Warm up quiz:

Example of operators

I. Multiplication: $\hat{A}\psi(x) = x\psi(x)$

II. Differentiation: $\hat{A}\psi(x) = \frac{d}{dx}\psi(x)$

III. Translation: $\hat{A}\psi(x) = e^{a\frac{d}{dx}}\psi(x) = \psi(x+a)$

IV. Projection: $\hat{A}\psi(x) = h(x)\psi(x)$, where

$$h(x) = \begin{cases} 0, & x<0 \\ 1, & x\geq 0 \end{cases}$$

① Which operator is linear?

A. I B. II C. III D. IV E. all

② Which operator is norm preserving?

A. I B. II C. III D. IV E. all

③ Which operator is Hermitian?

A. I B. I, II C. I, IV D. I, II, IV E. all

④ Which operator is unitary?

A. I B. II C. III D. IV E. all

Notes: to answer 3 & 4 we need to find adjoint operators, i.e. $\hat{A}^+$ such that

$$\int \varphi^*(x)\,\hat{A}\psi(x)\,dx = \int (\hat{A}^+\varphi(x))^*\,\psi(x)\,dx$$

I. $\hat{A} = x$, $\hat{A}^+ = x$

$$\int \varphi^*(x)\left(x\psi(x)\right)dx = \int \left(x\varphi(x)\right)^*\psi(x)\,dx$$

II. $\hat{A} = \frac{d}{dx}$, $\hat{A}^+ = -\frac{d}{dx}$

$$\int \varphi^*(x)\left(\frac{d}{dx}\psi(x)\right)dx = \cancel{\varphi^*(x)\psi(x)\Big|_{-\infty}^{\infty}}^{\,0} - \int \frac{d}{dx}\varphi^*(x)\,\psi(x)\,dx$$

$$= \int \left(-\frac{d}{dx}\varphi(x)\right)^* \psi(x)\,dx$$

III. $\hat{A} = e^{a\frac{d}{dx}}$, $\hat{A}^+ = e^{-a\frac{d}{dx}}$

$$\int \varphi^*(x)\left(\hat{A}\psi(x)\right)dx = \int \varphi^*(x)\,\psi(x+a)\,dx$$

$\rightarrow$ change of variables : $\left.\begin{matrix} x+a \\ x \\ dx \end{matrix}\right\} \rightarrow \left\{\begin{matrix} x \\ x-a \\ dx \end{matrix}\right.$

$$= \int \varphi^*(x-a)\,\psi(x)\,dx \; ; \quad \hat{A}^+ \text{ shifts } \underbrace{-a}$$

IV. $\hat{A} = h(x)$, $\hat{A} = h(x)$

$$\int_{-\infty}^{\infty} \varphi^*(x)\left(\hat{A}\psi(x)\right)dx = \int_0^{\infty} \varphi^*(x)\psi(x)\,dx = \int_{-\infty}^{\infty}\left(\hat{A}^+\varphi(x)\right)^*\psi(x)\,dx$$