Snake matching

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1 Introduction

Spin–orbit resonances in high energy accelerators arise when the electro–magnetic fields on synchro–betatron trajectories cause disturbances to the spin motion which build up coherently from turn to turn. These disturbances are described in first order by the spin–orbit coupling integrals to be described next. The integrals are especially big at first order resonances and when the spin disturbances from each FODO cell add up coherently [1].

In a perfectly flat ring, an initially vertical spin of a particle traveling on the closed orbit remains vertical during the particle motion. On a vertical betatron trajectory the particle traverses horizontal fields in quadrupoles and the spin no longer remains vertical. This disturbance of spin motion due to the vertical betatron motion is described in first order by the spin–orbit coupling integrals [2, 3] which take the form

$$I_y^{\pm} = \int_{l_0}^{l_0+L} k_y \mathrm{e}^{\mathrm{i}(-\Psi \pm \Psi_y)} dl \ . \tag{1}$$

where $k_y = k\sqrt{\beta_y}$ whereby k is the quadrupole strength and β_y is the vertical beta function; Ψ is the phase advance of the spin rotation around the vertical and Ψ_y is the vertical betatron phase. The integral is evaluated over the ring circumference L. Spin–orbit coupling integrals describing spin disturbances due to horizontal betatron motion and synchrotron motion can also be defined. But in a flat ring they vanish.

Thus in a first order approximation we only need the integrals $I_y^{\pm}$ and they yield the following important information: if the spin–orbit coupling integrals $I_y^{\pm}$ vanish, all initially vertical spins are again vertical after one turn, although they have traveled along different betatron trajectories. The ring is then said to be spin matched or spin transparent when viewed from l_0. In general the spin–orbit coupling integrals depend strongly on the beam energy and on the chosen optic. However, as expected and as implied by the numerical results presented earlier, the inclusion of Siberian Snakes can significantly weaken such dependencies. It is then interesting to see if snake configurations can be chosen by analytical means for which the spin–orbit coupling integrals can be made to vanish. We call this version of spin matching 'snake matching'. In large rings with many snakes, snake matching can be achieved for separate sections of the ring [2]. In HERA-p with a maximum of eight snakes this cannot be done in general. However, HERA-p has an approximate four fold symmetry and as we will see later, in rings with exact four fold symmetry all spin orbit integrals can be canceled completely even with only eight snakes. In fact, it is even possible to find snake axis orientations for these eight snakes which are independent of energy and nevertheless cancel all the spin orbit coupling integrals.

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2 Spin matching with Type III snakes

For a ring with super–period 4, the one turn spin integral starting at $l_0 = 0$ is

$$I_y^\pm = I_{\frac{1}{4}}^\pm (1 + \mathrm{e}^{\mathrm{i}(\nu\pm Q)/4} + \mathrm{e}^{\mathrm{i}2(\nu\pm Q)/4} + \mathrm{e}^{\mathrm{i}3(\nu\pm Q)/4}) \tag{2}$$

where $I_{\frac{1}{4}}^\pm = \int_0^{L/4} k_y \mathrm{e}^{\mathrm{i}(-\Psi\pm\Psi_y)} dl$ and where $\nu = -\Psi(L)$ and Q are the spin phase advance and the orbital phase advance during one turn. The sign convention is chosen so that $\nu = 2\pi G\gamma$ where G is the gyromagnetic anomaly and γ is the Lorentz factor. Spin transparency requires that I_y^+ as well as I_y^- vanish. This requires that the bracket in equation (2) vanishes which implies that $Q \mathrel{\hat{=}} \pi$ where the symbol $\mathrel{\hat{=}}$ indicates equivalence modulo 2π. However storage rings are never operated at such an orbital resonance so that the four–fold repetitive symmetry cannot be employed to impose spin transparency at any energy.

Type III snakes are devices which rotate spins around the vertical direction by some angle ϕ, while leaving the betatron motion unaltered. They can be used to manipulate the spin phase advance between quadrants. The spin disturbances of one quadrant can be made to cancel against the disturbances of another quadrant by choosing an appropriate spin phase advance between these quadrants. There are exactly three ways to arrange that the spin disturbances cancel during one turn. They are indicated in figure 1.

