ADIABATIC INVARIANTS FOR SPIN–ORBIT MOTION

G. H. Hoffstaetter*, DESY, Hamburg, Germany,
H. S. Dumas, UC, Cincinnati and J. A. Ellison, UNM, Albuquerque

Abstract

It has been predicted and found experimentally that the polarization direction of particles on the closed orbit can be manipulated, without a noticeable reduction of polarization, by a slow variation of magnetic £elds. This feature has been used to avoid imperfection resonances where the spin precession frequency is close to a multiple of the circulation frequency. We report here on a proof that relates this property to an adiabatic invariant of spin motion. The proof is relatively simple since only two frequencies, the spin rotation frequency and the particle's rotation frequency on the closed orbit, are involved. The invariant spin £eld describes a periodic polarization state of a beam's phase space distribution. This invariant spin £eld leads to a very useful parametrization of coupled spin and orbit dynamics. We also report on a proof showing that the invariant spin £eld gives rise to an adiabatic invariant of spin-orbit motion. The proof is now much more complicated since the orbital frequencies are involved. Due to this adiabatic invariance, the spin £eld of a polarized beam follows slow changes of the accelerator's invariant spin £eld that can occur during a slow acceleration cycle. This feature is essential when high-order spin orbit resonances are crossed since it allows a reduced degree of polarization at the resonance condition to recover, to a large degree, after the resonance has been crossed.

1 INTRODUCTION

The spins of particles which move through the magnetic £elds of a circular accelerator rotate according to the Thomas-BMT equation along their phase space trajectory $\vec{z}(\theta)$, i.e. $\dot{\vec{s}} = \vec{\Omega}(\vec{z}) \times \vec{s}$. After a particle has traveled one turn along the closed orbit from azimuth θ_0 to $\theta_0 + 2\pi$ the spin has rotated around some unit rotation axis $\vec{n}_0(\theta_0)$ by a rotation angle $2\pi\nu_0$, where ν_0 is called the closed orbit spin tune. In a ¤at accelerator without £eld errors, the closed orbit is in the horizontal plane and passes only through vertical £elds. Then $\vec{n}_0$ is vertical and $\nu_0 = G\gamma$ with the anomalous gyro-magnetic factor G and with the relativistic factor γ, which causes the number of spin rotations to increase with energy. When ν_0 is close to an integer, a case which is referred to as an imperfection resonance, the rotation matrix is close to the identity and spin directions have hardly changed after one turn. In this case £eld errors can dominate the rotation direction and can rotate spins away from the vertical. Therefore when ν_0 crosses an integer value during acceleration the rotation vector $\vec{n}_0$ can change signi£cantly. When the spin rotation is much faster than this change of the rotation vector, then a spin which is nearly parallel to $\vec{n}_0$ is dragged along with the evolving $\vec{n}_0$ [1]. To illustrate this fact, one can imagine that $\vec{n}_0$ changes away from the spin sometimes and towards the spin at other times while the spin rotates around the slowly changing $\vec{n}_0$. Due to this rapid rotation, both cases occur frequently and the total effect averages out. This causes the spin to follow the slow change of $\vec{n}_0$. and the projection of a spin on $\vec{n}_0$ hardly changes. It is conjectured that $\vec{s}(\theta) \cdot \vec{n}_0(\theta)$ is an adiabatic invariant. In section 2 we will show how such a statement may be proven [2].

When all particles of a beam are initially completely polarized parallel to each other, the polarization state of the beam is in general not 2π-periodic and the average beam polarization can change from turn to turn since spins evolve differently along the different phase space trajectories. Spin £elds describe the polarization direction for each phase space point of the beam and are propagated by a phase space dependent rotation matrix $\underline{R}(\vec{z}_i, \theta_0; \theta)$. A special spin £eld $\vec{n}(\vec{z}, \theta)$, which is 2π-periodic in θ, is called an invariant spin £eld. If the spin of each particle in a beam is initially polarized parallel to $\vec{n}(\vec{z}, \theta_0)$, particles get redistributed in phase space during one turn, but they will stay polarized parallel to the invariant spin £eld [3]. The invariant spin £eld was £rst introduced in [4] in the theory of radiative electron polarization and is often called the Derbenev-Kondratenko $\vec{n}$-axis. Note that it is usually not an eigenvector of the one-turn spin transport matrix $\underline{R}(\vec{z}, \theta_0; \theta_0 + 2\pi)$ at some phase space point, since the spin of a particle has changed after one turn around the ring, while the eigenvector does not change.

