Tune Shift with Amplitude

$$\frac{d}{d\vartheta}J = \cos(\tilde{\Psi}+\varphi)\sqrt{2J\beta}\,\Delta f\,\frac{L}{2\pi} \quad , \quad \frac{d}{d\vartheta}\varphi = \upsilon - \sin(\tilde{\Psi}+\varphi)\sqrt{\frac{\beta}{2J}}\,\Delta f\,\frac{L}{2\pi}$$

$$\frac{d}{d\vartheta}\varphi = \partial_J H \quad , \quad \frac{d}{d\vartheta}J = -\partial_\phi H \quad , \quad H(\varphi,J,\vartheta) = \upsilon\cdot J - \frac{L}{2\pi}\int_0^x \Delta f(\hat{x},s)\,d\hat{x}$$

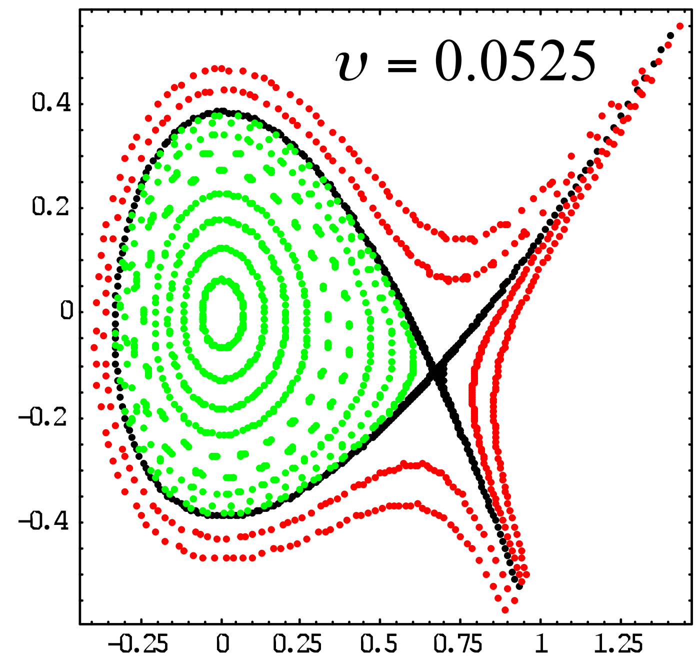
The motion remains Hamiltonian in the perturbed coordinates !

If there is a part in $\partial_J H$ that does not depend on φ, s $\Rightarrow$ Tune shift

The effect of other terms tends to average out.

$$\varphi(\vartheta) - \varphi_0 \approx \vartheta\cdot\partial_J\langle H\rangle_{\varphi,\vartheta}(J)$$

$$\upsilon(J) = \upsilon + \partial_J\langle\Delta H\rangle_{\varphi,\vartheta}(J)$$



Tune Shift Examples

$$H(\varphi,J)=\upsilon\cdot J-\tfrac{L}{2\pi}\int_0^x \Delta f(\hat{x},s)\,d\hat{x} \quad , \quad \Delta\upsilon(J)=\partial_J\langle\Delta H\rangle_{\varphi,\vartheta}$$

Quadrupole: $\Delta f=-\Delta k\,x$

$$\Delta H=\tfrac{L}{2\pi}\Delta k\tfrac{1}{2}x^2=\tfrac{L}{2\pi}\Delta k\,J\beta\sin^2(\tilde{\psi}+\varphi)$$

$$\langle\Delta H\rangle_{\varphi,\vartheta}=\tfrac{1}{2\pi}\int_0^{2\pi}\Delta k\beta\,d\vartheta\,L\frac{J}{4\pi}=\int_0^L\Delta k\beta\,ds\frac{J}{4\pi}\Rightarrow\Delta\upsilon=\tfrac{1}{4\pi}\oint\Delta k\beta\,ds$$

Sextupole: $\Delta f=-k_2\tfrac{1}{2}x^2$

$$\Delta H=\tfrac{L}{2\pi}k_2\tfrac{1}{3!}x^3=\tfrac{L}{2\pi}k_2\tfrac{1}{3!}\sqrt{2J\beta}^{\,3}\sin^3(\tilde{\psi}+\varphi)$$

$$\langle\Delta H\rangle_{\varphi,\vartheta}=0\quad\Rightarrow\quad\Delta\upsilon=0$$

Octupole: $\Delta f=-k_3\tfrac{1}{3!}x^3$

$$\Delta H=\tfrac{L}{2\pi}k_3\tfrac{1}{4!}x^4=\tfrac{L}{2\pi}k_3\tfrac{1}{3!}(J\beta)^2\sin^4(\tilde{\psi}+\varphi)$$

$$\langle\Delta H\rangle_{\varphi,\vartheta}=\tfrac{J^2}{3!2\pi}\oint k_3\beta^2ds\left\langle\tfrac{1}{2^4}(e^{i\varphi}-e^{-i\varphi})^4\right\rangle_\varphi\Rightarrow\Delta\upsilon=J\tfrac{1}{16\pi}\oint k_3\beta^2ds$$

Nonlinear Resonances

$$\frac{d}{d\vartheta}J = \cos(\tilde{\psi}+\varphi)\sqrt{2J\beta}\,\Delta f\,\frac{L}{2\pi} \quad , \quad \frac{d}{d\vartheta}\varphi = \upsilon - \sin(\tilde{\psi}+\varphi)\sqrt{\frac{\beta}{2J}}\,\Delta f\,\frac{L}{2\pi}$$

$$\frac{d}{d\vartheta}\varphi = \partial_J H \quad , \quad \frac{d}{d\vartheta}J = -\partial_\phi H \quad , \quad H(\varphi,J,\vartheta) = \upsilon\cdot J - \frac{L}{2\pi}\int_0^x \Delta f(\hat{x},s)\,d\hat{x}$$

The effect of the perturbation is especially strong when

$\cos(\tilde{\psi}+\varphi)\sqrt{\beta}\,\Delta f$ or $\sin(\tilde{\psi}+\varphi)\sqrt{\beta}\,\Delta f$

has contributions that hardly change, i.e. the change of

$\sqrt{\beta(\vartheta)}\Delta f(x(\vartheta),\vartheta)$ is in resonance with the rotation angle $\varphi(\vartheta)$.

