Proceedings EPAC98
Stockholm 1998

THE AMPLITUDE DEPENDENT SPIN TUNE AND THE INVARIANT SPIN FIELD IN HIGH ENERGY PROTON ACCELERATORS

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Abstract

There is some ambiguity in the definitions and use of the concepts of spin tune and spin-orbit resonance on synchro-betatron orbits. We clarify these issues and provide a numerical illustration of the internal consistency of our definitions using the computer code SPRINT for HERA with a standard Siberian Snake arrangement. Furthermore, we demonstrate the calculation of the invariant spin field using adiabatic anti-damping of the orbital motion and compare with the spin field obtained by stroboscopic averaging. In addition we study polarization loss on accelerating through resonances.

1 INTRODUCTION

To caste further light on the spin structure of nucleons, it would be very useful to run the HERA e$^{-/+}$–p collider in Hamburg with spin polarized protons to complement the spin polarized e$^{-/+}$ beam. This requires that the existing unpolarized H^- source is replaced by a polarized source and that the protons are accelerated up to 820 GeV without loss of polarization.

The motion of the centre-of-mass unit spin vector $\hat{S}$ of a relativistic charged particle travelling in electric and magnetic fields is governed by the Thomas-BMT equation. In a purely magnetic field with components $\vec{B}_\parallel$ and $\vec{B}_\perp$ parallel and perpendicular to the orbit this is

$$\frac{d\hat{S}}{dt} = \frac{e\hat{S}}{mc\gamma} \times ((1+G)\vec{B}_\parallel + (1+G\gamma)\vec{B}_\perp) \quad . (1)$$

The quantity $G = (g-2)/2$ is the gyromagnetic anomaly, for protons $G \approx 1.7928$. The other symbols have their usual meanings. Eq. (1) shows that for motion perpendicular to the field, the spin precesses around the field at a rate $1 + G\gamma$ faster than the corresponding rate of orbit deflection. At 820 GeV, $1 + G\gamma \approx$ 1568 so that a 1 mrad orbit deflection leads to about 90 degrees of spin precession! Thus at these high energies spins are very easily disturbed. The measure of the sensitivity is the quantity $G\gamma$ which we call the *spin enhancement factor*. At these high energies it is clearly essential for preservation of polarization that spin motion be very well understood. That in turn requires that clearly defined mathematical concepts and tools are available.

2 THE DEFINITION OF SPIN TUNE

The stable, i.e. equilibrium, direction for spins travelling along the design orbit of a perfectly aligned flat storage ring with no solenoids, is the vertical axis. For rings with vertical bends and/or solenoids and misalignments the corresponding direction is $\hat{n}_0(\theta)$, the 1–turn periodic real unit eigenvector of the orthogonal 1–turn spin transfer matrix for spin motion along the closed orbit. In general this is a function of the azimuth θ, and it is not vertical everywhere.

To describe equilibrium spin distributions of particles executing synchrobetatron motion in the quadrupole and other fields we introduce the *invariant spin field* of special solutions to the T–BMT equation $\hat{n}(\vec{z},\theta)$, with the property that

$$\hat{n}(\vec{z},\theta + 2\pi) = \hat{n}(\vec{z},\theta) \quad (2)$$

where $\vec{z}$ is the 6–D orbital phase space vector [1, 2].

For one turn around the ring the field vector at the final point $\vec{z}_f$ is related to the field at the initial point $\vec{z}_i$ by the relation

$$\underline{R}(\vec{z}_i,\theta)\hat{n}(\vec{z}_i,\theta) = \hat{n}(\vec{z}_f,\theta) \quad (3)$$

where $\underline{R}$ is the one turn spin map.

A distribution of spins set up so that each spin is parallel to the $\hat{n}(\vec{z},\theta)$ vector at its position in phase space does not change from turn to turn although after each turn the particles find themselves at new positions in phase space. Such a distribution is in equilibrium when viewed from a fixed azimuth. On the closed orbit $\hat{n}(\vec{z},\theta)$ reduces to $\hat{n}_0(\theta)$. By Eq. (1) the disturbance relative to $\hat{n}_0$ increases with $(G\gamma + 1)$ and the orbital amplitude so that at high energy and/or high amplitude the angle between $\hat{n}(\vec{z},\theta)$ and $\hat{n}_0(\theta)$ can be large. Indeed, for HERA at high energy for protons on the 1–sigma vertical phase space ellipse , angles of 60 degrees or more, between $\hat{n}(\vec{z},\theta)$ and $\hat{n}_0(\theta)$ are common. The angle can be particularly large and spin motion, in general, can be particularly disturbed if there is coherence between the spin motion and the oscillatory motion in the beam.

