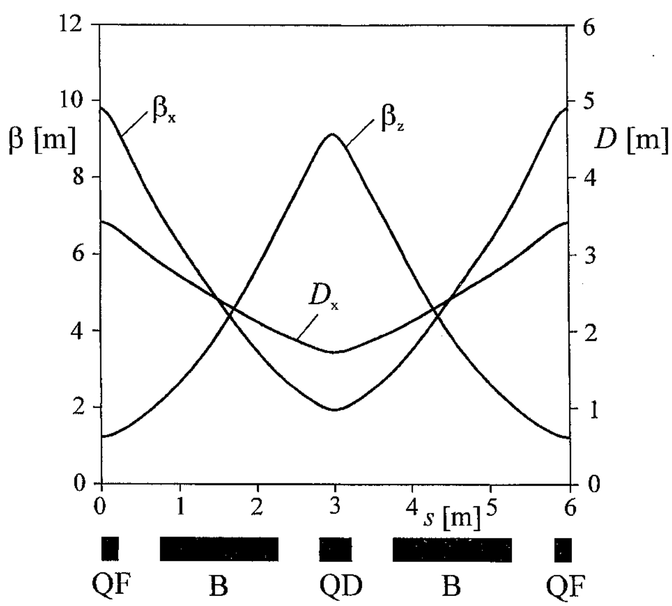
The FODO Cell

Alternating gradients allow focusing in both transverse plains. Therefore focusing and defocusing quadrupoles are usually alternated and interleaved with bending magnets.

$$L_{FoDo} \approx 6\text{m}\,, \quad \varphi \approx 22.5^\circ\,, \quad \mu_{FoDo} \approx \tfrac{\pi}{2}$$

$$\overline{\beta} \approx 3.8\text{m}$$

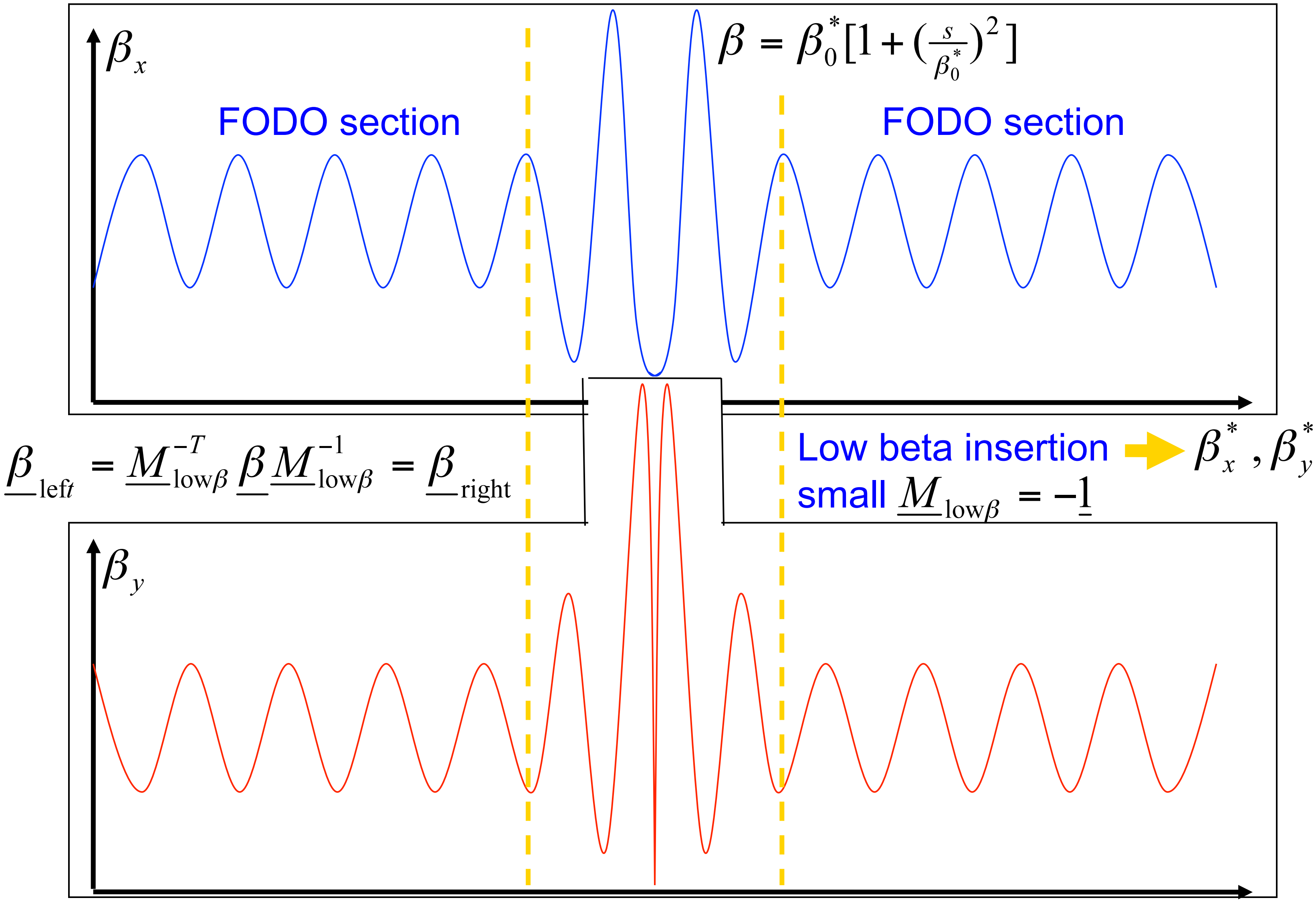
$$\beta_{\max} \approx 10.2\text{m}\,, \quad \beta_{\min} \approx 1.8\text{m}$$



