Diffraction (contd.)

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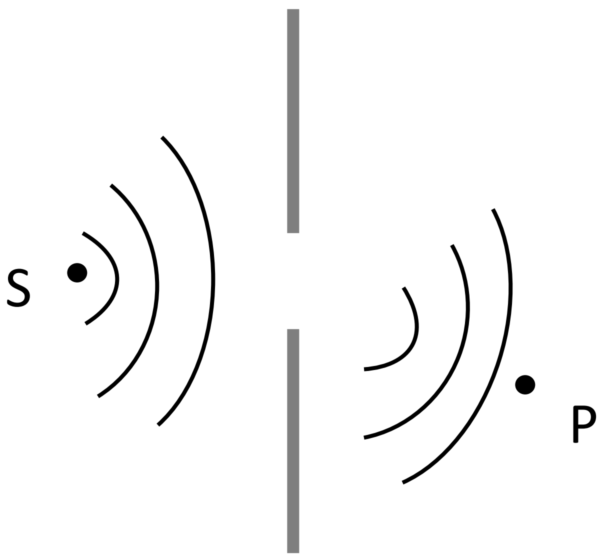
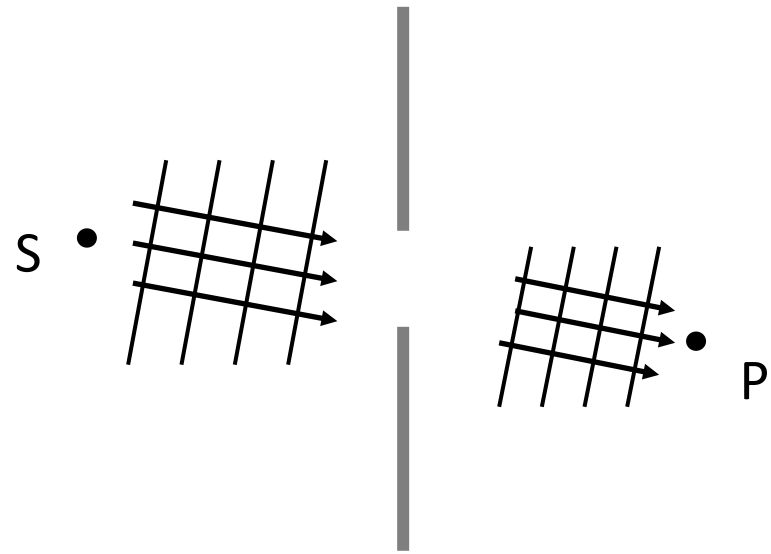
Outline

- **Fresnel vs. Fraunhofer diffraction**
- **Talbot effect**
- **Fresnel zones & plates**
- **Fraunhofer diffraction**
- **2D Fourier Transforms**

Fresnel vs. Fraunhofer diffraction





Fresnel:

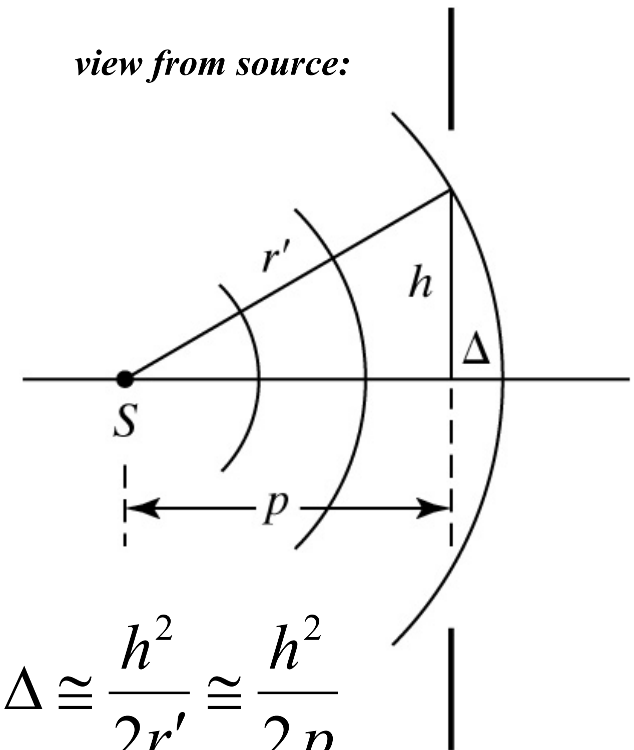
occurs when either S or P are close enough to the aperture that wavefront curvature is not negligible

Fraunhofer:

both incident and diffracted waves may be considered to be planar (i.e. both S and P are far from the aperture)

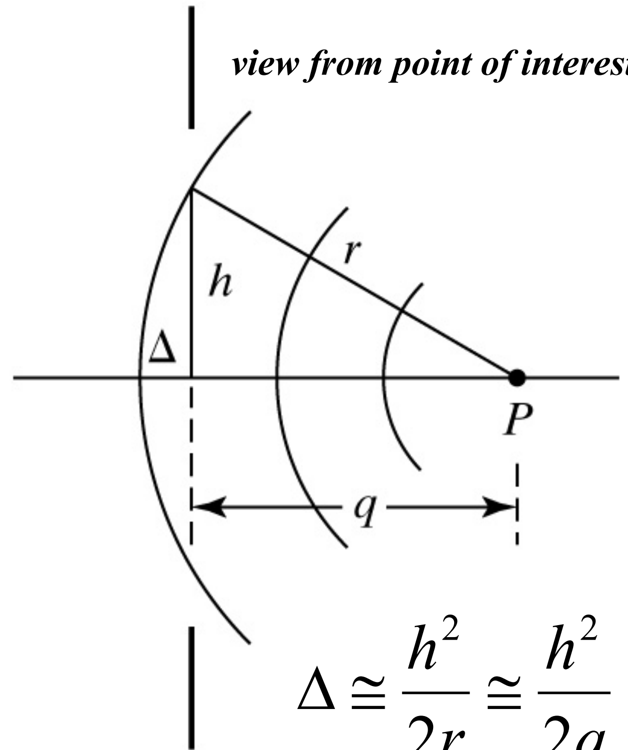
Fresnel vs. Fraunhofer criterion

view from source:



$$\Delta \cong \frac{h^2}{2r'} \cong \frac{h^2}{2p}$$

view from point of interest:



$$\Delta \cong \frac{h^2}{2r} \cong \frac{h^2}{2q}$$

near field $\equiv$

$$\Delta > \lambda$$

$$\frac{1}{2}\left(\frac{1}{p}+\frac{1}{q}\right)h^2 > \lambda$$

$$d < \frac{A}{\lambda}$$

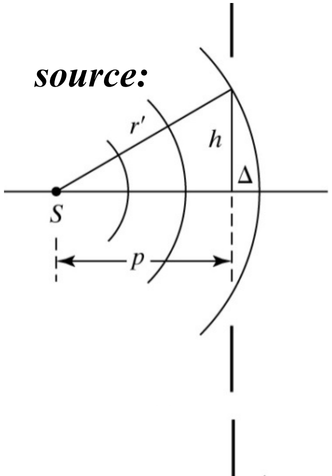
where d represents p or q (=distance from source or point to aperture)