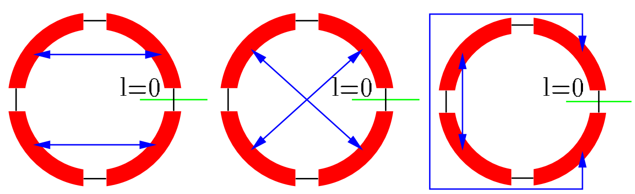

Figure 1: The three ways in which the depolarizing effects of the quadrants of a ring with super–period 4 can cancel each other. The arrows indicate which quadrants have canceled against each other after one turn.

When the Type III snake at $l = j\frac{L}{4}$, $j \in \{1-4\}$ has the rotation angle ϕ_j, then the spin–orbit coupling integrals are

$$\begin{aligned} I^\pm = I_{\frac{1}{4}}^\pm \{1 \; &+ \; \mathrm{e}^{\mathrm{i}[(\nu\pm Q)/4-\phi_1]} + \mathrm{e}^{\mathrm{i}[2(\nu\pm Q)/4-\phi_1-\phi_2]} \\ &+ \; \mathrm{e}^{\mathrm{i}[3(\nu\pm Q)/4-\phi_1-\phi_2-\phi_3]}\} \; . \end{aligned} \tag{3}$$

For a spin match, the bracket on the right hand side has to vanish for '+' and for '−'. A sum of four complex numbers with unit modulus can only vanish when it consists of two pairs of numbers which cancel each other. The three possibilities of cancelation are demonstrated in figure 1 and are described by the following three sets of equations:

1. $(\nu \pm Q)/4 - \phi_1 \mathrel{\hat{=}} \pi$ and $(\nu \pm Q)/4 - \phi_3 \mathrel{\hat{=}} \pi$,
2. $2(\nu \pm Q)/4 - \phi_1 - \phi_2 \mathrel{\hat{=}} \pi$ and $\phi_3 \mathrel{\hat{=}} \phi_1$,
3. $3(\nu \pm Q)/4 - \phi_1 - \phi_2 - \phi_3 \mathrel{\hat{=}} \pi$ and $(\nu \pm Q)/4 - \phi_2 \mathrel{\hat{=}} \pi$.

To spin match, one of these three conditions has to hold for '+', to make I^+ vanish, and another of the conditions has to hold for '$-$', to make I^- vanish. I^+ and I^- cannot vanish simultaneously within one of the three conditions, since this would lead to a restriction on the allowed orbital phase advance Q. Canceling one integral by condition 3 turns out to be incompatible with canceling the other integral by condition 1 or 2. When canceling I^+ by condition 1 and I^- by condition 2, the requirements are compatible and lead to $\phi_1 \overset{\circ}{=} \phi_3 \overset{\circ}{=} \pi + (\nu + Q)/4$ and $\phi_2 \overset{\circ}{=} (\nu - 3Q)/4$.

The rotation angle ϕ_4 of the Type III snake at $l_0 = 0$ is chosen in such a way that the spin tune of the ring is not changed by the snakes: $\phi_1 + \phi_2 + \phi_3 + \phi_4 \overset{\circ}{=} 0$. With $\Sigma = (\nu + Q)/4$ the required rotation angles are

$$\phi_1 \overset{\circ}{=} \phi_3 \overset{\circ}{=} \Sigma + \pi \ , \quad \phi_2 \overset{\circ}{=} \Sigma - Q \ , \quad \phi_4 \overset{\circ}{=} \Sigma - \nu \ . \tag{4}$$

A change in sign of Q cancels I^+ due to condition 2 and I^- due to condition 1. There are therefore exactly two possibilities of making a ring with super–period 4 spin transparent by means of four Type III snakes. These possibilities are shown in figure 2 where the longitudinal direction of particle motion is chosen to be clockwise. In passing we note that mirror symmetric