$$\underline{M}_0 = \underline{M}_{FoDo}^N$$

The periodic beta function and dispersion for each FODO is also periodic for an accelerator section that consists of many FODO cells. Often large sections of an accelerator consist of FODOs.

The Low Beta Insertion



Quadrupole Errors

$$\vec{z}' = \underline{L}(s)\vec{z} + \Delta\vec{f}(\vec{z},s)$$

$$\vec{z}(s) = \vec{z}_H(s) + \int_0^s \underline{M}(s,\hat{s})\Delta\vec{f}(\vec{z},\hat{s})\,d\hat{s} \approx \vec{z}_H(s) + \int_0^s \underline{M}(s,\hat{s})\Delta\vec{f}(\vec{z}_H,\hat{s})\,d\hat{s}$$

$$x'' = -(\kappa^2 + k)x - \Delta k(s)x \quad \Rightarrow \quad \begin{pmatrix} x' \\ a' \end{pmatrix} = \begin{pmatrix} a \\ -(\kappa^2 + k)x \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ \Delta k(s) & 0 \end{pmatrix}\begin{pmatrix} x \\ a \end{pmatrix}$$

$$\vec{z}(s) = \left\{ \underline{M}(s) - \int_0^s \underline{M}(s,\hat{s}) \begin{pmatrix} 0 & 0 \\ \Delta k(\hat{s}) & 0 \end{pmatrix} \underline{M}(\hat{s},0)\,d\hat{s} \right\} \vec{z}_0$$

One quadrupole error:

$$\underline{M}(s,\hat{s}) + \Delta\underline{M}(s,\hat{s}) = \underline{M}(s,\hat{s}) - \underline{M}(s,\hat{s}) \begin{pmatrix} 0 & 0 \\ \Delta kl(\hat{s}) & 0 \end{pmatrix}$$

Quadrupole Error and Phase advance

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$$\Delta \underline{M}(s,\hat{s}) = -\underline{M}(s,\hat{s})\begin{pmatrix} 0 & 0 \\ \Delta kl(\hat{s}) & 0 \end{pmatrix}$$

$$\underline{M}(s) = \begin{pmatrix} \sqrt{\frac{\beta}{\beta_0}}[\cos\tilde{\psi} + \alpha_0 \sin\tilde{\psi}] & \sqrt{\beta_0\beta}\sin\tilde{\psi} \\ \sqrt{\frac{1}{\beta_0\beta}}[(\alpha_0 - \alpha)\cos\tilde{\psi} - (1+\alpha_0\alpha)\sin\tilde{\psi}] & \sqrt{\frac{\beta_0}{\beta}}[\cos\tilde{\psi} - \alpha\sin\tilde{\psi}] \end{pmatrix}$$

