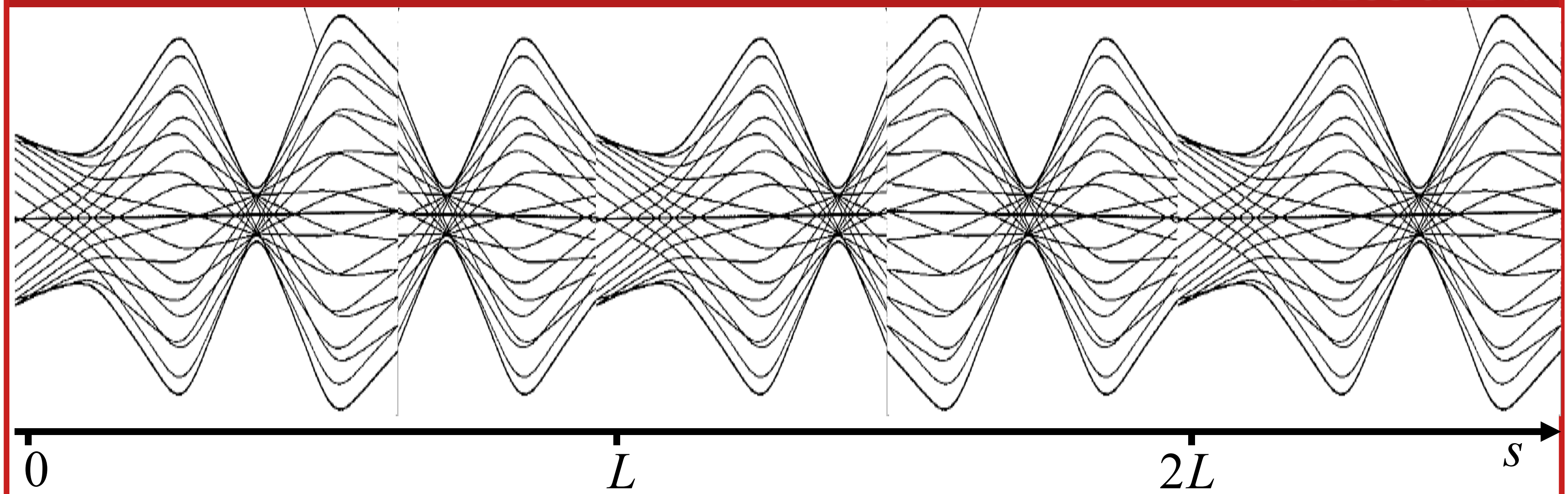
Periodic sections in an accelerator

CHESS & LEPP



$$\vec{z}(s) = \underline{M}(s,0)\vec{z}(0)$$

$$\vec{z}(L) = \underline{M}(L,0)\vec{z}(0)$$

$$\vec{z}(s+L) = \underline{M}_0(s)\vec{z}(s) \quad , \quad \underline{M}_0 = \underline{M}(s+L,s)$$

$$\vec{z}(s+nL) = \underline{M}_0^n(s)\vec{z}(s)$$

The Periodic Beta Function

If the particle distribution in a ring or any other periodic structure is stable, it is periodic from turn to turn.

$$\rho(x,x',s+L)=\rho(x,x',s)$$

To be matched to such a beam, the Twiss parameters α, β, γ must be the same after every turn.

$$\underline{M}(s,0)=\begin{pmatrix}\sqrt{\frac{\beta}{\beta_0}}[\cos\psi+\alpha_0\sin\psi] & \sqrt{\beta_0\beta}\sin\psi\\ \sqrt{\frac{1}{\beta_0\beta}}[(\alpha_0-\alpha)\cos\psi-(1+\alpha_0\alpha)\sin\psi] & \sqrt{\frac{\beta_0}{\beta}}[\cos\psi-\alpha\sin\psi]\end{pmatrix}$$

$$\underline{M}_p(s)=\begin{pmatrix}\cos\mu+\alpha\sin\mu & \beta\sin\mu\\ -\gamma\sin\mu & \cos\mu-\alpha\sin\mu\end{pmatrix}=\underline{1}\cos\mu+\begin{pmatrix}\alpha & \beta\\ -\gamma & -\alpha\end{pmatrix}\sin\mu$$