A is aperture area

Fresnel number

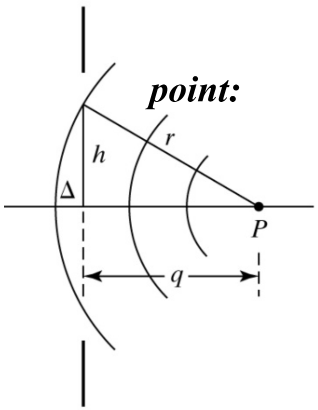
Fraunhofer diffraction occurs when:

$$F = \frac{h^2}{d\lambda} << 1$$



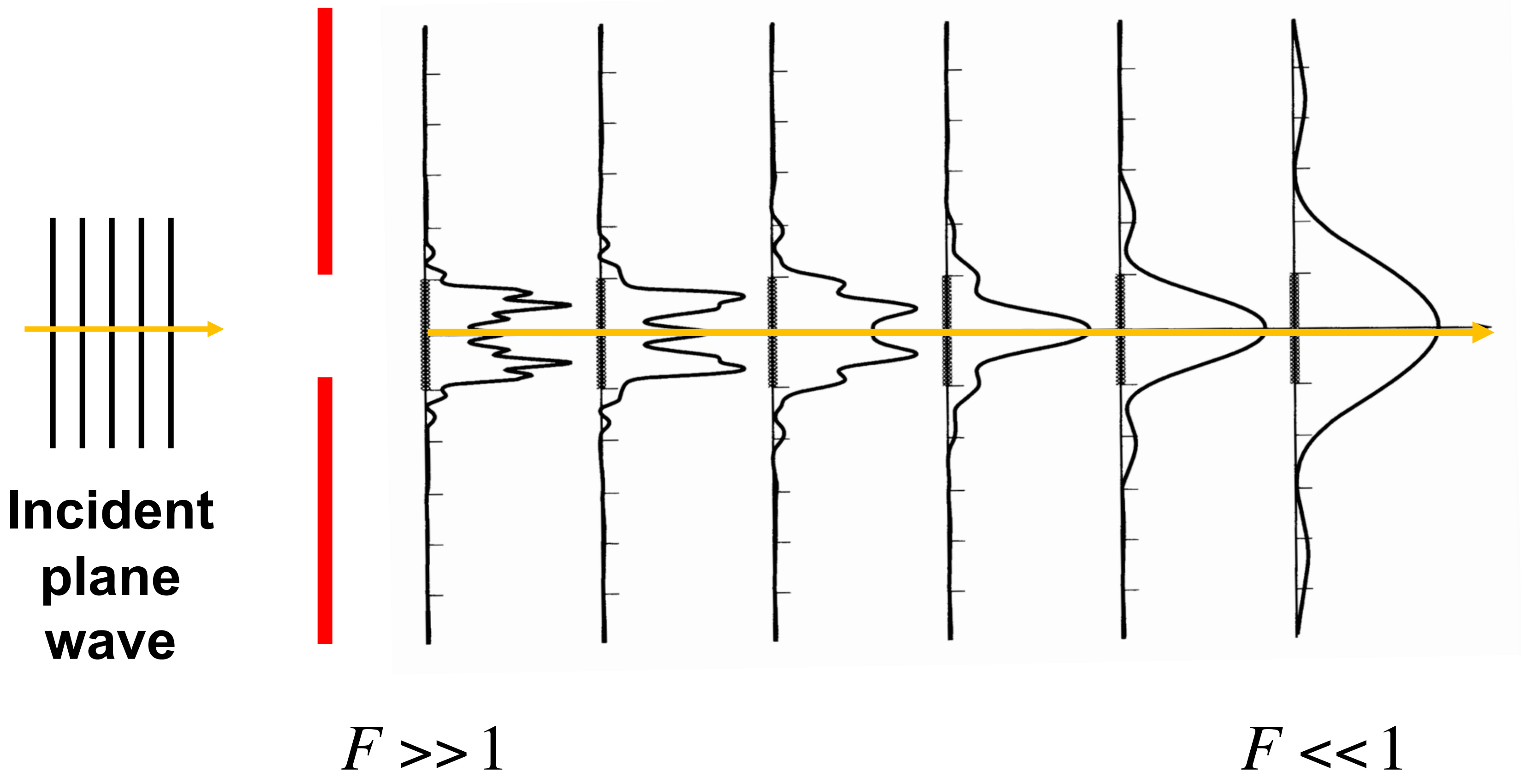
Fresnel diffraction occurs when:

$$F = \frac{h^2}{d\lambda} \geq 1$$

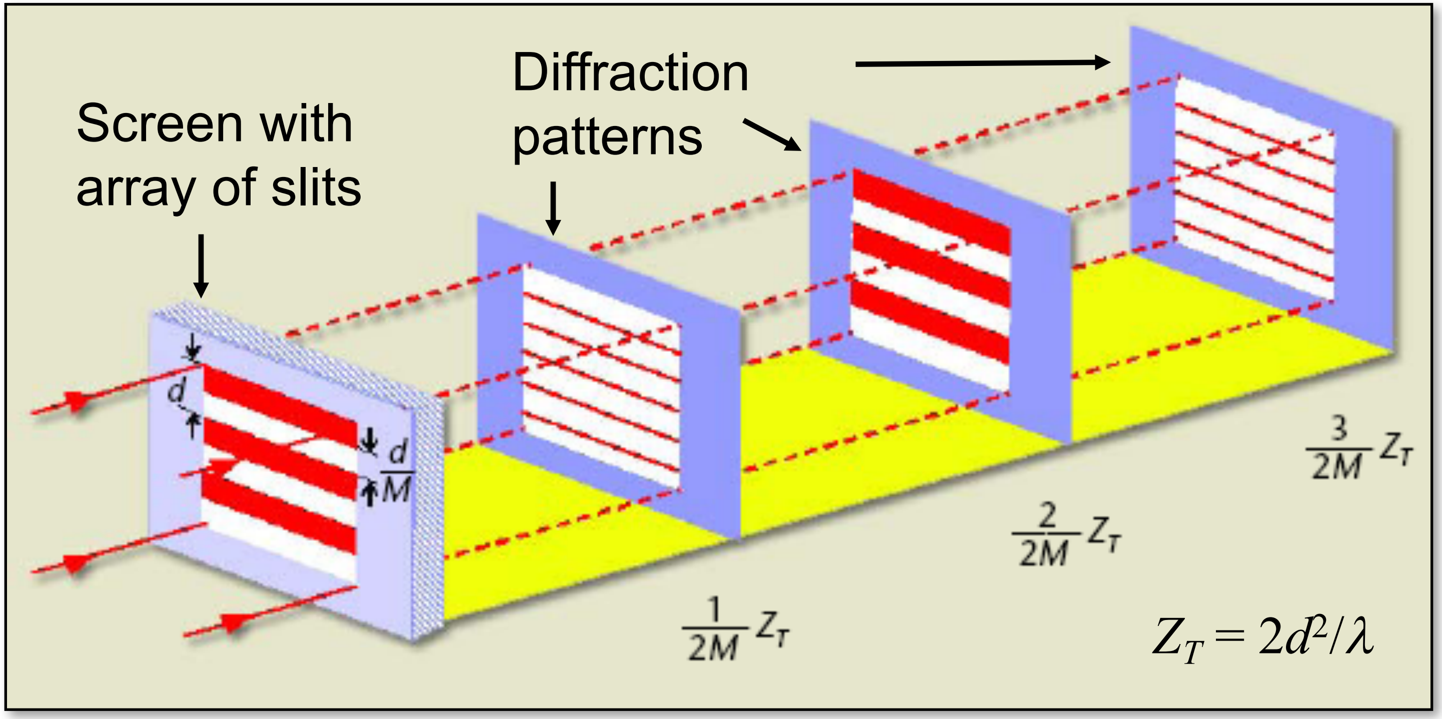


where h = aperture or slit size
λ = wavelength
d = distance from the aperture (p or q)