A convenient measure of the angle is $P_{\lim}(\vec{J},\theta) = \left\| \left\langle \hat{n}(\vec{J},\vec{\Psi},\theta=0) \right\rangle_{\vec{\Psi}} \right\|$. This is the polarization averaged over the surface of a 3–torus in phase space when the polarization in direction of $\hat{n}(\vec{z},\theta)$ is 100 % at each point on the surface. When the angle is large, $P_{\lim}(\vec{J},\theta)$ is small.

The degree of coherence is expressed in terms of the *spin tune*. To obtain this, one attaches unit vectors $\hat{u}_1(\vec{z},\theta)$ and

$\hat{u}_2(\vec{z},\theta)$ to each phase space point $\vec{z}$ in such a way that the set $(\hat{u}_1(\vec{z},\theta), \hat{u}_2(\vec{z},\theta), \hat{n}(\vec{z},\theta))$ forms an orthonormal coordinate system in which a spin that is not parallel to $\hat{n}(\vec{z},\theta)$ precesses uniformly around $\hat{n}(\vec{z},\theta)$ [3, 4, 2]. The vectors $\hat{u}_1$ and $\hat{u}_2$ obey periodicity conditions of the kind in Eq. (2) but do not obey the T-BMT equation. The spin tune ν is then the number of spin precessions around $\hat{n}$ in the $(\hat{u}_1, \hat{u}_2, \hat{n})$ system for one turn of a particle around the ring and the coherence is characterized by the spin–orbit resonance condition

$$\nu = k + k_x Q_x + k_y Q_y + k_s Q_s \tag{4}$$

where the k's are integers and the Q's are respectively the horizontal, vertical and longitudinal tunes of the orbital oscillations. The sum $|k_x| + |k_y| + |k_s|$ is called the order of the resonance.

On the closed orbit the spin tune is just the number of spin precessions around $\hat{n}_0$ for one turn around the ring and the fractional part can be extracted from the complex eigenvalues of the spin transport matrix on the closed orbit. It should be clear from the above, however, that for particles not on the closed orbit, $\hat{n}$ and ν *cannot* be extracted from a 1–turn eigenproblem since $\hat{n}$ changes when tracked for one turn. In general the spin tune depends on the orbital amplitudes $\vec{J}$ and is independent of the orbital phases $\vec{\Psi}$ and starting azimuth so that we denote it by $\nu(\vec{J})$. For a perfectly aligned flat storage ring with no solenoids the design orbit spin tune ν_0 is just the spin enhancement factor $G\gamma$.

Thus acceleration to 820 GeV implies that the beam must traverse several thousand first order spin–orbit depolarizing resonances and that therefore the polarization will be lost. The occurence of the large number of such resonances is confirmed by plotting the dependence of $P_{\rm lim}$ on energy for a normal HERA lattice. The standard response to this is to insert magnet systems called Siberian Snakes into the lattice [5]. These are designed to rotate spins by π around an axis in the horizontal plane independently of the reference energy and their synchrobetatron motion and when installed in the correct combinations the fractional part of the spin tune on the design orbit is 0.5 at all energies. It is thereby intended that by correct choice of orbital tunes, first order resonances can be avoided during acceleration so that the polarization is preserved. However, since spin tune depends on $\vec{J}$ there is no guarentee that this always works.

We now illustrate this for a perfectly aligned HERA proton lattice with a standard arrangement of Siberian Snakes. On the design orbit $\nu_0 = 0.5$ and the fractional part of $Q_{\rm y}$ is 0.273. The field $\hat{n}$ is calculated using *stroboscopic averaging* with the computer code SPRINT [2]. To obtain the spin tune ν, we register the spin phase advance in the $(\hat{u}_1, \hat{u}_2)$ plane for each turn starting from a fixed azimuth $\theta = 0$ and average the phase advance over 1000 turns. In Fig. 1 (upper) we plot $P_{\rm lim}(J_y, 0)$ versus emittance J_y for purely vertical betatron motion at 805 GeV (so that the surface of the 3–torus reduces to an ellipse). There is a clear minimum at about $J_{\rm y} = 27\,\pi$ mm mrad. In the lower figure we plot the corresponding ν versus the emittance. At zero emittance ν is 0.5 as required but it rises to 0.56 at $J_{\rm y} = 40\,\pi$ mm mrad which corresponds to about 3–sigma. At $J_{\rm y} = 27\,\pi$ mm mrad this shows a clear jump straddling the line marking a spin tune of $2Q_{\rm y}$. Thus we conclude that the dip in $P_{\rm lim}$ corresponds to the second order spin–orbit resonance $\nu = 2Q_{\rm y} + {\rm integer}$. Then, although particles at small amplitudes are not on resonance, particles near 27 π mm mrad are close to resonance. So snakes remove the energy variation of ν_0 but they do not sufficiently reduce the dependence of ν on $\vec{J}$. In spite of the comment above on