Since the particles oscillate with the orbital tunes around the closed orbit, the spin rotation vector $\vec{\Omega}(\vec{z})$ is modulated by these frequencies. Therefore the spin motion can be strongly disturbed when the spin tune is in resonance with the orbit tunes. These conditions are referred to as intrinsic resonances.

The strength of the spin precession and the precession axis in machine magnets depends on the trajectory and the energy of the particle. Thus in one turn around the ring an effective precession axis can deviate from the vertical and can strongly depend on the initial position of the particle in 6-dimensional phase space. From this it is clear that if an invariant spin £eld $\vec{n}(\vec{z}, \theta)$ exists, it can vary strongly across the orbital phase space. This phase space dependence is especially strong close to spin-orbit resonances. Therefore the invariant spin £eld can change signi£cantly while spin-orbit resonance conditions are crossed during beam acceleration. Similarly to $\vec{s} \cdot \vec{n}_0$, it can be conjectured that $\vec{s} \cdot \vec{n}(\vec{z}, \theta)$ is an adiabatic invariant, and we will show how such a statement may be proven. For the de£nition of

* Georg.Hoffstaetter@desy.de

adiabatic invariants we use [5, sec.8.1], [3].

De£nition: Adiabatic Invariants *Consider* $\frac{d}{d\theta}\vec{x} = \vec{g}(\vec{x},\tau)$ *with* $\tau = \varepsilon\theta$ *and* $\vec{x} \in I\!R^n$ *for a small parameter* ε *so that* $\vec{g}$ *is a slowly varying vector £eld. A function* $\vec{A}(\vec{x},\tau)$ *is said to be an adiabatic invariant of this system if its variation on the interval* $\theta \in [0, 1/\varepsilon]$ *(which implies* $\tau \in [0,1]$*) is small together with* ε*, except perhaps for a set of initial conditions whose measure goes to zero with* ε*. That is, for "most" initial conditions the following limit on the supremum over the interval* $[0, 1/\varepsilon]$ *holds:*

$$\lim_{\varepsilon\to 0} \sup_{\theta\in[0,1/\varepsilon]} |\vec{A}(\vec{x}(\theta),\varepsilon\theta) - \vec{A}(\vec{x}(0),0)| = 0 \; . \qquad (1)$$

2 ADIABATIC SPIN INVARIANT ON THE CLOSED ORBIT

To analyze whether $s_3 = \vec{s}\cdot\vec{n}_0$ is an adiabatic invariant when system parameters are changed, as conjectured in section 1, we simplify the equations for the £xed parameter system by introducing an orthonormal coordinate system which has $\vec{n}_0$ as its third coordinate axis. The other two axes are chosen to rotate around $\vec{n}_0$ in such a way that they are periodic with the ring azimuth and so that the rotation angle of a spin $\vec{s}$ around $\vec{n}_0$ in this coordinate system is linear with θ. If we now introduce the new variable ϕ which describes the angle of the spin in the coordinate plane perpendicular to $\vec{n}_0$, i.e. $s_1 = \sqrt{1-s_3^2}\cos\phi$ and $s_2 = \sqrt{1-s_3^2}\sin\phi$, then the equation of motion simply becomes $\frac{d}{d\theta}\phi = \nu_0$ and $\frac{d}{d\theta}s_3 = 0$ when no system parameters are changed. Thus s_3 is an invariant of the motion. To prove it is an adiabatic invariant we have to introduce slow variations of system parameters. When a parameter $\tau = \varepsilon\theta$ changes slowly, then also $\vec{n}_0(\tau)$ and $\nu_0(\tau)$ change. But for all τ the coordinate system remains an orthonormal frame, which means that all unit vectors can only rotate around a common vector $\vec{\eta}(\tau)$, i.e. $\partial_\tau\vec{n}_0 = \vec{\eta}(\tau)\times\vec{n}_0(\tau)$. The periodic dependence of $\vec{n}_0$ and $\vec{\eta}$ on θ is not indicated here. Using polar coordinates again, the equation of motion becomes [2]