$$\mu=\psi(s+L)-\psi(s)$$

One Turn Matrix to Periodic Twiss

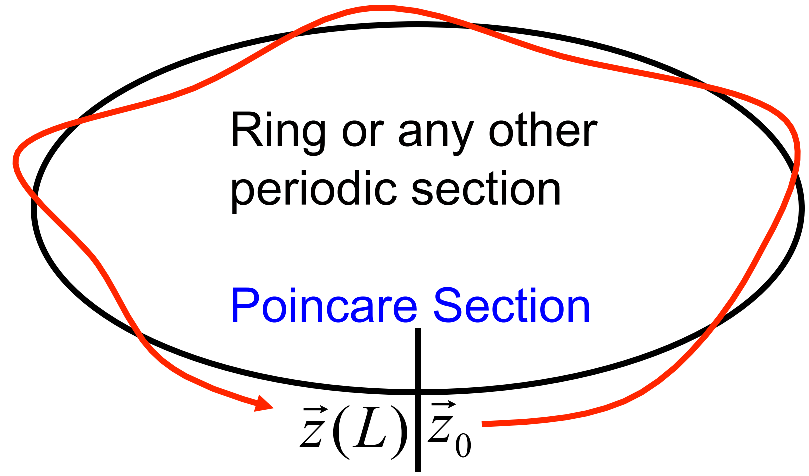
The periodic Twiss parameters are the solution of a nonlinear differential equation with periodic boundary conditions:

$$\beta' = -2\alpha \quad \text{with} \quad \beta(L) = \beta(0)$$

$$\alpha' = k\beta - \tfrac{1+\alpha^2}{\beta} \quad \text{with} \quad \alpha(L) = \alpha(0)$$

$$\mu = \int_0^L \tfrac{1}{\beta(\hat{s})} d\hat{s}$$

Note: $\beta(s) > 0$



$$\underline{M}_0(s) = \underline{1}\cos\mu + \underline{\beta}\sin\mu \; ; \; \underline{\beta} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$

$$\cos\mu = \tfrac{1}{2}\mathrm{Tr}[\underline{M}_0(s)]$$

$$\beta = \underline{M}_{0,12} \tfrac{1}{\sin\mu}$$

$$\alpha = (\underline{M}_{0,11} - \underline{M}_{0,22}) \tfrac{1}{2\sin\mu}$$

$$\gamma = \tfrac{1+\alpha^2}{\beta}$$

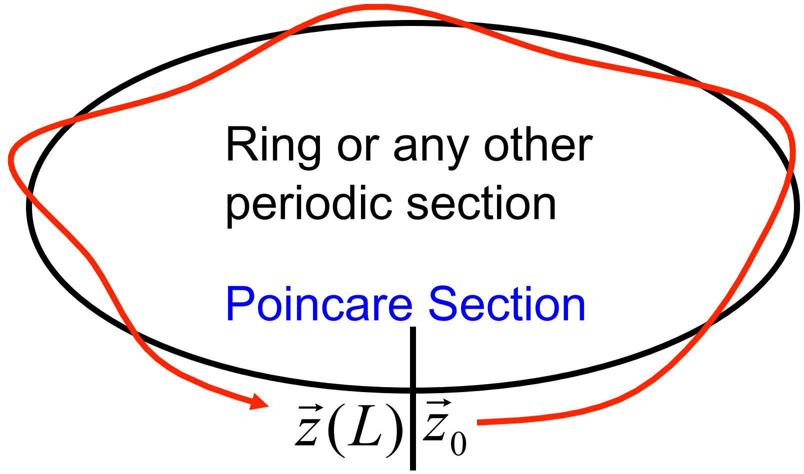
Stable beam motion and thus a periodic beta function can only exist when Tr[M]<2.

The Tune of a Periodic Accelerator

The betatron phase advance per turn devided by 2π is called the TUNE.