Periodicity allows Fourier expansion:

$$H(\varphi,J,\vartheta) = \sum_{n,m=-\infty}^{\infty}\widehat{H}_{nm}(J)\,e^{i[n\vartheta+m\varphi]} = \sum_{n,m=-\infty}^{\infty}H_{nm}(J)\cos(n\vartheta+m\varphi+\Psi_{nm}(J))$$

$H_{00}(J) = \langle H(\varphi,J,s)\rangle_{\varphi,s} \Rightarrow$ Tune shift

Nonlinear Motion

Sextupoles cause nonlinear dynamics, which can be chaotic and unstable.

$$\begin{pmatrix} x_{n+1} \\ x'_{n+1} \end{pmatrix} = \underline{M}_0 \left[\begin{pmatrix} x_n \\ x'_n \end{pmatrix} - \frac{k_2 l_s}{2} \begin{pmatrix} 0 \\ x_n^{\ 2} \end{pmatrix} \right] \qquad \begin{pmatrix} x_n \\ x'_n \end{pmatrix} = \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \hat{x}_n \\ \hat{x}'_n \end{pmatrix}$$

$$\begin{pmatrix} \hat{x}_{n+1} \\ \hat{x}'_{n+1} \end{pmatrix} = \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix} \left[\begin{pmatrix} \hat{x}_n \\ \hat{x}'_n \end{pmatrix} - \frac{k_2 l_s}{2} \sqrt{\beta} \begin{pmatrix} 0 \\ \beta\, \hat{x}_n^2 \end{pmatrix} \right]$$

$$\begin{pmatrix} \hat{x}_f \\ \hat{x}'_f \end{pmatrix} = \frac{k_2 l_s}{2} \beta^{\frac{3}{2}} \begin{pmatrix} 1-\cos\mu & \sin\mu \\ -\sin\mu & 1-\cos\mu \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \hat{x}_f^2 \end{pmatrix} = \frac{k_2 l_s}{2} \beta^{\frac{3}{2}} \frac{1}{2\sin\frac{\mu}{2}} \begin{pmatrix} -\cos\frac{\mu}{2} \\ \sin\frac{\mu}{2} \end{pmatrix} \hat{x}_f^2$$

$$\left. \begin{aligned} \hat{x}_f &= -\frac{4}{k_2 l_s} \beta^{-\frac{3}{2}} \tan\frac{\mu}{2} \\ \hat{x}'_f &= \frac{4}{k_2 l_s} \beta^{-\frac{3}{2}} \tan^2\frac{\mu}{2} \end{aligned} \right\} \hat{x} = \hat{x}_f + \Delta\hat{x} \qquad J_f = \tfrac{1}{2}(\hat{x}_f^2 + \hat{x}'^2_f) = \frac{1}{2\beta^3} \left(\frac{4}{k_2 l_s} \frac{\tan\frac{\mu}{2}}{\cos\frac{\mu}{2}} \right)^2$$

$$\begin{pmatrix} \Delta\hat{x}_{n+1} \\ \Delta\hat{x}'_{n+1} \end{pmatrix} = \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix} \left[\begin{pmatrix} \Delta\hat{x}_n \\ \Delta\hat{x}'_n \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{k_2 l_s}{2} \beta^{\frac{3}{2}} \Delta\hat{x}_n^2 - 4\tan\frac{\mu}{2} \Delta\hat{x}_n \end{pmatrix} \right]$$

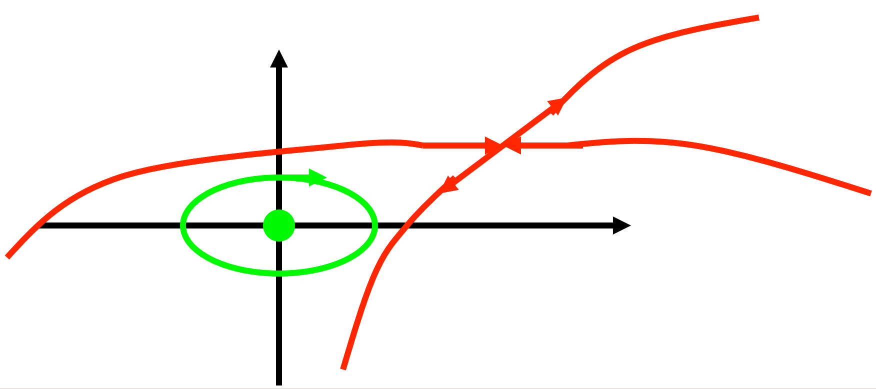
The Dynamic Aperture

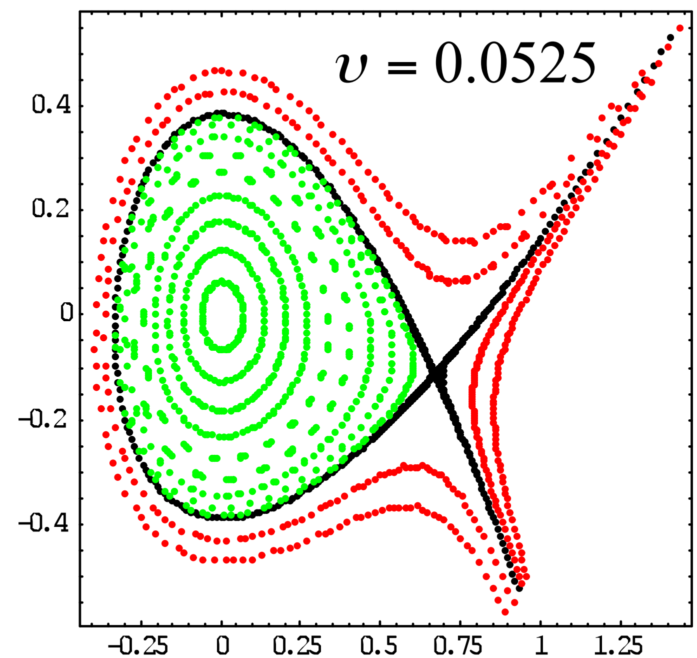
$$\begin{pmatrix}\Delta\hat{x}_{n+1}\\ \Delta\hat{x}'_{n+1}\end{pmatrix}=\begin{pmatrix}\cos\mu & \sin\mu\\ -\sin\mu & \cos\mu\end{pmatrix}\left[\begin{pmatrix}\Delta\hat{x}_n\\ \Delta\hat{x}'_n\end{pmatrix}-\begin{pmatrix}0\\ \frac{k_2 l_s}{2}\beta^{\frac{3}{2}}\Delta\hat{x}_n^2-4\tan\frac{\mu}{2}\Delta\hat{x}_n\end{pmatrix}\right]$$