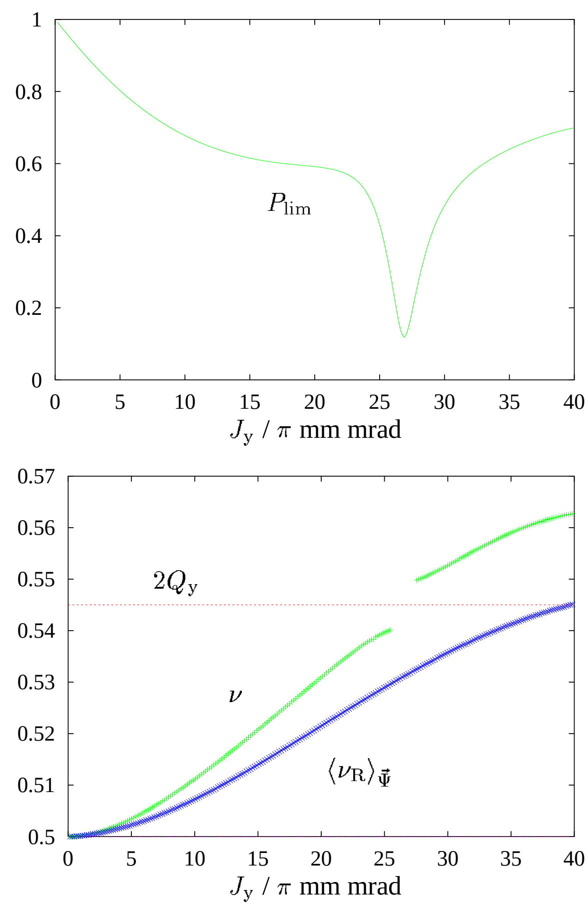

Figure 1: Upper: $P_{\rm lim}$ vs. vertical emittance. Lower: ν and $\langle \nu_{\rm R} \rangle_{\vec{\Psi}}$ vs. emittance

one turn eigensolutions, one often finds reference in the literature to the $\vec{J}$ and $\vec{\Psi}$ dependent quantity $\nu_{\rm R}$ obtained from the complex eigenvalues of $\underline{R}(\vec{J}, \vec{\Psi}, \theta)$. The lower curve in Fig. 1 (lower) is the average over $\Psi_{\rm y}$ of $\nu_{\rm R}(J_{\rm y}, \Psi_{\rm y}, \theta)$ plotted versus emittance. This average takes the value $2Q_{\rm y}$ at about $J_{\rm y} = 40\,\pi$ mm mrad, i.e. where $P_{\rm lim}$ is approaching a maximum. From this it is clear that the value of this "fake spin tune" gives no information about the expectation for $P_{\rm lim}(\vec{J}, \theta)$ or the stability of spin motion in general.

The locus of $\hat{n}$ on the unit sphere obtained by tracking and sampling at a fixed azimuth is a closed curve as required by Eq. (2) in the case of 2–D motion (see e.g. Fig.

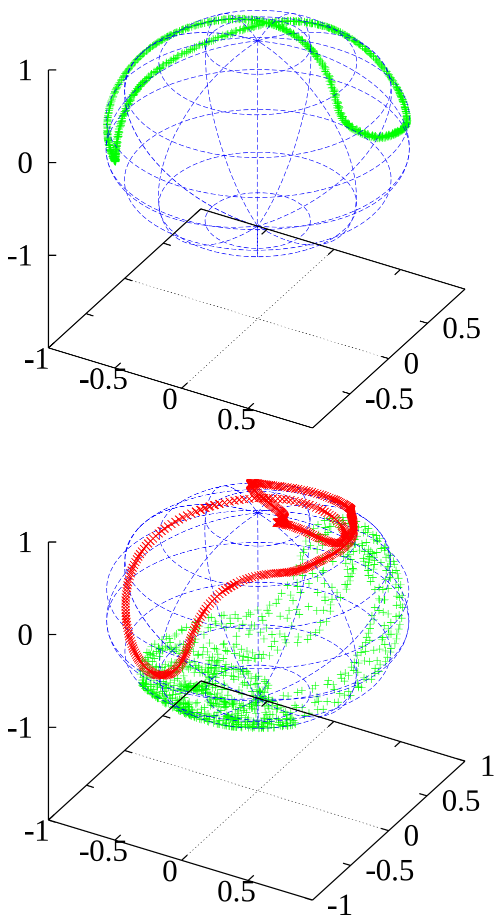

Figure 2: Upper: locus of $\hat{n}$ for 20 π mm mrad vertical emittance. Lower: locus of $\hat{n}$ for 40 π mm mrad and of the anti–damped spin after resonance crossing.