$$\mu = 2\pi\nu = \psi(s+L) - \psi(s)$$

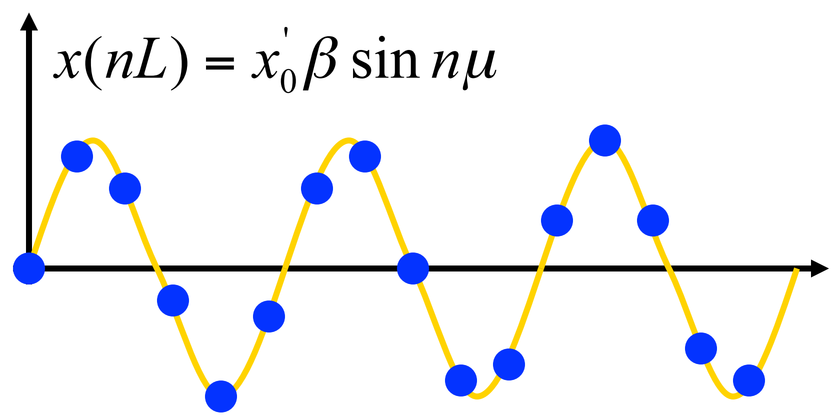
It is a property of the ring and does not depend on the azimuth s.



$$\underline{M}_0(s) = \underline{1}\cos\mu + \begin{pmatrix} \alpha(s) & \beta(s) \\ -\gamma(s) & -\alpha(s) \end{pmatrix} \sin\mu$$

$$2\cos\underline{\mu(s)} = \mathrm{Tr}[\underline{M}_0(s)] = \mathrm{Tr}[\underline{M}(s,0)\underline{M}_0(0)\underline{M}^{-1}(s,0)]$$

$$= \mathrm{Tr}[\underline{M}_0(0)] = 2\cos\underline{\mu(0)}$$



$$\underline{M}_0^n = \underline{1}\cos n\mu + \underline{\beta}\sin n\mu$$

Orbit Distortions for a One-pass Accelerator

$$x' = a$$

$$a' = -(\kappa^2 + k)x + \Delta f$$

The extra force can for example come from an erroneous dipole field or from a correction coil: $\Delta f = \frac{q}{p}\Delta B_y = \Delta\kappa$

Variation of constants: $\vec{z} = \underline{M}\vec{z}_0 + \Delta\vec{z}$ with $\Delta\vec{z} = \int_0^s \underline{M}(s-\hat{s})\begin{pmatrix} 0 \\ \Delta\kappa(\hat{s}) \end{pmatrix} d\hat{s}$

$$\Delta\vec{z} = \int_0^L \begin{pmatrix} -\sqrt{\beta\hat{\beta}}\sin\hat{\psi} \\ \sqrt{\frac{\hat{\beta}}{\beta}}[\cos\hat{\psi} + \alpha\sin\hat{\psi}] \end{pmatrix} \Delta\kappa(\hat{s})d\hat{s}$$

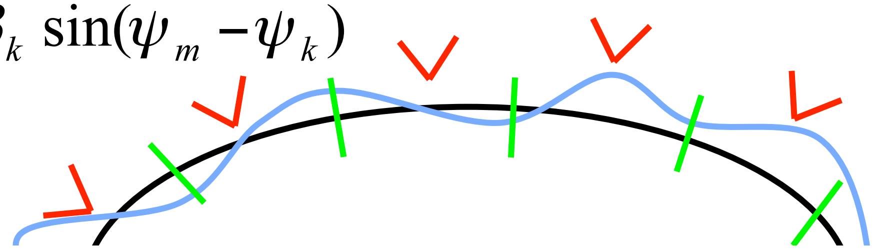
$$\boxed{\Delta x(s) = \sum_k \Delta\vartheta_k \sqrt{\beta(s)\beta_k}\sin(\psi(s) - \psi_k)}$$

Orbit Correction for a One-pass Accelerator

When the closed orbit $x_{\rm co}^{\rm old}(s_m)$ is measured at beam position monitors (BPMs, index m) and is influenced by corrector magnets (index k), then the monitor readings before and after changing the kick angles created in the correctors by $\Delta\vartheta_k$ are related by

$$x_{\rm co}^{\rm new}(s_m) = x_{\rm co}^{\rm old}(s_m) + \sum_k \Delta\vartheta_k \sqrt{\beta_m\beta_k}\sin(\psi_m - \psi_k)$$