$$\begin{pmatrix}\Delta\hat{x}_{n+1}\\ \Delta\hat{x}'_{n+1}\end{pmatrix}=\begin{pmatrix}\cos\mu+4\sin\mu\tan\frac{\mu}{2} & \sin\mu\\ -\sin\mu+4\cos\mu\tan\frac{\mu}{2} & \cos\mu\end{pmatrix}\left[\begin{pmatrix}\Delta\hat{x}_n\\ \Delta\hat{x}'_n\end{pmatrix}-\begin{pmatrix}0\\ \frac{k_2 l_s}{2}\beta^{\frac{3}{2}}\Delta\hat{x}_n^2\end{pmatrix}\right]$$

$$Tr[\underline{M}]=2\frac{\cos\frac{\mu}{2}(1+2\sin^2\frac{\mu}{2})}{\cos\frac{\mu}{2}}\geq 2$$

The additional fixed point is unstable !



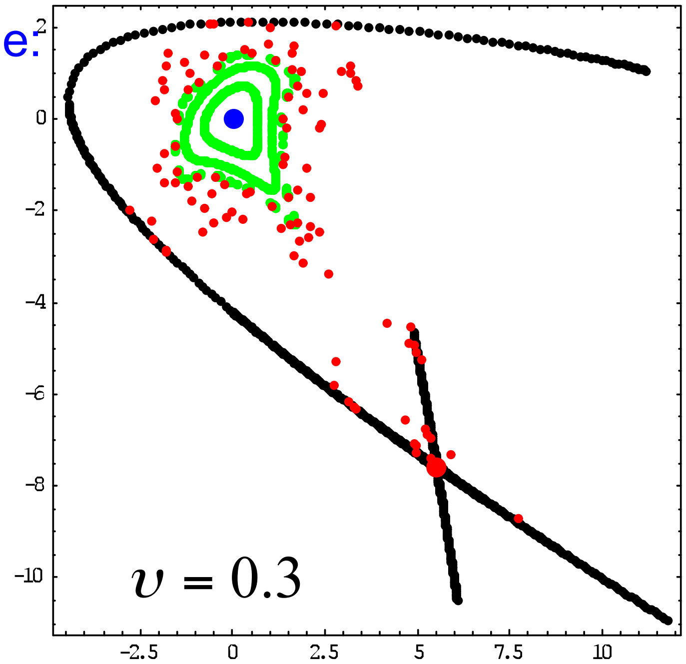


Sextupole Aperture

If the chormaticity is corrected by a single sextupole:

$$\xi_x = \xi_{0x} + \frac{1}{4\pi}\beta_x\eta_x k_2 l \approx 0$$

$$J_f = \frac{1}{2\beta^3}\left(\frac{4}{k_2 l_s}\frac{\tan\frac{\mu}{2}}{\cos\frac{\mu}{2}}\right)^2 \approx \frac{1}{2\beta}\left(\frac{\eta}{\xi_0\pi}\frac{\sin\frac{\mu}{2}}{\cos^2\frac{\mu}{2}}\right)^2$$



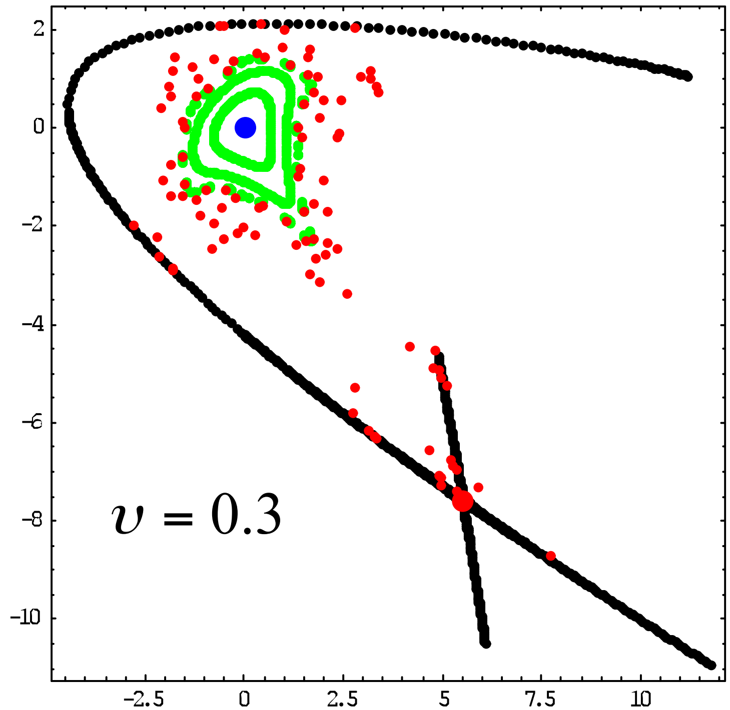
Often the dynamic aperture is much smaller than the fixed point indicates !