From Fresnel to Fraunhofer diffraction

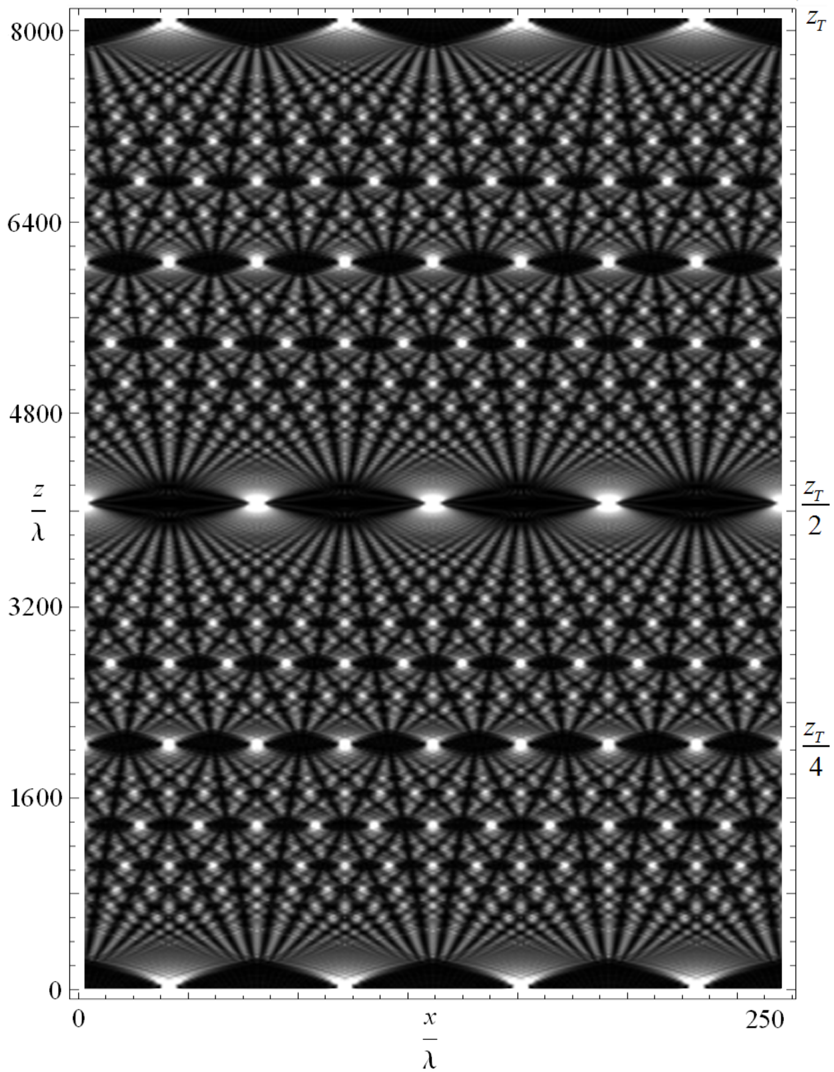


Fresnel diffraction from infinite array of slits: Talbot effect



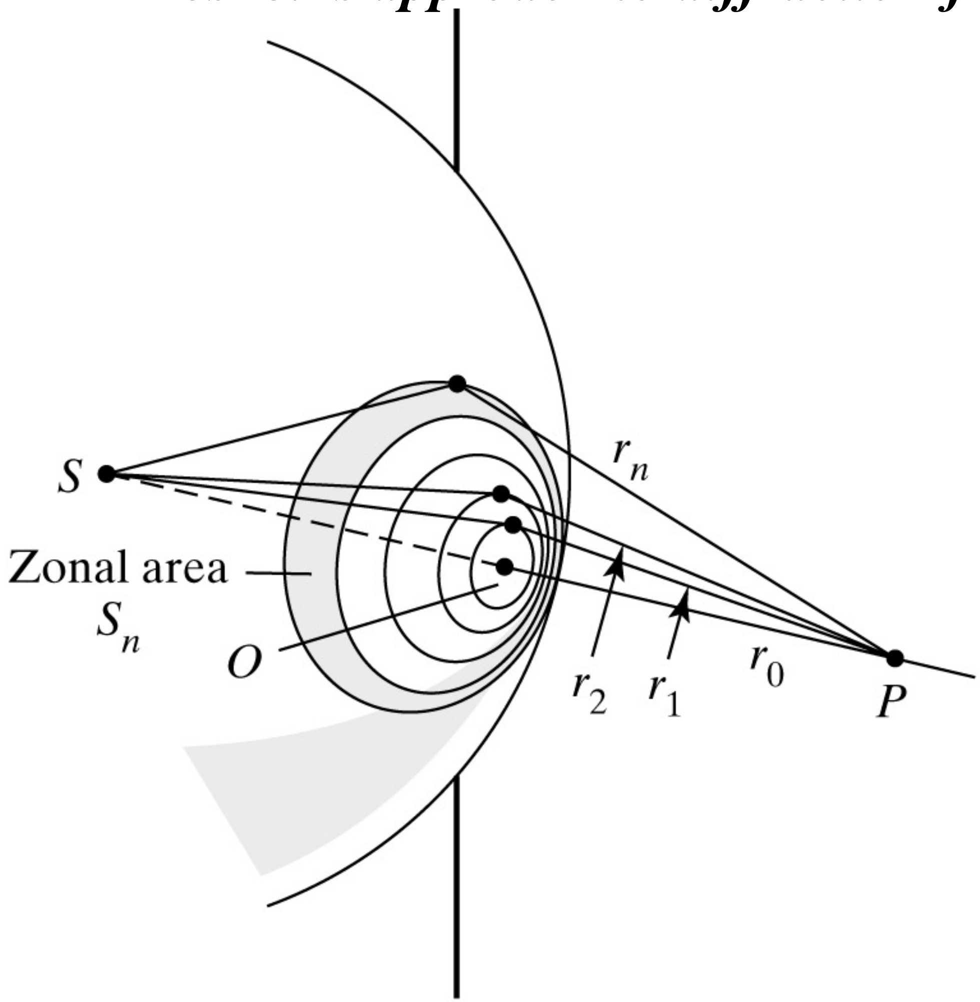
- **one of the few Fresnel diffraction problems that can be solved analytically**
- **beam pattern alternates between two different fringe patterns**

Talbot "carpet"



Fresnel zones (180º phase difference)

Fresnel's approach to diffraction from circular aperatures



zone spacing = $\lambda/2$:

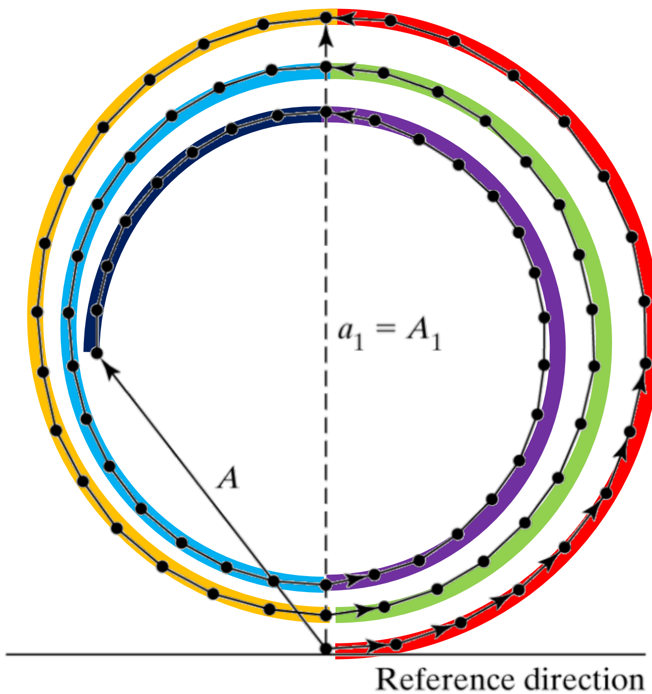
$$r_1 = r_0 + \lambda/2$$
$$r_2 = r_0 + \lambda$$
$$r_3 = r_0 + 3\lambda/2$$
$$\vdots$$
$$r_n = r_0 + n\lambda/2$$

these are called the *Fresnel zones*

(note: all zones have equal areas)

Adding up light from the zones

as we draw a phasor diagram where each zone is subdivided into 15 subzones



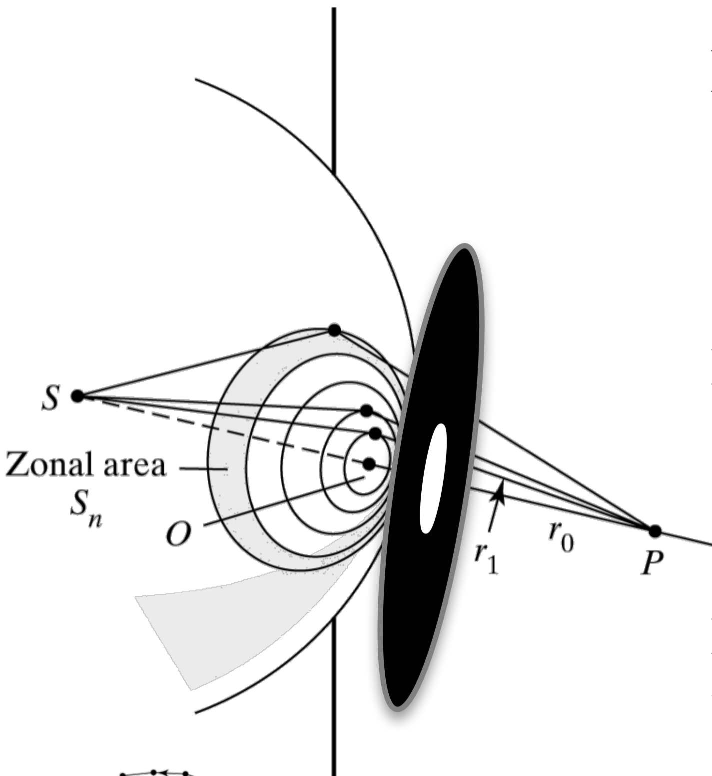
5½ half-period zones

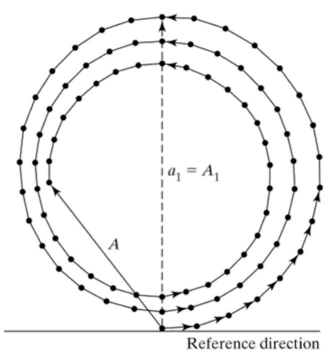
- **obliquity factor shortens successive phasors**
- **circles do not close, but spiral inwards**
- **amplitude $a_1 = A_1$: resultant of subzones in 1st half-period zone**
- **composite amplitude at P from n half-period zones:**

$$A_n = a_1 + a_2 e^{i\pi} + a_3 e^{i2\pi} + a_4 e^{i3\pi} + \ldots a_n e^{i(n-1)\pi}$$

$$A_n = a_1 - a_2 + a_3 - a_4 + \ldots a_n$$

Some interesting implications of Fresnel zones





A circular aperture is matched in size with the first Fresnel zone:

What is amplitude of the wavefront at P?

$$A_P = a_1$$

Now open the aperture wider to also admit zone 2:

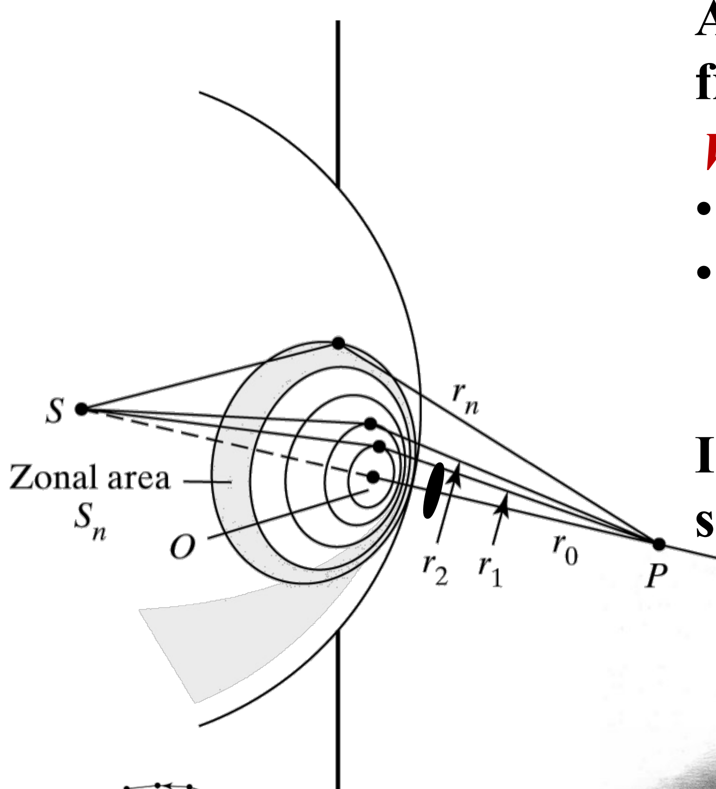
$$A_P \sim 0\ !$$

Now remove aperture, allowing all zones to contribute:

$$A_P = \tfrac{1}{2}\, a_1\ !!!$$

(To find intensity – square the amplitudes, i.e. it's only ¼ of the 1st zone!)

Some interesting implications of Fresnel zones



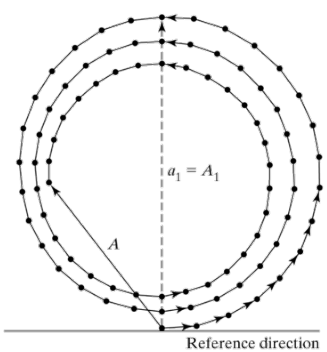
A circular <u>disk</u> is matched in size with the first Fresnel zone:

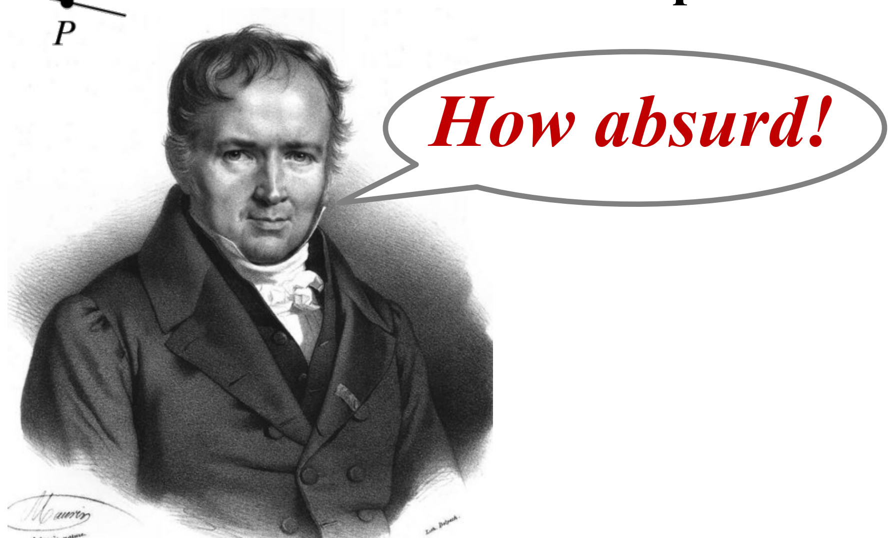
What is amplitude of the wavefront at P?

- **all zones *except* the first contribute**
- **first contributing zone is the second**

$$A_P = \tfrac{1}{2}\, a_2$$

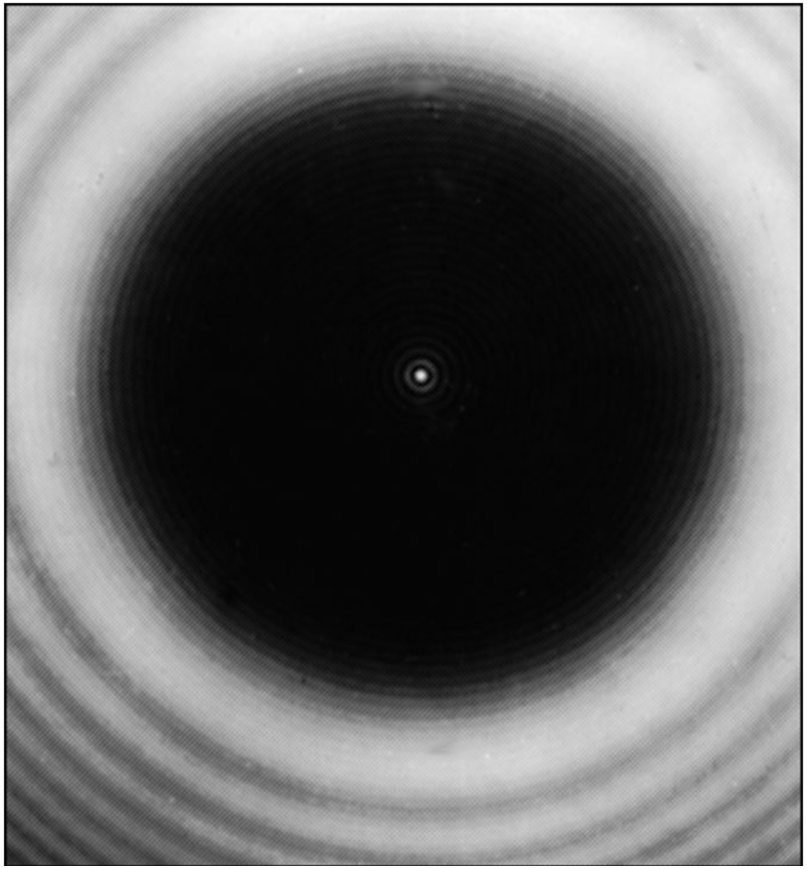
Irradiance at center of shadow nearly the same as without the disk present!