$$\Delta \underline{M}(s,\hat{s}) = -\Delta kl(\hat{s})\begin{pmatrix} \sqrt{\hat{\beta}\beta}\sin\psi & 0 \\ \sqrt{\frac{\hat{\beta}}{\beta}}[\cos\psi - \alpha\sin\psi] & 0 \end{pmatrix} \quad , \qquad \tilde{\psi} = \psi - \hat{\psi}$$

$$= \begin{pmatrix} \frac{\frac{1}{2}\Delta\beta[\cos\psi + \hat{\alpha}\sin\psi] + \Delta\psi\beta[\hat{\alpha}\cos\psi - \sin\psi]}{\sqrt{\hat{\beta}\beta}} & \sqrt{\hat{\beta}}\left(\frac{\frac{\Delta\beta}{2}\sin\psi + \Delta\psi\beta\cos\psi}{\sqrt{\beta}}\right) \\ \ldots & \ldots \end{pmatrix}$$

$$\Delta\psi = -\frac{\Delta\beta}{2\beta}\tan\tilde{\psi}$$

$$\tfrac{1}{2}\Delta\beta\cos\tilde{\psi} + \tfrac{1}{2}\Delta\beta\frac{\sin^2\tilde{\psi}}{\cos\tilde{\psi}} = \tfrac{1}{2}\Delta\beta\frac{1}{\cos\tilde{\psi}} = -\Delta kl(\hat{s})\beta\hat{\beta}\sin\tilde{\psi}$$

Quadrupole Error correction

$$\Delta\psi = \Delta k l(\hat{s})\hat{\beta}\sin^2(\psi - \hat{\psi})$$

→ More focusing always increases the tune

$$\frac{\Delta\beta}{\beta} = -\Delta k l(\hat{s})\hat{\beta}\sin(2[\psi - \hat{\psi}])$$

→ Beta beat oscillates twice as fast as orbit.

$$\Delta\psi = \sum_j \Delta k l_j \beta_j \tfrac{1}{2}[1 - \cos(2[\psi - \psi_j])]$$

$$\frac{\Delta\beta}{\beta} = -\sum_j \Delta k l_j(\hat{s})\beta_j \sin(2[\psi - \psi_j])$$

When beta functions and betatron phases have been measured at many places, quadrupoles can be changed with these formulas to correct the Twiss errors.

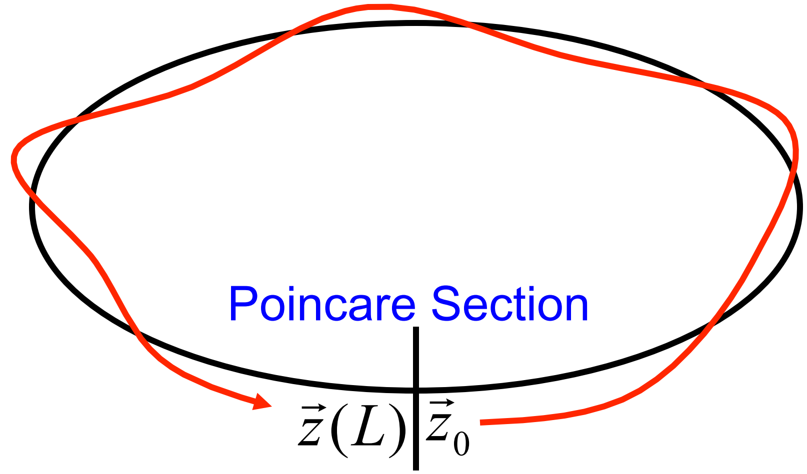
Tune shift in a ring due to quadrupole error

CHESS & LEPP

Average phase advance change per turn:

$$\overline{\Delta\psi} = \tfrac{1}{2}\Delta kl(\hat{s})\overline{\hat{\beta}} = \tfrac{1}{2}\Delta kl(\hat{s})\beta_0$$