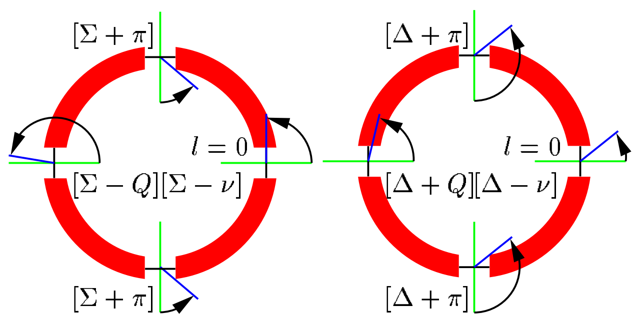

Figure 2: The two only ways to spin match a ring with super–period four by four Type III snakes. The number between 0 and 2π which equals x modulo 2π is written as $[x]$, $\Sigma = (\nu + Q)/4$, $\Delta = (\nu - Q)/4$.

arcs present no advantages for spin matching a ring by this method.

3 Energy independent spin matching

Spin matching with four Type III snakes has the great disadvantage that the rotation angles change with the energy dependent spin phase advance ν and thus have to be ramped in order to spin match at each energy. So we now consider what can be achieved with snakes whose rotation axes lie in the horizontal plane ('horizontal snakes').

Siberian Snakes [4] other than Type III rotate all spins by π around their horizontal rotation axes. If the axis is at an angle $\alpha/2$ with respect to the radial direction in the longitudinal backward direction, a 'horizontal' snake is said to have a snake angle of α [2]. Its effect on spins is equivalent to that of a snake with a radial axis followed immediately by a Type III snake with a rotation angle α. Therefore also snakes with horizontal axes can be used to manipulate the spin phase advance between parts of the ring.

We now assume that there are n horizontal snakes in the ring. The snake angle of the j-th snake is α_j. Furthermore we assume that these snakes have been chosen in such a way that $\vec{n}$ on the closed orbit, which is denoted by $\vec{n}_0$ and is periodic, is aligned along the vertical in all of the ring and that the spin tune on the closed orbit is 1/2. The positions of these snakes are denoted by l_j and the advance of the spin phase beyond the jth snake is $\Psi_j(l)$ where $\Psi_j(l_j) = 0$.

For simplicity we set $l_0 = 0$, $l_{n+1} = L$, and denote the spin phase advance between snake j and snake $j+1$ as Ψ_j. The orbital phase advance Ψ_{yj} between snake j and snake $j+1$ is denoted by Ψ_{yj}. Taking $\vec{n}_0$ to point initially vertically upwards we then obtain the spin–orbit coupling integrals [2]

$$
\begin{aligned}
I^{\pm} &= \sum_{j=0}^{n} \mathrm{e}^{\mathrm{i}\sum_{k=0}^{j-1}[-(-1)^k(\alpha_k+\Psi_k)\pm\Psi_{yk}]} \\
&\cdot \int_{l_j}^{l_{j+1}} k_y \mathrm{e}^{\mathrm{i}[-(-1)^j(\Psi_j(l)+\alpha_j)\pm\Psi_{yj}(l)]} dl \ . \qquad (5)
\end{aligned}
$$

3.1 Four snake schemes

For a four–fold symmetric ring with a horizontal snake between each arc the spin–orbit coupling integrals (5) reduce to

$$
\begin{aligned}
I^{\pm} &= I^{\pm}_{\frac{1}{4}}\{1 + e^{\mathrm{i}(\alpha_1-\alpha_2\pm 2Q/4)}\} \\
&+ (I^{\mp}_{\frac{1}{4}})^* e^{\mathrm{i}[(\nu\pm Q)/4+\alpha_1]}\{1 + e^{\mathrm{i}(-\alpha_2+\alpha_3\pm 2Q/4)}\} \ .
\end{aligned}
$$

Spin transparency of the ring is therefore in general only obtained when

$$
\alpha_1 - \alpha_2 \pm 2Q/4 \overset{\circ}{=} \pi \text{ and } -\alpha_2 + \alpha_3 \pm 2Q/4 \overset{\circ}{=} \pi \ . \qquad (6)
$$

Choosing the '+' combination for one equation and the '−' combination for the other implies $Q \overset{\circ}{=} 0$. A synchrotron is never operated with this orbital phase advance. Therefore four horizontal snakes cannot be used to eliminate all spin–orbit coupling integrals.