Siméon Denis Poisson (1781-1840)

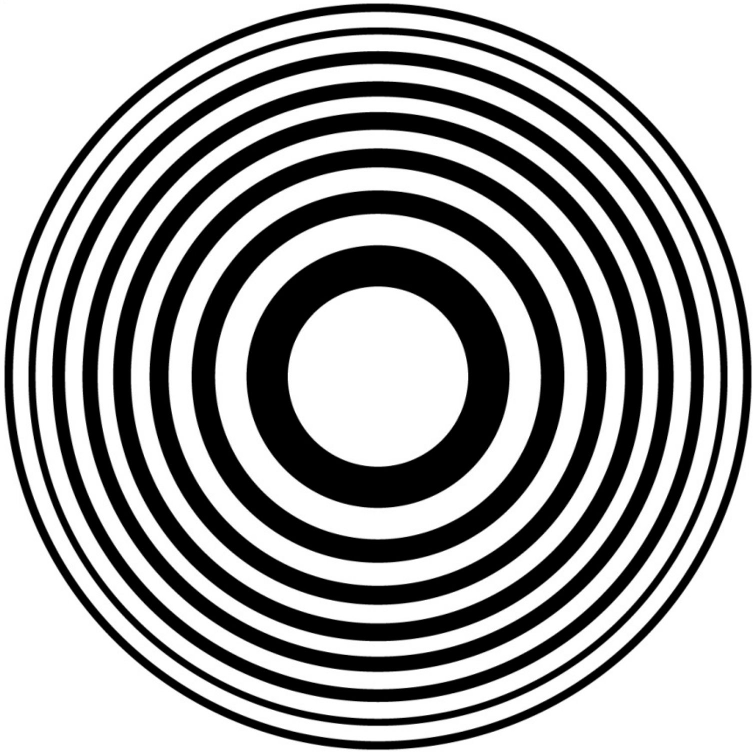
Poisson/Arago spot



François Arago
(1786-1853)

The Fresnel zone plate

16 zones



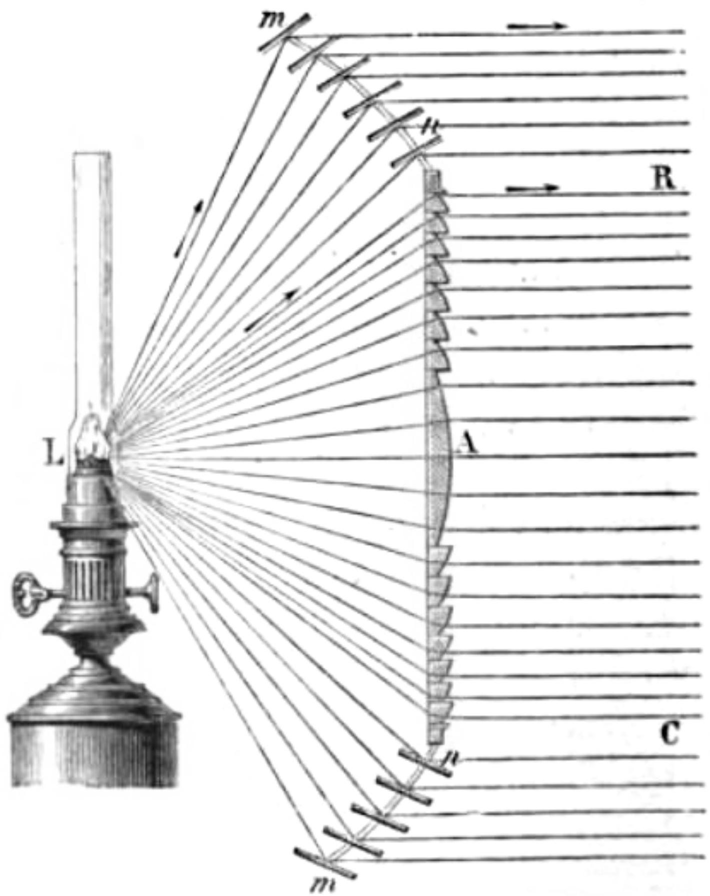
$$A_n = a_1 - a_2 + a_3 - a_4 + \ldots a_n$$

If the 2nd, 4th, 6th, etc. zones are blocked, then:

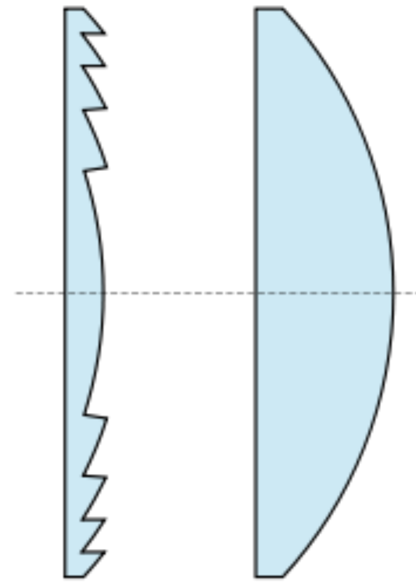
$$A_{16} = a_1 + a_3 + a_5 + a_7 + a_9 + a_{11} + a_{13} + a_{15}$$

Amplitude at P is 16 times the amplitude of $a_1/2$
Irradiance at P is $(16)^2$ times! (a.k.a. focusing)