Tune change: $\Delta\nu = \frac{1}{4\pi}\Delta kl(\hat{s})\beta_0$



$$\cos(\mu+\Delta\mu) \approx \cos\mu - \Delta\mu\sin\mu =$$

$$\tfrac{1}{2}\mathrm{Tr}[\begin{pmatrix} 1 & 0 \\ -\Delta kl(\hat{s}) & 1 \end{pmatrix}\begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}] = \cos\mu - \tfrac{1}{2}\Delta kl(\hat{s})\beta\sin\mu$$

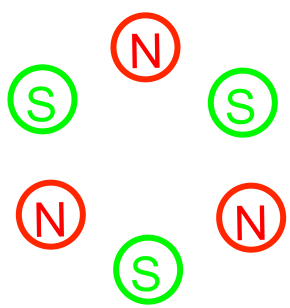
Oscillation frequencies can be measured relatively easily and accurately.

Measurement of beta function: Change k and measure tune.

Sextupoles (revisited)

$$\psi = \Psi_3 \operatorname{Im}\{(x-iy)^3\} = \Psi_3 \cdot (y^3 - 3x^2 y) \quad \Rightarrow \quad \vec{B} = -\vec{\nabla}\psi = \Psi_3\, 3 \begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix}$$

C_3 Symmetry



i) Sextupole fields hardly influence the particles close to the center, where one can linearize in x and y.

ii) In linear approximation a by Δx shifted sextupole has a quadrupole field.

iii) When Δx depends on the energy, one can build an energy dependent quadrupole.

$$\vec{B} = -\vec{\nabla}\psi = \Psi_3\, 3 \begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix}$$

$$x \mapsto \Delta x + x$$

$$\vec{B} \approx \Psi_3\, 3 \begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix} + 6\Psi_3 \Delta x \begin{pmatrix} y \\ x \end{pmatrix} + O(\Delta x^2)$$

$$k_2 = 3!\,\Psi_3 \Rightarrow k_1 = k_2 \Delta x$$

Chromaticity and its Correction

Chromaticity ξ = energy dependence of the tune

$$\upsilon(\delta) = \upsilon + \frac{\partial \upsilon}{\partial \delta}\delta + \dots$$

$$\xi = \frac{\partial \upsilon}{\partial \delta} \quad \text{with} \quad \upsilon = \frac{\mu}{2\pi}$$

Natural chromaticity ξ_0 = energy dependence of the tune due to quadrupoles only

$$\xi_{x0} = -\frac{1}{4\pi}\oint \beta_x(\hat{s})k_1(\hat{s})d\hat{s} \qquad \xi_{y0} = \frac{1}{4\pi}\oint \beta_y(\hat{s})k_1(\hat{s})d\hat{s}$$

Particles with energy difference oscillate around the periodic dispersion leading to a quadrupole effect in sextupoles that also shifts the tune:

$$\xi_x = \frac{1}{4\pi}\oint \beta_x(-k_1 + \eta_x k_2)d\hat{s} \qquad \xi_y = \frac{1}{4\pi}\oint \beta_y(k_1 - \eta_x k_2)d\hat{s}$$

Typically the the chormaticity ξ is chosen to be slightly positive, between 0 and 3.

Perturbations

$$\begin{pmatrix} x' \\ a' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -K & 0 \end{pmatrix} \begin{pmatrix} x \\ a \end{pmatrix} + \begin{pmatrix} 0 \\ \Delta f \end{pmatrix}$$

$$\begin{pmatrix} x \\ a \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \sin(\psi + \phi_0) \\ \cos(\psi + \phi_0) \end{pmatrix} = \sqrt{2J}\, \underline{\beta}\, \vec{S}$$

This would be a solution with constant J and ϕ when $\Delta f=0$.

Variation of constants:

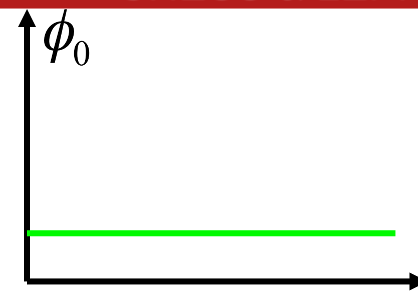
$$\frac{J'}{\sqrt{2J}} \underline{\beta}\, \vec{S} + \sqrt{2J}\, \phi_0' \begin{pmatrix} 0 & \sqrt{\beta} \\ -\frac{1}{\sqrt{\beta}} & -\frac{\alpha}{\sqrt{\beta}} \end{pmatrix} \vec{S} = \begin{pmatrix} 0 \\ \Delta f \end{pmatrix}$$

$$\frac{J'}{\sqrt{2J}} \vec{S} + \sqrt{2J}\, \phi_0' \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \vec{S} = \underline{\beta}^{-1} \begin{pmatrix} 0 \\ \Delta f \end{pmatrix} \quad \text{with} \quad \underline{\beta}^{-1} = \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix}$$