$$= x_{\rm co}^{\rm old}(s_m) + \sum_k O_{mk}\Delta\vartheta_k$$



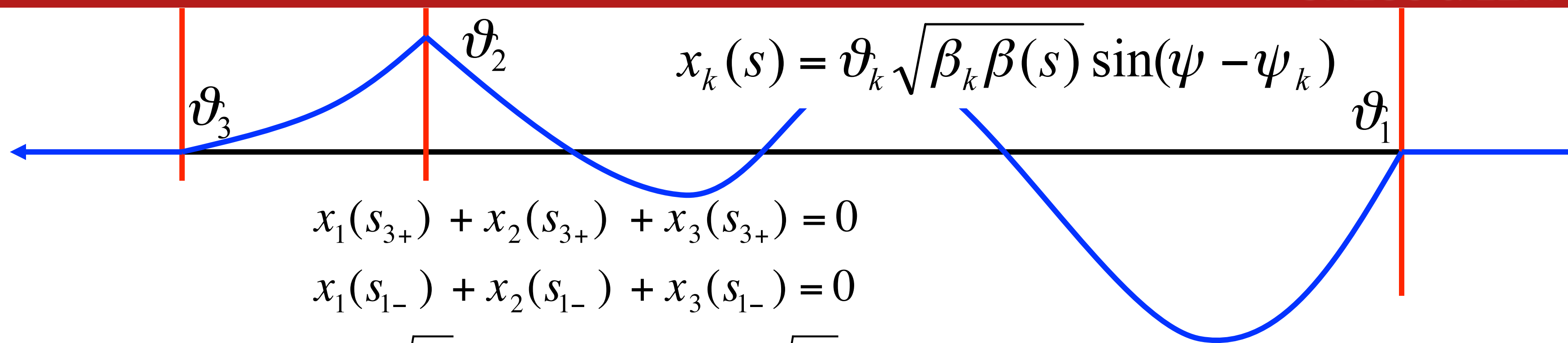
$$\vec{x}_{\rm co}^{\rm new} = \vec{x}_{\rm co}^{\rm old} + \underline{O}\Delta\vec{\vartheta}$$

$$\Delta\vec{\vartheta} = -\underline{O}^{-1}\vec{x}_{\rm co}^{\rm old} \quad\Rightarrow\quad \vec{x}_{\rm co}^{\rm new} = 0$$

It is often better not to try to correct the closed orbit at the the BPMs to zero in this way since

1. computation of the inverse can be numerically unstable, so that small errors in the old closed orbit measurement lead to a large error in the corrector coil settings.
2. A zero orbit at all BPMs can be a bad orbit inbetween BPMs

Closed Orbit Bumps



$$x_k(s) = \vartheta_k \sqrt{\beta_k \beta(s)} \sin(\psi - \psi_k)$$

$$x_1(s_{3+}) + x_2(s_{3+}) + x_3(s_{3+}) = 0$$

$$x_1(s_{1-}) + x_2(s_{1-}) + x_3(s_{1-}) = 0$$

$$\vartheta_1 \sqrt{\beta_1} \sin(\psi_3 - \psi_1) + \vartheta_2 \sqrt{\beta_2} \sin(\psi_3 - \psi_2) = 0$$

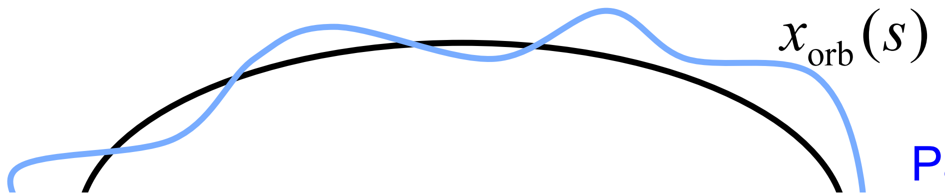
$$\vartheta_3 \sqrt{\beta_3} \sin(\psi_3 - \psi_1) + \vartheta_2 \sqrt{\beta_2} \sin(\psi_2 - \psi_1) = 0$$

$$\frac{\vartheta_1}{\vartheta_2} = -\frac{\sin(\psi_3 - \psi_2)/\sqrt{\beta_1}}{\sin(\psi_3 - \psi_1)/\sqrt{\beta_2}}$$

$$\frac{\vartheta_2}{\vartheta_3} = -\frac{\sin(\psi_3 - \psi_1)/\sqrt{\beta_2}}{\sin(\psi_2 - \psi_1)/\sqrt{\beta_3}}$$

$$\vartheta_1 : \vartheta_2 : \vartheta_3 = \beta_1^{-\frac{1}{2}} \sin\psi_{32} : -\beta_2^{-\frac{1}{2}} \sin\psi_{31} : \beta_3^{-\frac{1}{2}} \sin\psi_{21}$$