An alternative to blocking zones



Fresnel lens vs. *plano-convex lens*



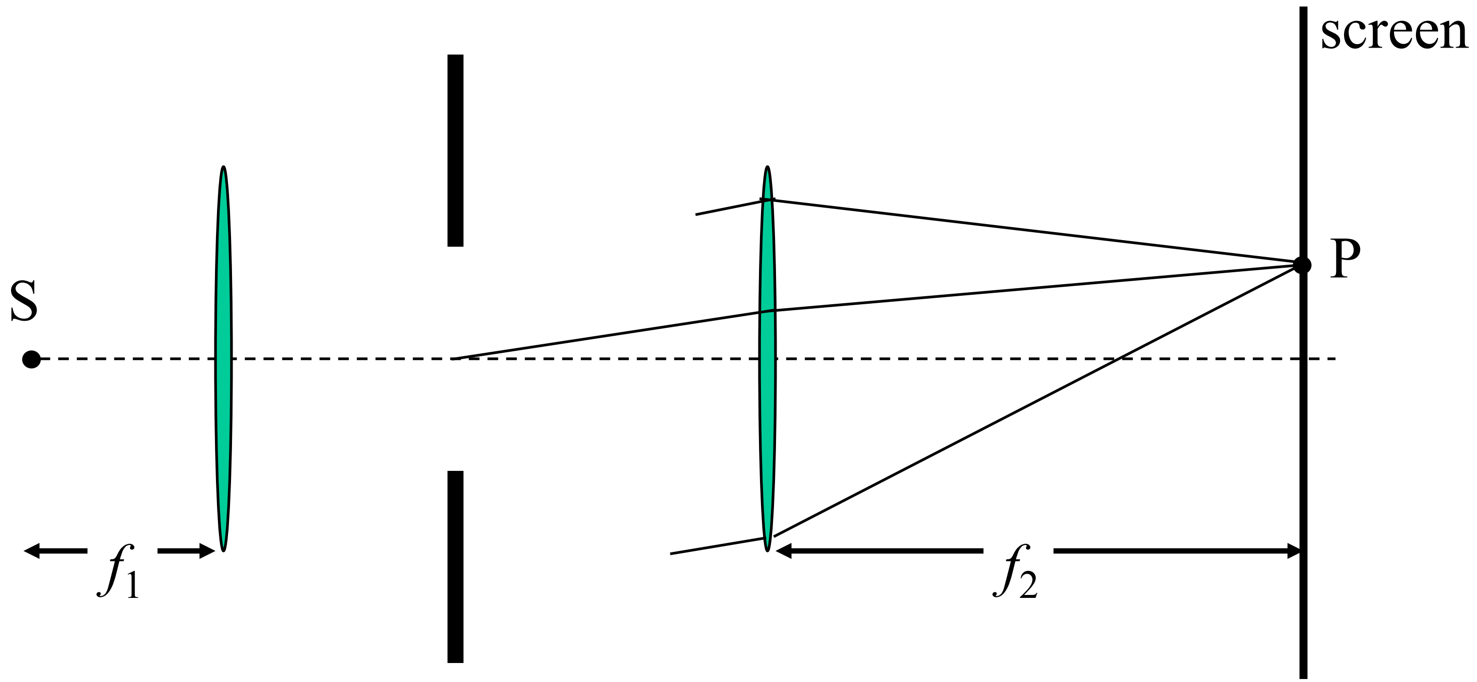
phases of adjacent Fresnel zones changed by π

Fresnel lighthouse lens

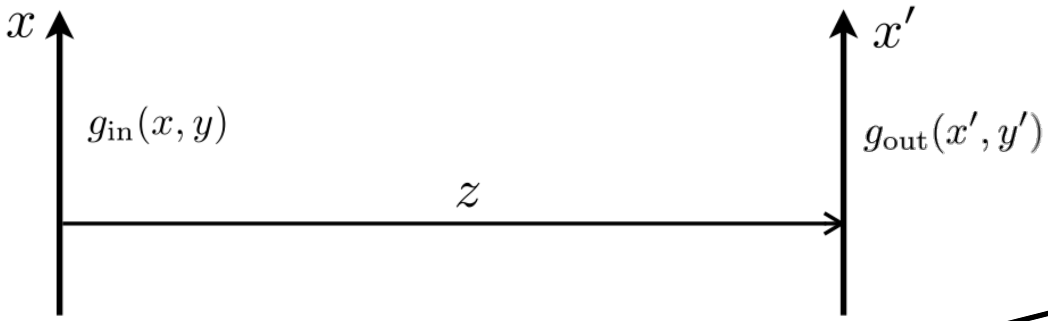
other applications: *overhead projectors*
automobile headlights
solar collectors
traffic lights

Back to Fraunhofer diffraction

- Typical arrangement (or use laser as a source of plane waves)
- Plane waves in, plane waves out



Fraunhofer diffraction



Fresnel-Kirchhoff integral
(free space propagation)

$$g_{\text{out}}(x', y'; z) = \frac{1}{i\lambda z} \exp\left\{i2\pi\frac{z}{\lambda}\right\} \iint g_{\text{in}}(x, y) \exp\left\{i\pi\frac{(x'-x)^2 + (y'-y)^2}{\lambda z}\right\} \mathrm{d}x\mathrm{d}y$$

$$\begin{aligned} g_{\text{out}}(x', y'; z) &= \frac{1}{i\lambda z} \exp\left\{i2\pi\frac{z}{\lambda}\right\} \iint g_{\text{in}}(x, y) \exp\left\{i\pi\frac{x'^2 + x^2 - 2xx' + y'^2 + y^2 - 2yy'}{\lambda z}\right\} \mathrm{d}x\mathrm{d}y \\ &\approx \underbrace{\exp\left\{i2\pi\frac{z}{\lambda} + i\pi\frac{x'^2 + y'^2}{\lambda z}\right\}}_{|\ldots| = 1} \iint g_{\text{in}}(x, y) \exp\left\{-i2\pi\frac{xx' + yy'}{\lambda z}\right\} \mathrm{d}x\mathrm{d}y \end{aligned}$$

$$u \equiv \frac{x'}{\lambda z} \qquad v \equiv \frac{y'}{\lambda z}$$

$$g_{\text{out}}(x', y'; z) \approx \exp\left\{i2\pi\frac{z}{\lambda} + i\pi\frac{x'^2 + y'^2}{\lambda z}\right\} \iint g_{\text{in}}(x, y) \exp\left\{-i2\pi\left(ux + vy\right)\right\} \mathrm{d}x\mathrm{d}y$$