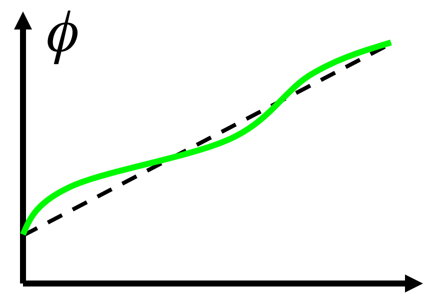
$$\frac{J'}{\sqrt{2J}} = \cos(\psi + \phi_0)\sqrt{\beta}\,\Delta f \quad , \quad \sqrt{2J}\, \phi_0' = -\sin(\psi + \phi_0)\sqrt{\beta}\,\Delta f$$

Simplification of linear motion

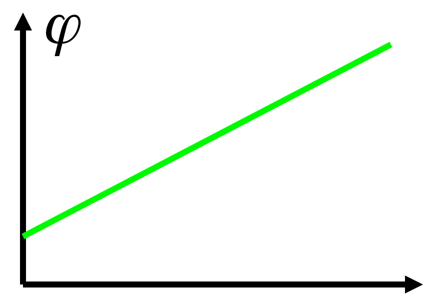
$$\begin{pmatrix} x \\ a \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \sin(\psi + \phi_0) \\ \cos(\psi + \phi_0) \end{pmatrix} \Rightarrow \begin{matrix} J' = 0 \\ \phi_0' = 0 \end{matrix}$$



$$\begin{pmatrix} x \\ a \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \sin\phi \\ \cos\phi \end{pmatrix} \Rightarrow \begin{matrix} J' = 0 \\ \phi' = \frac{1}{\beta} \end{matrix}$$



$$\begin{pmatrix} x \\ a \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \sin(\psi - \mu \frac{s}{L} + \varphi) \\ \cos(\psi - \mu \frac{s}{L} + \varphi) \end{pmatrix} \Rightarrow \begin{matrix} J' = 0 \\ \varphi' = \mu \frac{1}{L} \end{matrix}$$



$$\tilde{\psi} = \psi - \mu \tfrac{s}{L} \Rightarrow \tilde{\psi}(s + L) = \tilde{\psi}(s)$$

Corresponds to Floquet's Theorem

Quasi-periodic Perturbation

$$J' = \cos(\psi + \phi_0)\sqrt{2J\beta}\,\Delta f \quad , \quad \phi_0' = -\sin(\psi + \phi_0)\sqrt{\tfrac{\beta}{2J}}\,\Delta f$$

$$J' = \cos(\tilde{\psi} + \varphi)\sqrt{2J\beta}\,\Delta f \quad , \quad \varphi' = \mu\tfrac{1}{L} - \sin(\tilde{\psi} + \varphi)\sqrt{\tfrac{\beta}{2J}}\,\Delta f$$

New independent variable $\vartheta = 2\pi\frac{s}{L}$

$$\tfrac{d}{d\vartheta}J = \cos(\tilde{\psi} + \varphi)\sqrt{2J\beta}\,\Delta f\,\tfrac{L}{2\pi} \quad , \quad \tfrac{d}{d\vartheta}\varphi = \upsilon - \sin(\tilde{\psi} + \varphi)\sqrt{\tfrac{\beta}{2J}}\,\Delta f\,\tfrac{L}{2\pi}$$

$$\Delta f(x) = \Delta f(\sqrt{2J\beta}\sin(\tilde{\psi} + \varphi))$$

The perturbations are 2π periodic in ϑ and in φ

φ is approximately $\varphi \approx \upsilon \cdot \vartheta$

For irrational ν, the perturbations are quasi-periodic.