Oscillations around a distorted Orbit



Particles oscillate around this periodic orbit, not around the design orbit.

$$\vec{z} = \vec{z}_{\beta} + \vec{z}_{\text{orb}}$$

$$\vec{z}_{\text{orb}}(s) = \underline{M}\vec{z}_{\text{orb}}(0) + \Delta\vec{z}(s)$$

$$\vec{z}_{\beta}(s) + \vec{z}_{\text{orb}}(s) = \vec{z}(s) = \underline{M}\vec{z}(0) + \Delta\vec{z}(s) = \underline{M}[\vec{z}_{\beta}(0) + \vec{z}_{\text{orb}}(0)] + \Delta\vec{z}(s)$$

$$= \underline{M}\vec{z}_{\beta}(0) + \vec{z}_{\text{orb}}(s)$$

$$\boxed{\vec{z}_{\beta}(L) = \underline{M}_0\vec{z}_{\beta}(0)}$$

The distorted orbit does not change the linear transport matrix.

Dispersion Integral for One-pass Accelerators

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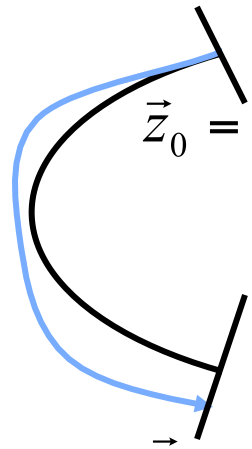
$$x' = a$$

$$a' = -(\kappa^2 + k)x + \kappa\delta$$



$$\vec{z} = \underline{M}\vec{z}_0 + \int_0^s \underline{M}(s-\hat{s})\begin{pmatrix} 0 \\ \delta\kappa(\hat{s}) \end{pmatrix} d\hat{s}$$

$$\Rightarrow \quad \vec{D}(s) = \int_0^s \underline{M}(s-\hat{s})\begin{pmatrix} 0 \\ \kappa(\hat{s}) \end{pmatrix} ds'$$

$$\Delta\kappa = \delta\kappa$$

$$D(s) = \sqrt{\beta(s)}\int_0^s \kappa(\hat{s})\sqrt{\beta(\hat{s})}\sin(\psi(s) - \psi(\hat{s}))d\hat{s}$$

The Closed Orbit of a Periodic Accelerator

$$x' = a$$

$$a' = -(\kappa^2 + k)x + \Delta f$$

The extra force can for example come from an erroneous dipole field or from a correction coil: $\Delta f = \frac{q}{p}\Delta B_y = \Delta\kappa$

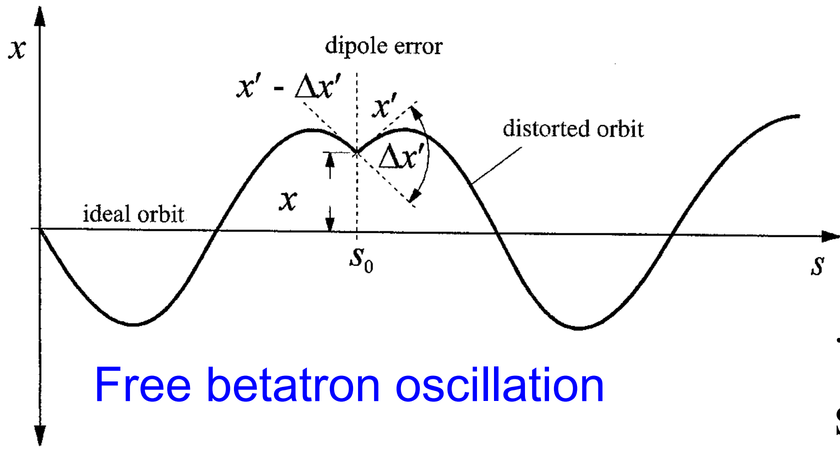
Variation of constants: $\vec{z} = \underline{M}\vec{z}_0 + \Delta\vec{z}$ with $\Delta\vec{z} = \int_0^s \underline{M}(s-\hat{s})\begin{pmatrix}0\\ \Delta\kappa(\hat{s})\end{pmatrix}d\hat{s}$