2). However, although the locus of $\hat{u}_{\rm R}(J_{\rm y}, \Psi_{\rm y}, \theta)$, the unit eigenvector with $\hat{u}_{\rm R}(\vec{J}, \vec{\Psi}, \theta) = \underline{R}(\vec{J}, \vec{\Psi}, \theta)\hat{u}_{\rm R}(\vec{J}, \vec{\Psi}, \theta)$ as a function of $\Psi_{\rm y}$ is also a closed curve, it differs from that for $\hat{n}$ and if a $\hat{u}_{\rm R}(J_{\rm y}, \Psi_{\rm y}, \theta)$ for some initial $\Psi_{\rm y}$ is subsequently tracked, it does not form a closed curve. In summary, $\hat{u}_{\rm R}(\vec{J}, \vec{\Psi}, \theta)$ and $\nu_{\rm R}(\vec{J}, \vec{\Psi}, \theta)$ seem to have little relevance.

3 ADIABATIC ANTI–DAMPING

For integrable orbital motion the *spin action* $\hat{S} \cdot \hat{n}$ for an arbitrary spin $\hat{S}$ is an adiabatic invariant away from resonances. Thus by setting the spin $\hat{S}_i$ of a particle which is almost on the closed orbit parallel to $\hat{n}_0$ and tracking it around the ring while adiabatically increasing the orbital amplitude, the final spin $\hat{S}_f$, should be parallel to $\hat{n}$ at the final phase space position. This then provides an alternative to stroboscopic averaging for constructing $\hat{n}$. Fig. 2 (upper), shows the locus on the unit sphere of the $\hat{n}$ reached at $J_{\rm y} = 20\ \pi$ mm mrad. This is indistinguishable from the locus of the $\hat{n}$ obtained by stroboscopic averaging. With a further adiabatic increase through the resonance at 27 π mm mrad out to $J_{\rm y} = 40\ \pi$ mm mrad the method fails: Fig. 2 (lower) shows the locus of $\hat{n}$ and the curve for $\hat{S}_f$ at 40 π mm mrad.

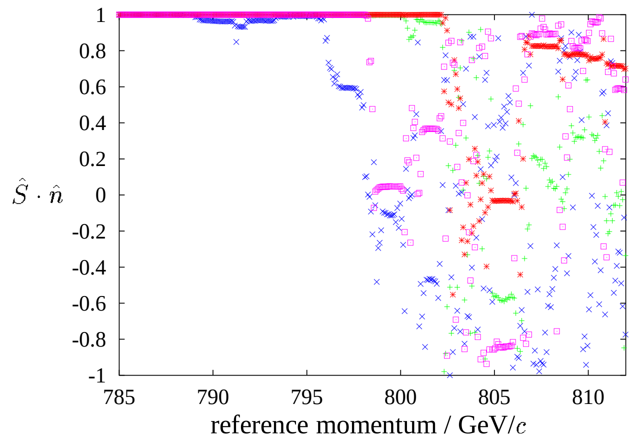

Figure 3: Spin action $\hat{n} \cdot \hat{S}$ during acceleration for 4 representative spins on the 3–torus described in the text.

4 ACCELERATION

When accelerating through energy regions with strong resonances, the spin action $\hat{S} \cdot \hat{n}$, is no longer adiabatically invariant and although below and above the resonances $P_{\rm lim}$ is large, the spins do not necessarily follow $\hat{n}$ so that polarization is lost. This is illustrated in Fig 3 which shows the behaviour of the $\hat{S} \cdot \hat{n}$ for each of 4 particles during an energy ramp from 785 GeV to 812 GeV in which the particles are distributed on the invariant 3–torus with each spin initially parallel to its $\hat{n}$. The emittances are 1.5 times the r.m.s. values of the beam width/energy spread. At various points in the resonance region around 804 GeV [6] the spin actions begin to oscillate wildly so that the polarization of the set finally vanishes.

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