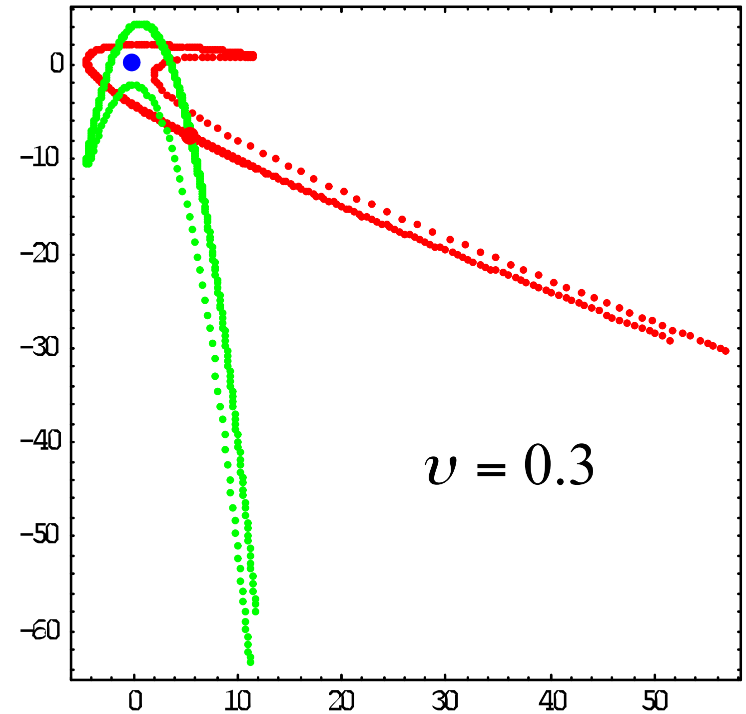
When many sextupoles are used: $$\xi_{0x} + \frac{N}{4\pi}\beta_x\eta_x k_2 l \approx 0$$

The sum of all k_2^2 is then reduced to about $$\sum (k_2 l\beta)^2 \approx N(k_2 l\beta)^2 \approx \frac{1}{N}\left(\frac{4\pi}{\eta}\xi_0\right)^2$$

The dynamic aperture is therefore greatly increased when distributed sextupoles are used.

Sextupole Extraction



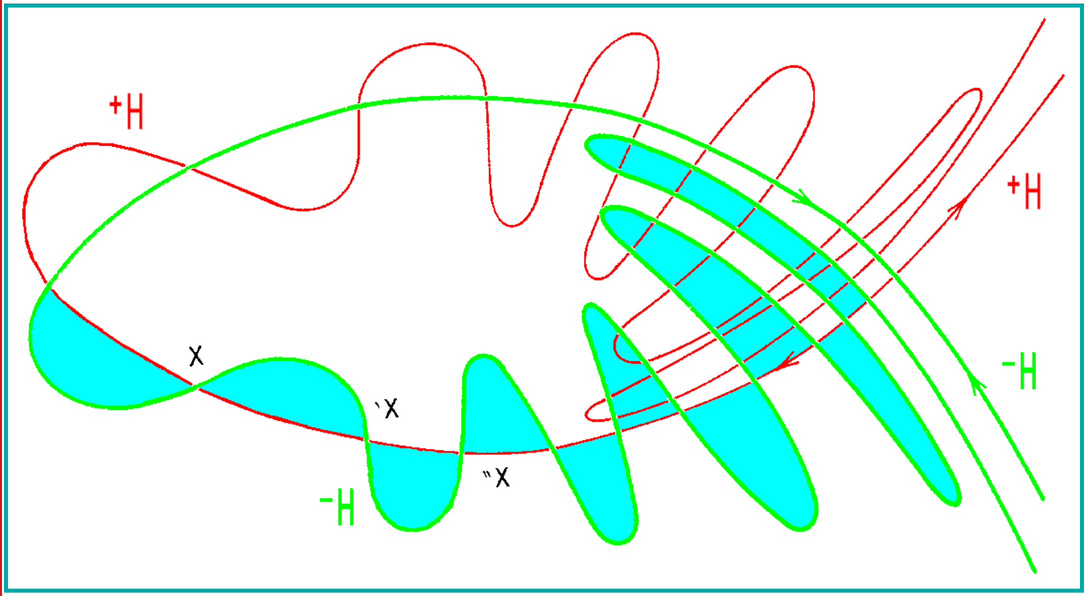


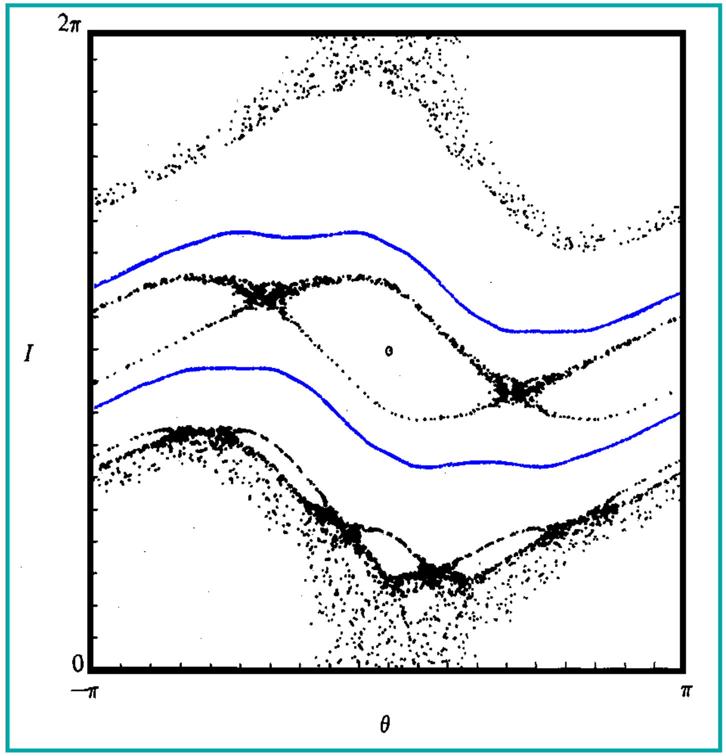
Due to the narrow region of unstable trajectories, sextupoles are used for slow particle extraction at a tune of 1/3.

The intersection of stable and unstable manifolds is a certain indication of chaos.

Homoclinic Points

- At instable fixed points, there is a stable and an instabile invariant curve.
- Intersections of these curves (homoclinic points) lead to chaos.