For the periodic or closed orbit: $\vec{z}_{co} = \underline{M}_0\vec{z}_{co} + \underline{M}_0\int_0^L \underline{M}^{-1}(\hat{s})\begin{pmatrix}0\\ \Delta\kappa(\hat{s})\end{pmatrix}d\hat{s}$

$$\vec{z}_{co} = [\underline{M}_0^{-1} - \underline{1}]^{-1}\int_0^L \underline{M}^{-1}(\hat{s})\begin{pmatrix}0\\ \Delta\kappa(\hat{s})\end{pmatrix}d\hat{s}$$

$$= \tfrac{1}{2-2\cos\mu}[(\cos\mu - 1)\underline{1} + \sin\mu\,\underline{\beta}]\int_0^L \begin{pmatrix}-\sqrt{\beta\hat{\beta}}\sin\hat{\psi}\\ \sqrt{\frac{\hat{\beta}}{\beta}}[\cos\hat{\psi} + \alpha\sin\hat{\psi}]\end{pmatrix}\Delta\kappa(\hat{s})d\hat{s}$$

Periodic Closed Orbit from One Kick



Free betatron oscillation

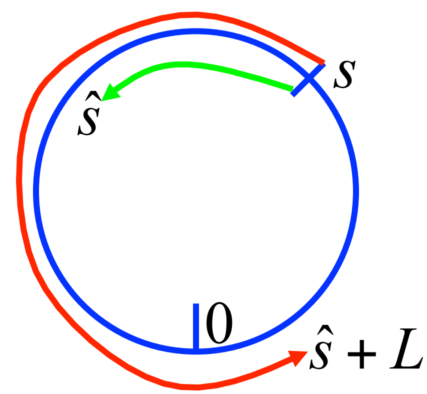
The oscillation amplitude J diverges when the tune ν is close to an integer.

$$x_{co}(s) = \text{sig}\Delta\vartheta_k A\sqrt{\beta}\sin(\psi-\psi_k+\tfrac{\pi}{2}-\tfrac{\mu}{2})$$

$$\text{sig} = \text{Sign(fractionl part of } \mu)$$

$$x'_{co}(s_k) - x'_{co}(s_k+L) = \Delta\vartheta_k$$

$$\text{sig}A\sin\tfrac{\mu}{2} = -\text{sig}A\sin\tfrac{\mu}{2} + \sqrt{\beta_k}$$



$$x_{co+}(s) = \Delta\vartheta_k \frac{\sqrt{\beta\beta_k}}{2\sin\frac{\mu}{2}}\cos(\psi-\psi_k-\tfrac{\mu}{2})$$

$$x_{co-}(s) = \Delta\vartheta_k \frac{\sqrt{\beta\beta_k}}{2\sin\frac{\mu}{2}}\cos(\psi-\psi_k+\tfrac{\mu}{2}) = \Delta\vartheta_k \frac{\sqrt{\beta\beta_k}}{2\sin\frac{\mu}{2}}\cos(|\psi-\psi_k|-\tfrac{\mu}{2})$$

Closed Orbit Correction

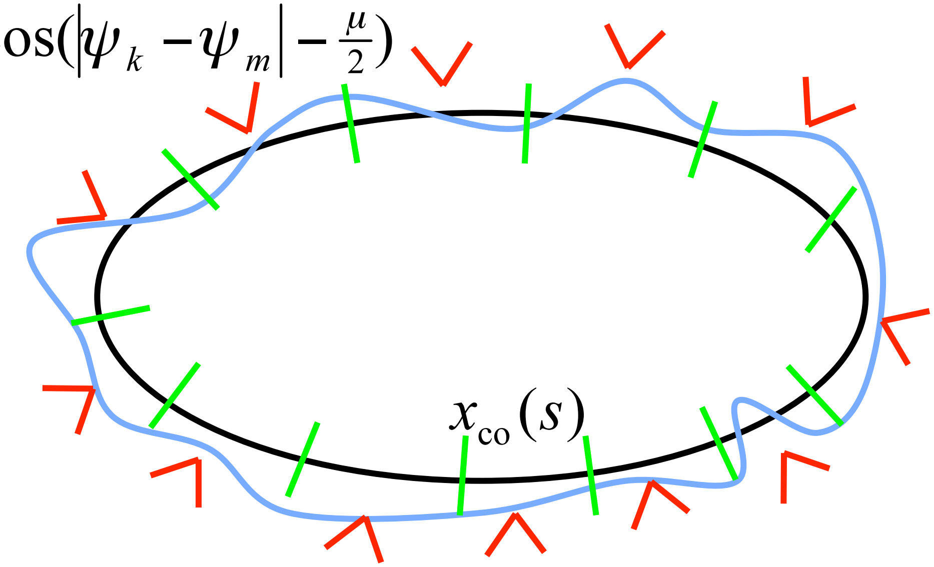
When the closed orbit $x_{co}^{old}(s_m)$ is measured at beam position monitors (BPMs, index m) and is influenced by corrector magnets (index k), then the monitor readings before and after changing the kick angles created in the correctors by $\Delta\vartheta_k$ are related by