$$\frac{d}{d\theta}\begin{pmatrix} s_3 \\ \phi \end{pmatrix} = \begin{pmatrix} & & \varepsilon f_3 \\ \nu_0(\tau) & + & \varepsilon f_\phi \end{pmatrix} \; , \qquad (2)$$

$$f_3 = [\eta_2(\tilde{\theta},\tau)\cos\phi - \eta_1(\tilde{\theta},\tau)\sin\phi]\sqrt{1-s_3^2} \; ,$$

$$f_\phi = [\eta_2(\tilde{\theta},\tau)\sin\phi + \eta_1(\tilde{\theta},\tau)\cos\phi]\frac{s_3}{\sqrt{1-s_3^2}} - \eta_3(\tilde{\theta},\tau) \; .$$

For small ε and $|\nu_0|$ large compared to ε, s_3 is slowly varying and the phase ϕ is rapidly varying. It is therefore suitable for averaging methods and can be brought into a standard form for averaging theorems of two frequency systems,

$$\frac{d}{d\theta}\begin{pmatrix} s_3 \\ \tau \\ \phi \\ \tilde{\theta} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \nu_0(\tau) \\ 1 \end{pmatrix} + \varepsilon \begin{pmatrix} f_3(s_3,\phi,\tau,\tilde{\theta}) \\ 1 \\ f_\phi(s_3,\phi,\tau,\tilde{\theta}) \\ 0 \end{pmatrix} \; . \qquad (3)$$

Here we will state an abbreviated form of theorem 3 of [5, sec.4.1] which is attributed to [6]. The application of two-phase averaging to the simple problem of spin motion on the closed orbit might seem more complicated than necessary but the effects of resonances cannot be ignored and, moreover, the stage is set for adiabatic invariants in the case of spin motion on a general trajectory.

Theorem (Averaging for two frequency systems): *Consider a system of the form*

$$\frac{d}{d\theta}\vec{I} = \varepsilon\vec{f}(\vec{I},\phi,\tilde{\theta},\varepsilon) \; , \qquad (4)$$

$$\frac{d}{d\theta}\phi = \nu(\vec{I}) + \varepsilon g(\vec{I},\phi,\tilde{\theta},\varepsilon) \; , \quad \frac{d}{d\theta}\tilde{\theta} = 1 \; , \qquad (5)$$

where several mild conditions are required on the domain and properties of the involved functions. The associated averaged system is

$$\frac{d}{d\theta}\vec{\bar{I}} = \varepsilon\vec{\bar{f}}(\vec{\bar{I}}) \; , \quad \vec{\bar{f}}(\vec{\bar{I}}) = \frac{1}{(2\pi)^2}\iint_0^{2\pi}\vec{f}(\vec{\bar{I}},\phi,\tilde{\theta},0)d\phi d\tilde{\theta} \; , \qquad (6)$$

with the initial condition $\vec{\bar{I}}(0) = \vec{I}(0)$*. Suppose every trajectory of the exact system for which* $\vec{I}$ *stays in the range of de£nition for* $\theta \in [0,1/\varepsilon]$ *has a strictly monotonic variation of* $\nu(\vec{I})$ *with* $|\frac{d}{d\theta}\nu| > c_1\varepsilon$*. Then: On all these trajectories there exists* $c > 0$ *such that for suf£ciently small* $\varepsilon > 0$

$$\sup_{\theta\in[0,1/\varepsilon]} |\vec{I}(\theta) - \vec{\bar{I}}(\theta)| < c\sqrt{\varepsilon} \; , \; c_1, c \in I\!R^+ \; . \qquad (7)$$

In general the solution of the averaged system does not approximate the original system well if $\tilde{\theta}$ and ϕ are in resonance, which means here that the closed-orbit spin tune $\nu_0(\tau)$ is an integer m. The simplest way to avoid perturbations which accumulate at resonances is to consider only systems in which resonances are traversed quickly. This is the reason for the condition $|\frac{d}{d\theta}\nu| > c_1\varepsilon$.