Hyperion: rotation around the vertical

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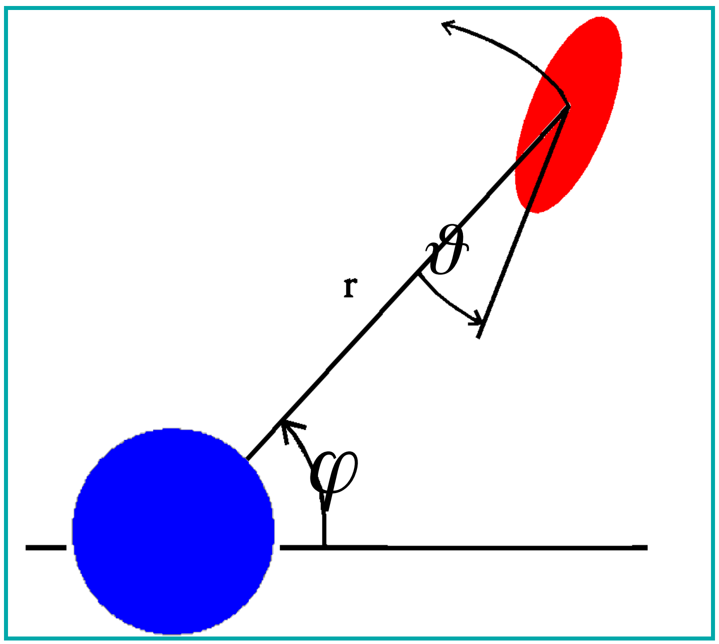
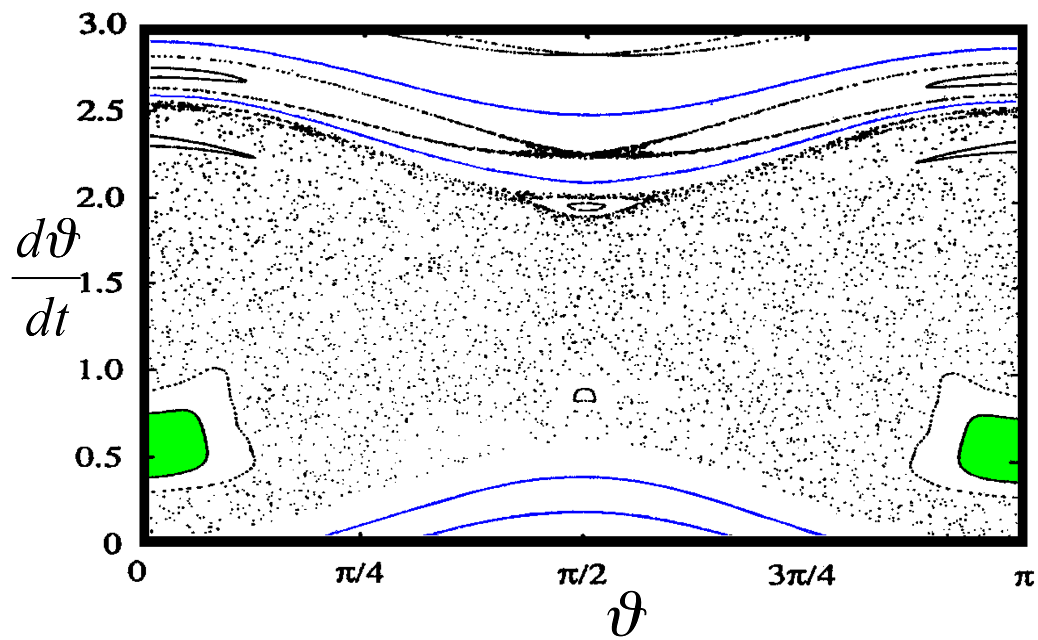
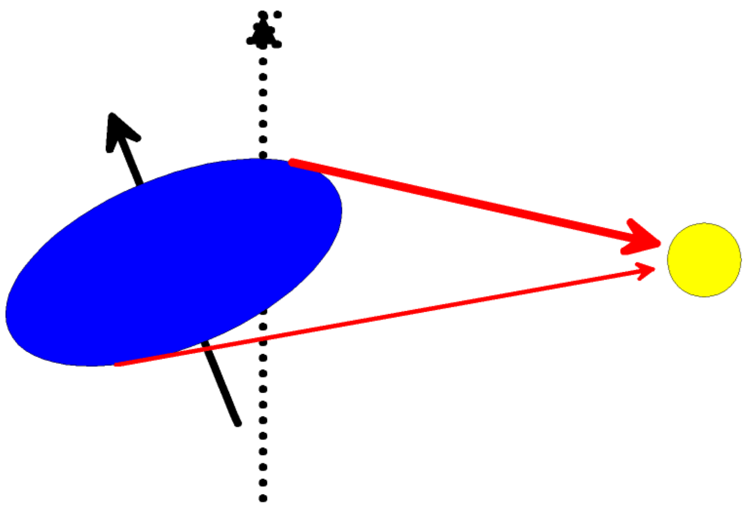
$$\frac{d^2(\vartheta+\varphi(t))}{dt^2} = -\alpha\left(\frac{a}{r(t)}\right)^3 \sin 2\vartheta$$