$$x_{co}^{new}(s_m) = x_{co}^{old}(s_m) + \sum_k \Delta\vartheta_k \frac{\sqrt{\beta_m\beta_k}}{2\sin\frac{\mu}{2}}\cos(|\psi_k - \psi_m| - \tfrac{\mu}{2})$$

$$= x_{co}^{old}(s_m) + \sum_k O_{mk}\Delta\vartheta_k$$

$$\vec{x}_{co}^{new} = \vec{x}_{co}^{old} + \underline{O}\Delta\vec{\vartheta}$$

$$\Delta\vec{\vartheta} = -\underline{O}^{-1}\vec{x}_{co}^{old} \quad \Rightarrow \quad \vec{x}_{co}^{new} = 0$$

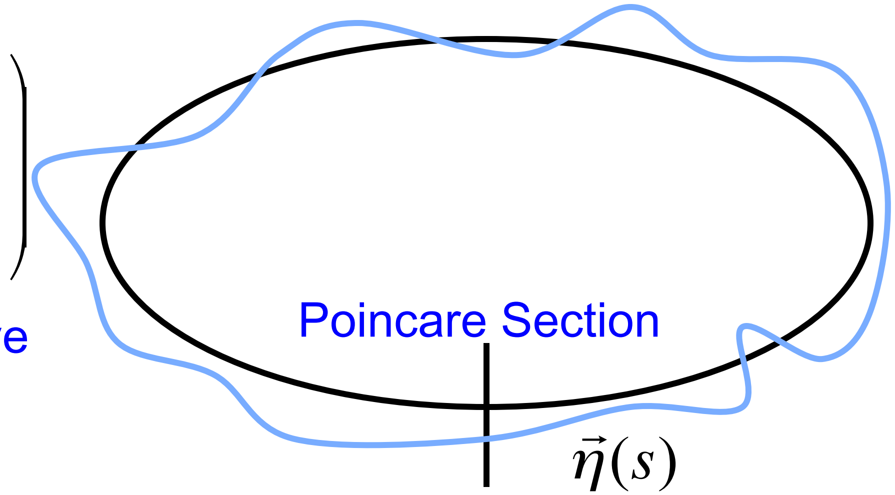


It is often better not to try to correct the closed orbit at the the BPMs to zero in this way since

1. computation of the inverse can be numerically unstable, so that small errors in the old closed orbit measurement lead to a large error in the corrector coil settings.
2. A zero orbit at all BPMs can be a bad orbit inbetween BPMs

The Periodic Dispersion

$$\begin{pmatrix} \underline{M}_{0x}\vec{z}_0 + \vec{D}(L)\delta \\ M_{56}\delta \\ \delta \end{pmatrix} = \begin{pmatrix} \underline{M}_{0x} & \vec{0} & \vec{D}(L) \\ \vec{T}^T & 1 & M_{56} \\ \vec{0}^T & 0 & 1 \end{pmatrix} \begin{pmatrix} \vec{z}_0 \\ 0 \\ \delta \end{pmatrix}$$



The periodic orbit for particles with relative energy deviation δ is

$$\vec{\eta}(0) = \underline{M}_0\vec{\eta}(0) + \vec{D}(L) \qquad \vec{\eta}(L) = \underline{M}_0\vec{\eta}(0) + \vec{D}(L) \quad \text{with} \quad \vec{\eta}(L) = \vec{\eta}(0)$$

$$\Downarrow$$

$$\boxed{\vec{\eta}(0) = [\underline{1} - \underline{M}_0(0)]^{-1}\vec{D}(L)}$$

Particles with energy deviation δ oscillates around this periodic orbit.