Since $\int_0^{2\pi} f_3 d\phi = 0$, $\bar{s}_3(\theta) = s_3(0)$ and the scalar product s_3 is an adiabatic invariant. A direct proof of this using the method of stationary phase will be presented in [8].

3 ADIABATIC SPIN INVARIANT ON PHASE SPACE TRAJECTORIES

Since the spin £eld of a beam is only invariant with time when it is parallel to $\vec{n}(\vec{z})$, this invariant spin £eld has the highest possible phase space averaged polarization [3] and is therefore referred to as $P_{lim} = <\vec{n}>$, where the average is taken over the phase space coordinates $\vec{z}$. Also here the dependence on θ is not indicated. At resonances where $\vec{n}$ varies strongly over phase space, P_{lim} can be very small. Thus, how can a polarized proton beam be transported with little loss of polarization from low energy through regions with small P_{lim}, and therefore small beam polarization, to a suitable energy where P_{lim} is acceptable? Can the beam polarization recover to large values at this suitable energy after it was much smaller at lower energies?

This is possible if the spins which are initially parallel to $\vec{n}(\vec{z})$ remain close to the invariant spin £eld along its trajectory, even when parameters of particle motion, for example the energy, are slowly changed and the invariant spin £eld changes.

Here we will sketch a proof that spins follow slow changes of the invariant spin £eld by showing that the product $J_s = \vec{s}\cdot\vec{n}(\vec{z})$ is an adiabatic invariant. We use a coordinates system with $\vec{n}(\vec{z})$ as the third unit vector. Along each particle trajectory, the other two unit vectors $\vec{u}_1(\vec{z})$ and $\vec{u}_2(\vec{z})$ are rotated in the plane perpendicular to $\vec{n}(\vec{z})$ so that they are periodic in θ and such that components of a spin $\vec{s}$ in that plane rotate linearly with time. It turns out that the rotation angle only depends on the action but not on the phase variables of phase space motion. When polar coordinates are used in this plane, then the equation of spin motion agrees with equation (3), only now s_3 is replaced by the spin action J_s, ν_0 is replaced by the amplitude-dependent spin tune $\nu(\vec{J},\tau)$ and the vector $\vec{\eta}(\vec{z},\tau)$ that rotates the frame when τ changes is a function of phase space. With de£nitions of f_{J_s} and f_ϕ that correspond to equation (2), the equation of spin-orbit motion becomes

$$\frac{d}{d\theta}\begin{pmatrix}\vec{J}\\ J_s\\ \tau\\ \vec{\Phi}\\ \phi\\ \tilde{\theta}\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\\ \vec{Q}(\vec{J},\tau)\\ \nu(\vec{J},\tau)\\ 1\end{pmatrix} + \varepsilon\begin{pmatrix}p_J\\ f_{J_s}\\ 1\\ p_\Phi\\ f_\phi\\ 0\end{pmatrix}\ . \qquad (8)$$

The small perturbations εp_J and εp_ϕ to the motion of the action and angle variables are due to the variation of the equation of phase space motion with the parameter τ. When the 6-dimensional phase space motion in accelerators is considered, this system has 5 slowly and 5 rapidly changing variables for small ε. It is written in the standard form of multi-phase averaging theorems [7, sec.1.9], [5, chapter 6], here abreviated as:

Theorem (Averaging for N frequency systems): *Consider a system of the form*

$$\frac{d}{d\theta}\vec{I} = \varepsilon\vec{f}(\vec{I},\vec{\phi},\varepsilon)\ , \qquad (9)$$

$$\frac{d}{d\theta}\vec{\phi} = \vec{\nu}(\vec{I}) + \varepsilon g(\vec{I},\vec{\phi},\varepsilon)\ , \qquad (10)$$

with some mild restrictions on the domain and the properties of the involved functions. The associated averaged system is

$$\frac{d}{d\theta}\bar{\vec{I}} = \varepsilon\bar{\vec{f}}(\bar{\vec{I}})\ ,\ \ \bar{\vec{f}}(\bar{\vec{I}}) = \frac{1}{(2\pi)^n}\int_0^{2\pi}\vec{f}(\bar{\vec{I}},\vec{\phi},0)d\vec{\phi}\ , \qquad (11)$$

with $\bar{\vec{I}}(0) = \vec{I}(0)$. Let the following non-degeneracy condition (called Arnold's condition) be satis£ed: "Assuming the frequency $\nu_n(\vec{I}) \neq 0$, then the map $\vec{I} \to (\nu_1(\vec{I}),\ldots,\nu_{n-1}(\vec{I}))/\nu_n(\vec{I})$ has maximal rank, equal to $n-1$". Then the set of allowed initial conditions V is partitioned as $V = V'(\varepsilon)\bigcup V''(\varepsilon)$ for suf£ciently small $\varepsilon > 0$ such that

$$\sup_{\theta\in[0,1/\varepsilon]}|\vec{I}(\theta) - \bar{\vec{I}}(\theta)| < \varepsilon^{\frac{1}{4}} \qquad (12)$$

for initial conditions in V'. That is the separation between the exact solution and the solution of the averaged system is less than $\varepsilon^{\frac{1}{4}}$. Moreover, the measure of V'' is smaller than $C\varepsilon^{\frac{1}{4}}$ for some $C > 0 \in \mathbb{R}^+$.

The frequency of $\tilde{\theta}$ in equation (8) is 1 and may therefore be used as ν_n of Arnold's condition. The 4 frequencies $(\vec{Q}(\vec{J},\tau),\nu(\vec{J},\tau))$ depend on 4 of the 5 slowly changing variables and we assume that the rank is 4 so that the Jacobian matrix of the 4 frequencies has non-vanishing determinant, $\det[\partial_{(\vec{J},\tau)}(\vec{Q},\nu)] \neq 0$.

When the frequencies are in resonance, the slowly changing variables $\vec{I}$ can accumulate large changes and the solution of the averaged system does not approximate the original system well. In the above theorem, Arnold's condition ensures that no slowly changing variable I_j can change at a resonance without moving the system out of this resonance.

For Hamiltonian systems, p_J is a derivative of the periodic Hamiltonian with respect to $\vec{\Phi}$ and therefore $\bar{p}_J = 0$. This shows that the action variables $\vec{J}$ are adiabatic invariants, which is a well known fact. Due to $\bar{f}_{J_s} = 0$, one £nds that $\bar{J}_s(\theta) = J_s(0)$ and therefore $J_s = \vec{s}\cdot\vec{n}(\vec{z})$ is an adiabatic invariant as de£ned in section 1.

Here we have simply relied on general theorems to show that the derived equations of motion give rise to adiabatic invariants. For the two-frequency case it is quite straightforward to prove the adiabatic invariance directly. With signi£cantly more effort, but less than proving the above theorems, this can also be done for the multiple frequency case [8]. In addition, this will give more insight into the spin case.

4 REFERENCES

[1] M. Froissart, R. Stora, Nucl. Instr. Meth., 7:297–305 (1960)

[2] G. H. Hoffstaetter, habilitation, Darmstadt Univ. (2000)

[3] G. H. Hoffstaetter, M. Vogt, and D. P. Barber, Phys. Rev. ST-AB, 2(11):1–15 (1999)

[4] Ya. S. Derbenev, A. M. Kondratenko, Sov. Phys. JETP, 35:230 (1972)

[5] P. Lochak and C. Meunier, volume 72 of Applied Mathematical sciences, Springer (1988)

[6] A. I. Neistadt, Soviet Phys. Doklady, 20(3): 189–191 (1975) (1975)

[7] V. I., Arnold, Chap. 5, Dynamical Systems III of Encyclopedia of Mathematical Sciences; V. Springer (1988)

[8] H. S. Dumas, J. A. Ellison, and G. H. Hoffstaetter, in preparation.