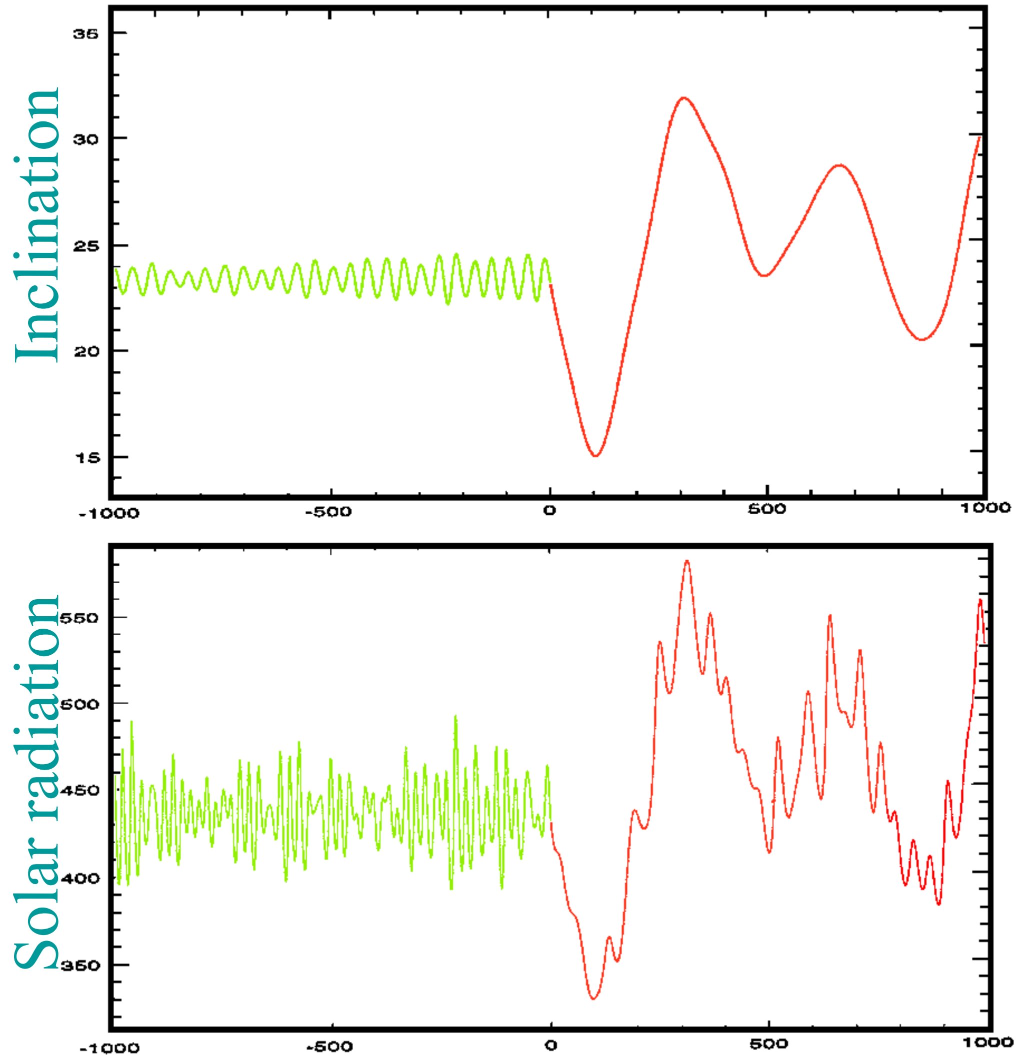




- On the path from Rotation to Libration around the Spin-Orbit-Coupling is a strong chaotic region.

Tilt of the earth





-1Ma with and without moon 1M a

- Tidal forces from moon and sun cause a stabilization of the rotation axis.

The Single Resonance Model

$$\frac{d}{d\vartheta} J = \sum_{n,m=-\infty}^{\infty} m H_{nm}(J) \sin(n\vartheta + m\varphi + \Psi_{nm}(J))$$

$$\frac{d}{d\vartheta} \varphi = \upsilon + \partial_J \sum_{n,m=-\infty}^{\infty} H_{nm}(J) \cos(n\vartheta + m\varphi + \Psi_{nm}(J))$$

Strong deviation from: $J = J_0 \,, \quad \varphi = \upsilon\vartheta + \varphi_0$

Occur when there is coherence between the perturbation and the phase space rotation: $n + m\frac{d}{ds}\varphi \approx 0$

Resonance condition: tune is rational $\boxed{n + m\,\upsilon = 0}$

On resonance the integral would increases indefinitely !

Neglecting all but the most important term

$$H(\varphi, J, \vartheta) \approx \upsilon J + H_{00}(J) + H_{nm}(J)\cos(n\vartheta + m\varphi + \Psi_{nm}(J))$$

Fixed points

$$\tfrac{d}{d\vartheta} J = mH_{nm}(J)\sin(n\vartheta + m\varphi + \Psi_{nm}(J))$$

$$\tfrac{d}{d\vartheta} \varphi = \upsilon + \Delta\upsilon(J) + \partial_J [H_{nm}(J)\cos(n\vartheta + m\varphi + \Psi_{nm}(J))]$$

$$\Phi = \tfrac{1}{m}[n\vartheta + m\varphi + \Psi_{nm}(J)] \ , \quad \delta = \upsilon + \tfrac{n}{m}$$

$$\tfrac{d}{d\vartheta} J = mH_{nm}(J)\sin(m\Phi) \ , \quad \tfrac{d}{d\vartheta} \Phi = \delta + \Delta\upsilon(J) + H'_{nm}(J)\cos(m\Phi)$$

$$H(\varphi, J, \vartheta) \approx \delta J + H_{00}(J) + H_{nm}(J)\cos(m\Phi)$$

Fixed points: $\tfrac{d}{d\vartheta} J = mH_{nm}(J_f)\sin(m\Phi_f) = 0 \quad \Rightarrow \quad \Phi_f = \tfrac{k}{m}\pi$

If $\delta + \Delta\upsilon(J_f) \pm H'_{nm}(J_f) = 0$ has a solution.

$$\tfrac{d}{d\vartheta} \Delta J = \pm m^2 H_{nm}(J_f)\Delta\Phi \ , \quad \tfrac{d}{d\vartheta} \Delta\Phi = [\Delta\upsilon'(J_f) \pm H''_{nm}(J_f)]\Delta J$$

Stable fixed point for: $H_{nm}(J_f)[H''_{nm}(J_f) \pm \Delta\upsilon'(J_f)] < 0$

Third Integer Resonances

Sextupole: $\Delta f = -k_2 \frac{1}{2} x^2$

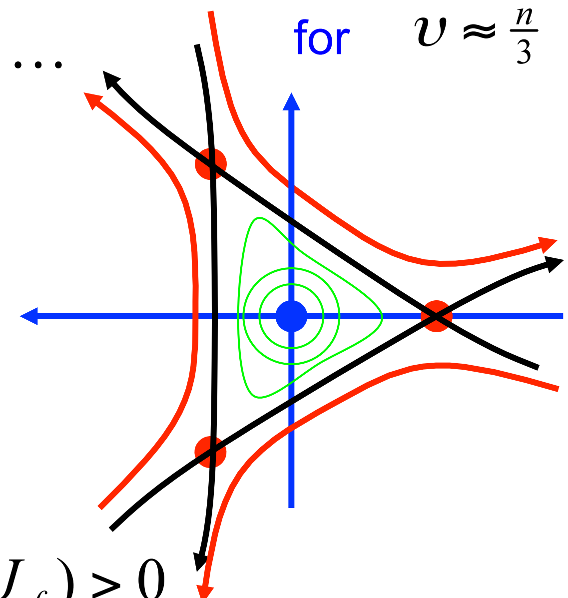
$$\Delta H = \frac{L}{2\pi} k_2 \frac{1}{3!} x^3 = \frac{L}{2\pi} k_2 \frac{1}{3!} \sqrt{2J\beta}^3 \sin^3(\tilde{\psi} + \varphi)$$

$$= \frac{L}{2\pi} k_2 \frac{1}{3!4} \sqrt{2J\beta}^3 [\sin(3[\tilde{\psi} + \varphi]) + 3\sin(\tilde{\psi} + \varphi)]$$

