cos\phi_0 \end{pmatrix}$$

$$\underline{M}(s) = \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \cos\psi(s) & \sin\psi(s) \\ -\sin\psi(s) & \cos\psi(s) \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\beta_0}} & 0 \\ \frac{\alpha_0}{\sqrt{\beta_0}} & \sqrt{\beta_0} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{\frac{\beta}{\beta_0}}[\cos\psi + \alpha_0 \sin\psi] & \sqrt{\beta_0 \beta}\sin\psi \\ \sqrt{\frac{1}{\beta_0 \beta}}[(\alpha_0 - \alpha)\cos\psi - (1+\alpha_0\alpha)\sin\psi] & \sqrt{\frac{\beta_0}{\beta}}[\cos\psi - \alpha\sin\psi] \end{pmatrix}$$