In passing we note again that an additional mirror symmetry does not simplify the compensation of the spin–orbit integrals.

3.2 Eight snake schemes

The same procedure can now be repeated with eight snakes. To do that we place four more horizontal snakes at the locations $kL/4 + \Delta l_k$, $k \in \{0,1,2,3\}$, i.e. at a distance Δl_k downstream from the four snakes treated above. The complete spin phase advance of the ring is $\sum_{j=0}^{7}(-1)^j(\Psi_j + \alpha_j)$ and this should be set to π independently of the energy. Therefore $\sum_{j=0}^{7}(-1)^j\Psi_j$ must vanish at all energies so that the Δl_k must be chosen to ensure that $\Psi_j = \Psi_0 \equiv \Psi$, $j \in \{1-7\}$. From equation (5) and with $I_0^{\pm} = \int_0^{\Delta l} k_y e^{\mathrm{i}(-\Psi\pm\Psi_y)} dl$ and $I_1^{\pm} = \int_{\Delta l}^{\frac{L}{4}} k_y e^{\mathrm{i}(\Psi\pm\Psi_y)} dl$, the spin–orbit coupling integrals are

$$
\begin{aligned}
I^{\pm} &= I_0^{\pm}\{1 + e^{\mathrm{i}[\alpha_1-\alpha_2\pm(\Psi_{y0}+\Psi_{y1})]} + \ldots \qquad (7)\\
&+ e^{\mathrm{i}[\alpha_1-\alpha_2\mp\ldots-\alpha_{n-2}\pm(\Psi_{y0}+\ldots+\Psi_{yn-3})]}\} \\
&+ I_1^{\pm} e^{\mathrm{i}(-\Psi_0+\alpha_1\pm\Psi_{y0})}\{1 + e^{\mathrm{i}[-\alpha_2+\alpha_3\pm(\Psi_{y1}+\Psi_{y2})]} \\
&+ \ldots + e^{\mathrm{i}[-\alpha_2+\alpha_3\pm\ldots+\alpha_{n-1}\pm(\Psi_{y1}+\ldots+\Psi_{yn-2})]}\} \ .
\end{aligned}
$$

where most of the spin phases $\Psi_j \equiv \Psi$ have canceled. In terms of the difference angles $\Delta_{jk} = \alpha_j - \alpha_k$, spin matching the ring therefore requires

$$\begin{aligned} 1 &+ e^{\mathrm{i}(\pm Q/4+\Delta_{12})} + e^{\mathrm{i}(\pm 2Q/4+\Delta_{12}+\Delta_{34})} \\ &+ e^{\mathrm{i}(\pm 3Q/4+\Delta_{12}+\Delta_{34}+\Delta_{56})} = 0 \ , \end{aligned} \tag{8}$$

$$\begin{aligned} 1 &+ e^{\mathrm{i}(\pm Q/4-\Delta_{23})} + e^{\mathrm{i}(\pm 2Q/4-\Delta_{23}-\Delta_{45})} \\ &+ e^{\mathrm{i}(\pm 3Q/4-\Delta_{23}-\Delta_{45}-\Delta_{67})} = 0 \ . \end{aligned} \tag{9}$$

These equations have the same structure as the matching conditions of equations (3) and we can therefore use the relations (4) to obtain the following two ways to satisfy equation (8):

$$\Delta_{12} \ \overset{\circ}{=} \ \Delta_{56} \overset{\circ}{=} \pi - Q/4 \ , \quad \Delta_{34} \overset{\circ}{=} 3Q/4 \ , \tag{10}$$

$$\Delta_{12} \ \overset{\circ}{=} \ \Delta_{56} \overset{\circ}{=} \pi + Q/4 \ , \quad \Delta_{34} \overset{\circ}{=} -3Q/4 \ . \tag{11}$$