Simplification: one sextupole $k_2(\vartheta) = k_2 \delta(\vartheta) = k_2 \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \cos(n\vartheta)$

$$\Delta H = \frac{L}{2\pi} k_2 \frac{1}{3!4} \sqrt{2J\beta}^3 \frac{1}{2\pi} \cos(-n\vartheta + 3\varphi + 3\tilde{\psi} - \frac{\pi}{2}) + \dots$$

for $\upsilon \approx \frac{n}{3}$



$$\Delta H \approx A_2 \sqrt{J}^3 \cos(3\Phi)$$

$$\left.\begin{array}{l} \Phi_f = 0, \frac{1}{3}\pi, \frac{2}{3}\pi, \dots \\ \delta \pm A_2 \frac{3}{2}\sqrt{J} = 0 \end{array}\right\} \begin{array}{l} \Phi_f = \frac{1}{3}\pi, \pi, \frac{5}{3}\pi \\ \text{for } \delta > 0 \end{array}$$

All these fixed points are instable since $H_{nm}(J_f) H''_{nm}(J_f) > 0$

Fourth Integer Resonances

Octupole: $\Delta f = -k_3 \frac{1}{3!} x^3$, $\Delta H = \frac{L}{2\pi} k_3 \frac{1}{4!} x^4 = \frac{L}{2\pi} k_3 \frac{1}{3!} J^2 \beta^2 \sin^4(\tilde{\psi} + \varphi)$

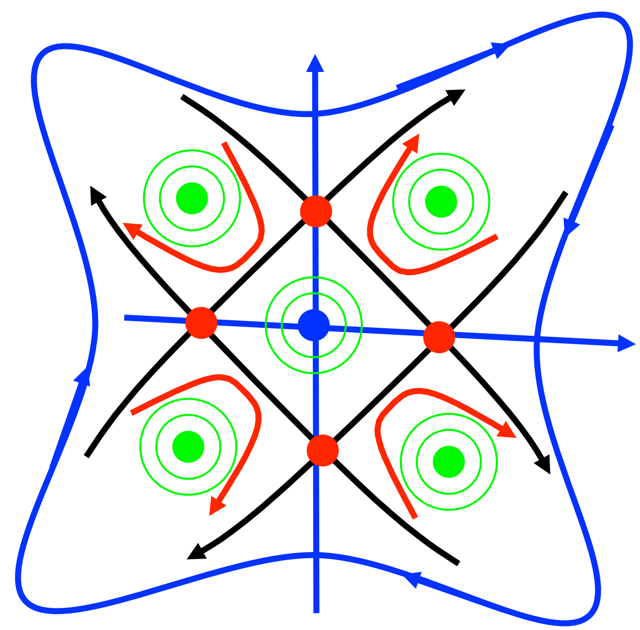
$$= \frac{L}{2\pi} k_3 \frac{1}{3!8} J^2 \beta^2 [\cos(4[\tilde{\psi} + \varphi]) - 4\cos(2[\tilde{\psi} + \varphi]) + 3]$$

Simplification: one octupole $k_3(\vartheta) = k_3 \delta(\vartheta) = k_3 \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \cos(n\vartheta)$

$\Delta H \approx A_3 J^2 [3 + \cos(4\Phi)]$ for $\upsilon \approx \frac{n}{4}$

$\Phi_f = 0, \frac{1}{4}\pi, \frac{2}{4}\pi, \ldots$ Either 8 fixed points: $\delta < 0$

$\delta + A_3 2J(3 \pm 1) = 0$ or none for: $\delta > 0$



$$H_{nm}(J_f)[H''_{nm}(J_f) \pm \Delta\upsilon'(J_f)] < 0$$

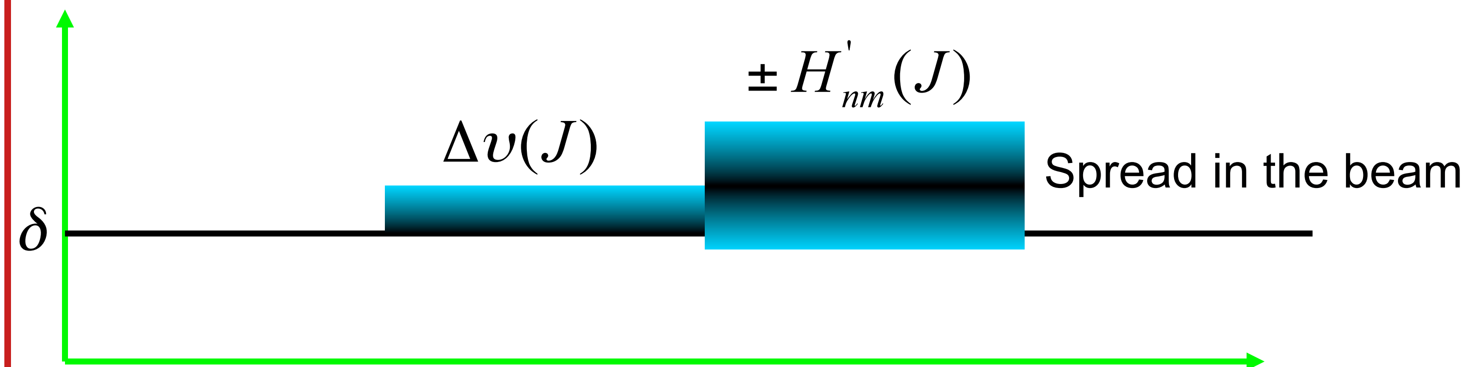
Stability for $(2A_3 J)^2 [1 \pm 3] < 0$,
i.e. for the 4 outer fixed points.