$$\vec{z} = \vec{z}_\beta + \delta\vec{\eta}$$

$$\vec{z}_\beta(L) + \delta\vec{\eta}(L) = \vec{z}(L) = \underline{M}_0\vec{z}(0) + \vec{D}(L)\delta = \underline{M}_0[\vec{z}_\beta(0) + \delta\vec{\eta}(0)] + \vec{D}(L)\delta$$

$$= \underline{M}_0\vec{z}_\beta(0) + \delta\vec{\eta}(L)$$

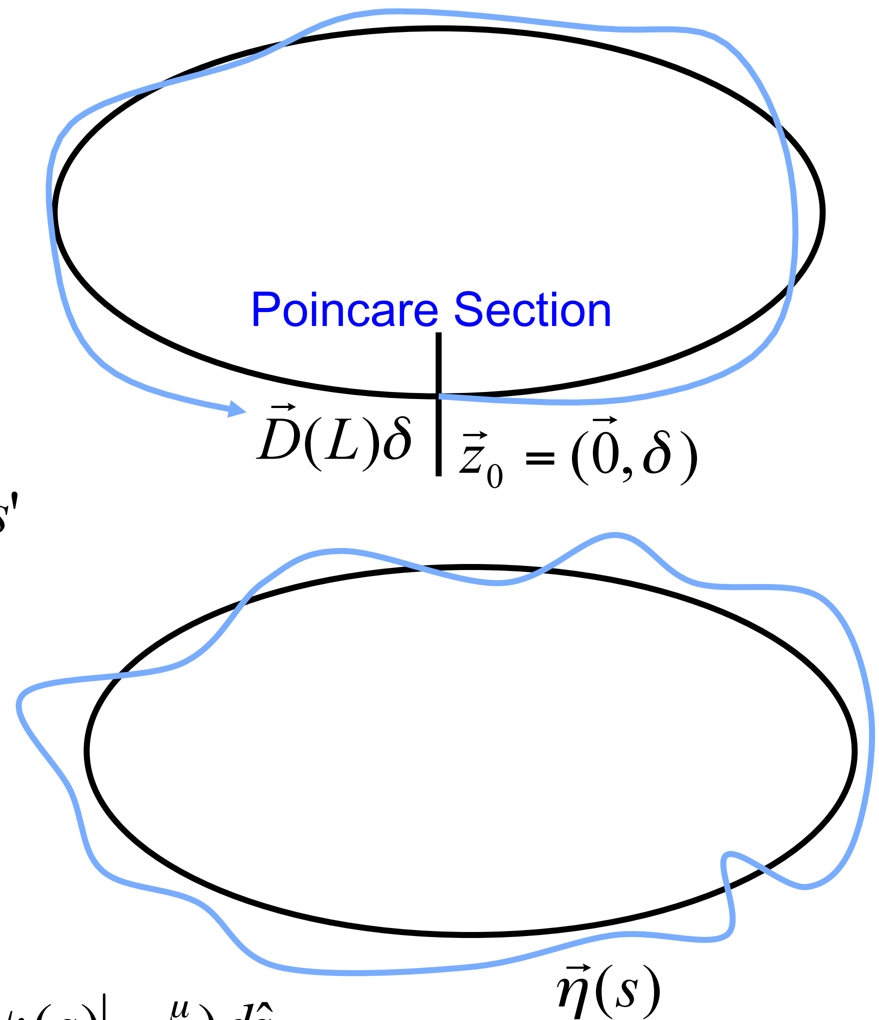
Periodic dispersion Integral

$$x' = a$$

$$a' = -(\kappa^2 + k)x + \kappa\delta$$

$$\vec{z} = \underline{M}\vec{z}_0 + \int_0^s \underline{M}(s-\hat{s})\begin{pmatrix} 0 \\ \delta\kappa(\hat{s}) \end{pmatrix} d\hat{s}$$

$$\Rightarrow \quad \vec{D}(L) = \int_0^L \underline{M}(L-\hat{s})\begin{pmatrix} 0 \\ \kappa(\hat{s}) \end{pmatrix} ds'$$



$$\Delta\kappa = \delta\kappa$$

$$\eta(s) = \frac{\sqrt{\beta(s)}}{2\sin\frac{\mu}{2}} \oint \kappa(\hat{s})\sqrt{\beta(\hat{s})}\cos(|\psi(\hat{s}) - \psi(s)| - \tfrac{\mu}{2}) d\hat{s}$$