Sept 4.

Quantum motion in phase space

Motivation

(1) class. mechanics: part. state (x,p) + Hamiltonian $\Rightarrow$ full knowledge (including past & future)

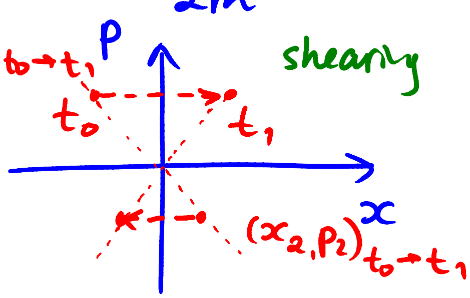
Time evolution: Hamilton's eqns (1D)

$$\begin{cases} \dot{p} = -\dfrac{\partial \mathcal{H}}{\partial x} \\ \dot{x} = \dfrac{\partial \mathcal{H}}{\partial p} \end{cases}$$

Ex1: simple motion (drift)

$$\mathcal{H}(x,p) = KE + PE = \frac{p^2}{2m}$$

$$\begin{cases} \dot{p} = 0 \\ \dot{x} = \dfrac{p}{m} \end{cases}$$



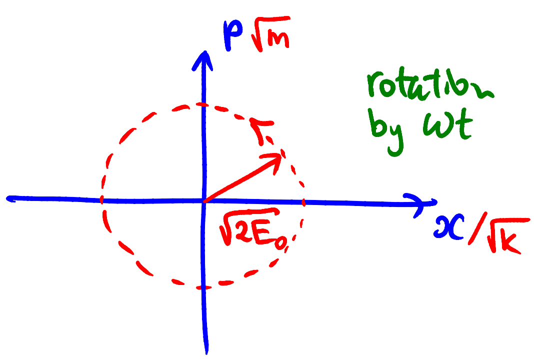
Ex2: simple motion (linear restor. force)

$$\mathcal{H} = \frac{p^2}{2m} + \frac{kx^2}{2} = E_0$$

$$\begin{cases} \dot{p} = -kx \\ \dot{x} = \dfrac{p}{m} \end{cases}$$

$$\ddot{p} + \underbrace{\frac{k}{m}}_{\omega^2} p = 0$$

$$\ddot{x} + \frac{k}{m} x = 0$$



② QM state: $|\psi\rangle$, $\hat{\mathcal{H}}$ = full knowledge

position $\quad\quad\quad\quad\quad$ momentum

$$\psi(x) \equiv \langle x|\psi\rangle \xleftrightarrow{FT} \tilde{\psi}(p) \equiv \langle p|\psi\rangle$$

$|\psi(x)|^2 dx$ $\quad\quad\quad\quad\quad$ $|\tilde{\psi}(p)|^2 dp$

$p(x, x+dx)$ $\quad\quad\quad\quad\quad$ $p(p, p+dp)$

probability density $W(x,p)$ – ?

$[\hat{x}, \hat{p}] = i\hbar \neq 0$: incompatible operators
$\Rightarrow$ no joint eigenbasis

But can still define $W(x,p)$ – [quasi] probability

③ Wigner distribution:

$$W(x,p) = \int_{-\infty}^{\infty} \langle\psi|x+\tfrac{x'}{2}\rangle \underbrace{\langle x+\tfrac{x'}{2}|p\rangle\langle p|x-\tfrac{x'}{2}\rangle}_{A} \langle x-\tfrac{x'}{2}|\psi\rangle dx'$$

Dirac notation of integrand: quantum trajectory from $x-\frac{x'}{2}$ to $x+\frac{x'}{2}$ with momentum p

$\int_{-\infty}^{\infty} \ldots dx'$: superposition of all possible trajectories

Expanding A:

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{ipx}{\hbar}} \leftarrow \text{HW1}$$

$$A: \frac{1}{2\pi\hbar} e^{\frac{ip}{\hbar}(x+\frac{x'}{2})} e^{-\frac{ip}{\hbar}(x-\frac{x'}{2})} = \frac{1}{2\pi\hbar} e^{\frac{ipx'}{\hbar}}$$

$$\boxed{W(x,p) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \psi^*(x+\tfrac{x'}{2})\, \psi(x-\tfrac{x'}{2})\, e^{\frac{ipx'}{\hbar}} dx'}$$

④ Can write the same thing in momentum space

$$W(x,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} \langle\psi|p+\tfrac{p'}{2}\rangle\langle p+\tfrac{p'}{2}|x\rangle\langle x|p-\tfrac{p'}{2}\rangle\langle p-\tfrac{p'}{2}|\psi\rangle\, dp'$$

$$\boxed{W(x,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} \tilde{\psi}\left(p+\frac{p'}{2}\right)\tilde{\psi}\left(p-\frac{p'}{2}\right)e^{-\frac{ip'x}{\hbar}}\, dp'}$$

where $\tilde{\psi}(p) = \frac{1}{\sqrt{2\pi\hbar}}\int \psi(x)\, e^{-\frac{ipx}{\hbar}}\, dx$

$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}}\int \tilde{\psi}(p)\, e^{\frac{ipx}{\hbar}}\, dp$

FT

⑤ Properties of Wigner distribution function WDF

a) $W^*(x,p) = W(x,p)$ - or $W(x,p) \in \mathbb{R}$

b) $\iint W(x,p)\, dx\, dp = 1$ - normalization

c) $\left.\begin{aligned} \int W(x,p)\, dx &= |\tilde{\psi}(p)|^2 \\ \int W(x,p)\, dp &= |\psi(x)|^2 \end{aligned}\right\}$ marginals

d) $|W(x,p)| \leq \frac{1}{\pi\hbar}$ - boundedness

⑥ Expectation values:

operator $\hat{A} \longrightarrow A(x,p)$

Wigner-Weyl transformation

$$\langle A\rangle = \iint A(x,p)\, W(x,p)\, dx\, dp$$

Some nice examples:

$$\hat{x}^n \to x^n$$
$$\hat{p}^n \to p^n$$
$$\frac{1}{2}(\hat{x}\hat{p} + \hat{p}\hat{x}) \to xp$$

⑦ Time evolution of WDF $W(x,p;t)$

$$\frac{\partial W}{\partial t} + \frac{p}{m}\frac{\partial W}{\partial x} + \frac{i}{\hbar}\left[V\left(x + \frac{i\hbar}{2}\frac{\partial}{\partial p}\right) - V\left(x - \frac{i\hbar}{2}\frac{\partial}{\partial p}\right)\right] W = 0$$

HW proof

let $V(x) = V_0 - F_0 x + \frac{1}{2}kx^2$

or $F(x) = F_0 - kx$,

$$\Rightarrow \quad \frac{\partial W}{\partial t} + \frac{p}{m}\frac{\partial W}{\partial x} + F\frac{\partial W}{\partial p} = 0$$

same as Liouville's eqn!

show that :

$$\frac{i}{\hbar}\Big[V_0 - F_0\left(x + \frac{i\hbar}{2}\frac{\partial}{\partial p}\right) + \frac{1}{2}k\left(x + \frac{i\hbar}{2}\frac{\partial}{\partial p}\right)^2$$
$$- V_0 + F_0\left(x - \frac{i\hbar}{2}\frac{\partial}{\partial p}\right) - \frac{1}{2}k\left(x - \frac{i\hbar}{2}\frac{\partial}{\partial p}\right)^2\Big]$$

$$= \frac{i}{\hbar}\left[-F_0\, i\hbar\frac{\partial}{\partial p} + kx\, i\hbar\frac{\partial}{\partial p}\right]$$

$$= (F_0 - kx)\frac{\partial}{\partial p} = F(x)\frac{\partial}{\partial p} \quad \therefore$$

⑧ Implications

* WDF evolves classically with time for linear forces
* WDF can be measured using tomography method (same as measuring $\psi(x)$, not $|\psi(x)|^2$!)
* "squeezed" states, "coherent" states