Resonance Width (Strength)

Fixed points: $\frac{d}{d\vartheta}J = mH_{nm}(J_f)\sin(m\Phi_f) = 0 \quad \Rightarrow \quad \Phi_f = \frac{k}{m}\pi$

If $\delta + \Delta\upsilon(J_f) \pm H'_{nm}(J_f) = 0$ has a solution.

δ has to avoid the region $\delta + \Delta\upsilon(J) \pm H'_{nm}(J) = 0$ for all particles.



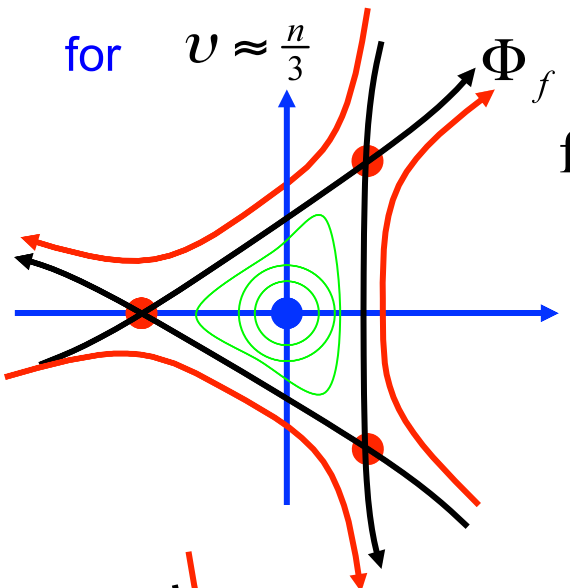
Assuming that the tune shift and perturbation are monotonous in J:

This tune region has the width $\Delta_{nm} = 2\,|\,H'_{nm}(J_{\max})|$ for strong resonances.

Δ_{nm} Is called Resonance Width, Resonance Strength, or Stop-Band Width

Particle motion in the single resonance model

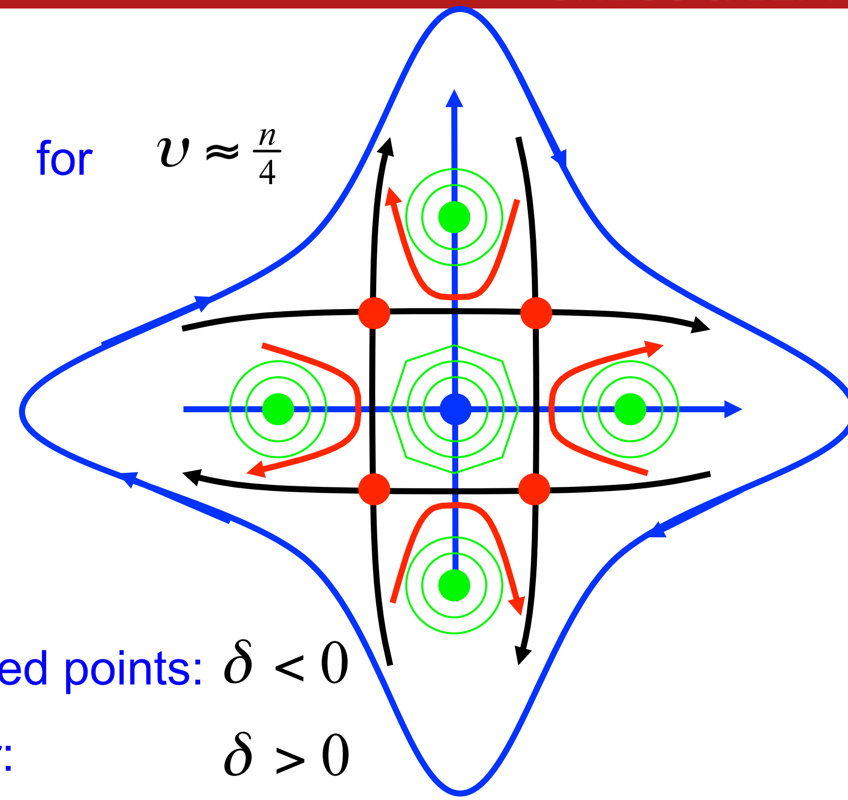
for $\upsilon \approx \frac{n}{3}$

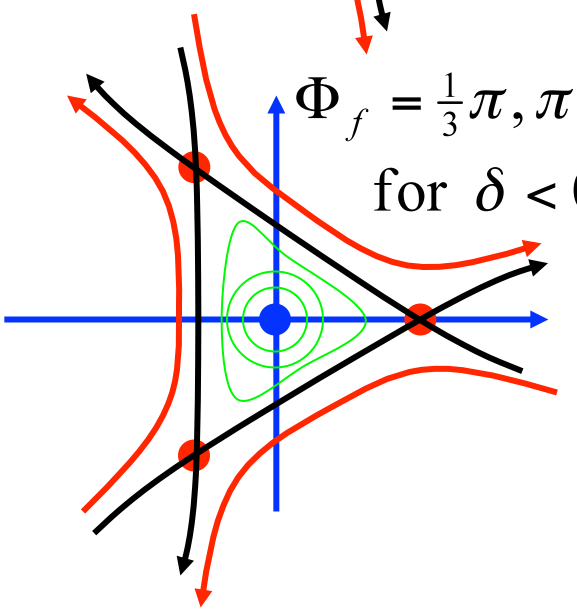


$\Phi_f = \frac{1}{3}\pi, \pi, \frac{5}{3}\pi$

for $\delta > 0$

for $\upsilon \approx \frac{n}{4}$





$\Phi_f = \frac{1}{3}\pi, \pi, \frac{5}{3}\pi$

for $\delta < 0$

Either 8 fixed points: $\delta < 0$

or none for: $\delta > 0$

How can the motion inside the fixed points be simplified for a real accelerator ?

→ Normal From Theory