There are also exactly two ways to solve equation (9),

$$\Delta_{23} \overset{\circ}{=} \Delta_{67} \overset{\circ}{=} \pi + Q/4 \ , \quad \Delta_{45} \overset{\circ}{=} -3Q/4 \ , \tag{12}$$

$$\Delta_{23} \overset{\circ}{=} \Delta_{67} \overset{\circ}{=} \pi - Q/4 \ , \quad \Delta_{45} \overset{\circ}{=} 3Q/4 \ . \tag{13}$$

There are now four ways to spin match the ring; these are obtained by combining the equations (10)&(12), (10)&(13), (11)&(12), or (11)&(13), where the last two possibilities result from the first two by reversing the sign of Q.

Since only differences in the snake angles appear, one of the angles can be chosen arbitrarily. All other snake angles are then fixed. Combining the equations (10) and (12) and choosing $\alpha_1 = 0$ leads to

$$\begin{aligned} \alpha_1 \ &\overset{\circ}{=} \ \alpha_3 \overset{\circ}{=} \alpha_5 \overset{\circ}{=} \alpha_7 \overset{\circ}{=} 0 \ , \quad \alpha_2 \overset{\circ}{=} \alpha_6 \overset{\circ}{=} \pi + Q/4 \ , \\ \alpha_4 \ &\overset{\circ}{=} \ -3Q/4 \ , \quad \alpha_8 \overset{\circ}{=} \pi + Q/4 \ . \end{aligned} \tag{14}$$

Combining the equations (10) and (13) and choosing $\alpha_1 = -Q/2$ leads to

$$\begin{aligned} \alpha_1 \ &\overset{\circ}{=} \ -2Q/4 \ , \quad \alpha_2 \overset{\circ}{=} \alpha_6 \overset{\circ}{=} \pi - Q/4 \ , \\ \alpha_3 \ &\overset{\circ}{=} \ \alpha_5 \overset{\circ}{=} \alpha_7 \overset{\circ}{=} 0 \ , \quad \alpha_4 \overset{\circ}{=} -3Q/4 \ , \quad \alpha_8 \overset{\circ}{=} \pi + 3Q/4 \ . \end{aligned} \tag{15}$$

These snake schemes are shown in figure 3. Advantage has been taken of the fact that the angle $\alpha/2$ between the radial direction and the rotation axis only needs to be known modulo π.

Here it is very important to note that the snake angles are independent of $\nu = 2\pi G\gamma$ and therefore that a spin match has been achieved for all energies for the chosen Q.

At high energies, for example at 820GeV in HERA-p, the polarization limit P_{lim} determined by the $\vec{n}$-axis can be problematically small [5]. In first order the proposed spin matching increases the polarization limit to 100% since all spins return to the vertical on all betatron orbits as long as no misalignments and deviations from the four–fold symmetry are present in the ring.

Note that the set of snake angles depicted in figure 3 is somewhat unconventional. Naturally this set of snake angles differs from that obtained by filtering as described in [6] but nevertheless this analysis indicates why filtering will select such exotic sets of snake angles. Of course a similar kind of analysis could be used to explain why some snake schemes can be particularly bad.

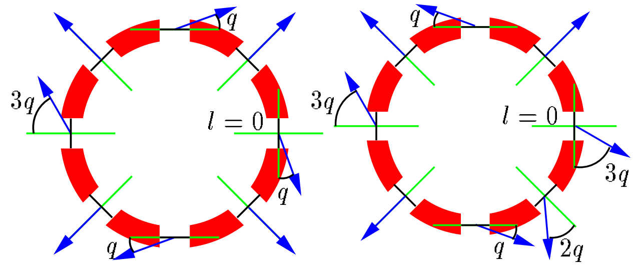

Figure 3: Two of the four possible ways to spin match a ring with super–period four using eight horizontal snakes. q is $Q/8$ modulo π and describes the angle between the radial direction and the snake axis. The other two ways are obtained by reversing the sign of Q. Note that the snake angles are independent of ν and thus of energy.

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