Fraunhofer diffraction ∝ Fourier Transform: Rectangular aperture

Fourier transform

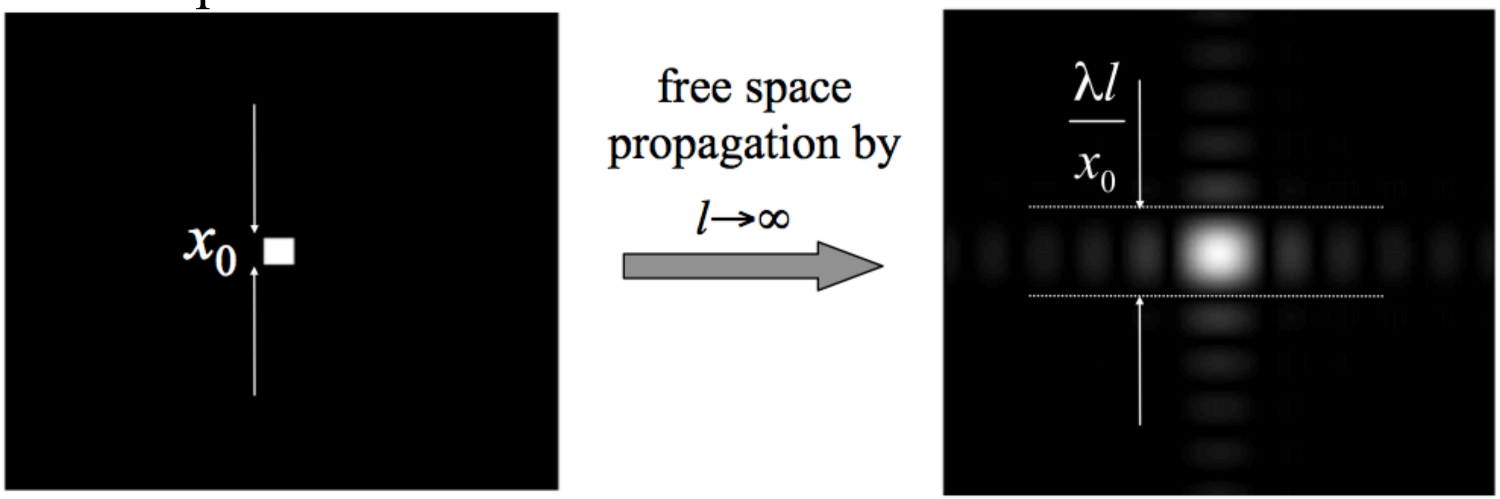
$$g_{\text{in}}(x, y) = \text{rect}\left(\frac{x}{x_0}\right)\text{rect}\left(\frac{y}{y_0}\right)$$

$$G_{\text{in}}(u, v) = x_0 y_0 \text{sinc}\left(x_0 u\right)\text{sinc}\left(y_0 v\right)$$

$$g_{\text{out}}(x', y'; z \to \infty) \propto \text{sinc}\left(\frac{x_0 x'}{\lambda z}\right)\text{sinc}\left(\frac{y_0 y'}{\lambda z}\right)$$

Input field

Far-field



Circular aperature

$$g_{\text{in}}(x,y) = \text{circ}\left(\frac{\sqrt{x^2+y^2}}{r_0}\right)$$

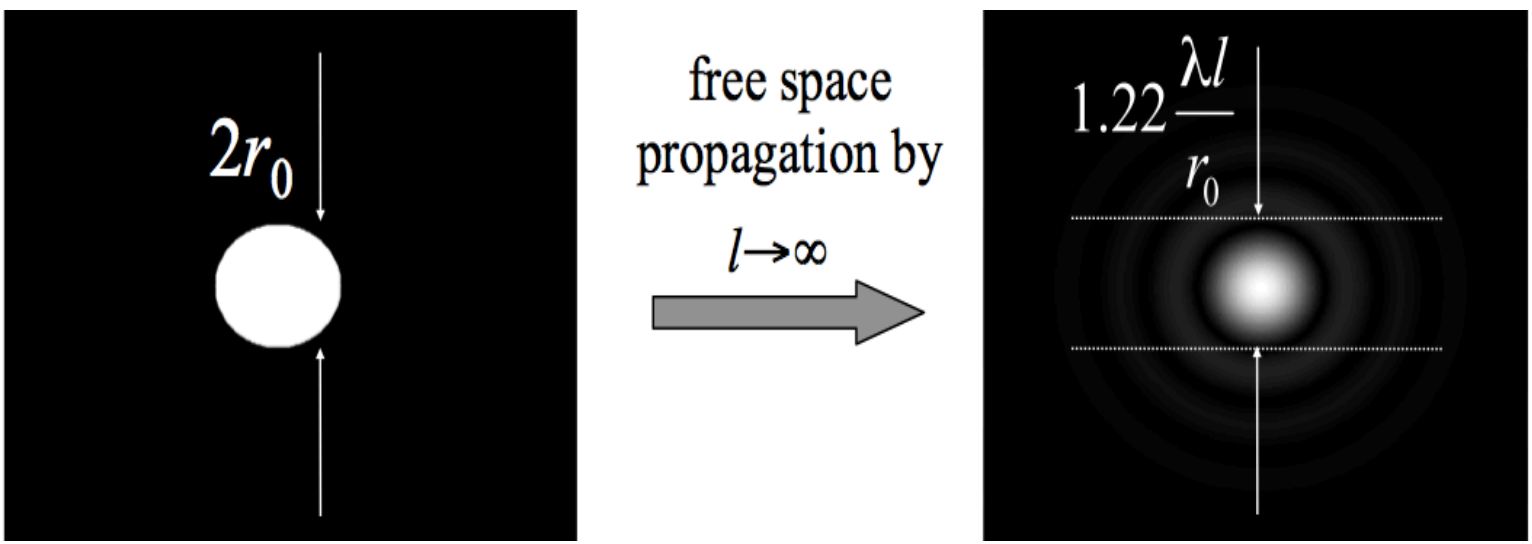
$$G_{\text{in}}(u,v) = r_0^2 \,\text{jinc}\left(r_0\sqrt{u^2+v^2}\right)$$

$$\equiv r_0 \frac{\text{J}_1\left(2\pi\sqrt{u^2+v^2}\right)}{\sqrt{u^2+v^2}}$$

$$g_{\text{out}}(x',y';z\to\infty) \propto \text{jinc}\left(\frac{2\pi r_0\sqrt{x'^2+y'^2}}{\lambda z}\right)$$

Input field

Far-field
Airy pattern



Fourier transform pair

$$G(\nu) = \int_{-\infty}^{+\infty} g(t) \exp\{-i2\pi\nu t\}\,\mathrm{d}t.$$

1D

$$g(t) = \int_{-\infty}^{+\infty} G(\nu) \exp\{i2\pi\nu t\}\,\mathrm{d}\nu.$$

$$G(u,v) = \iint_{-\infty}^{+\infty} g(x,y) \exp\{-i2\pi(ux+vy)\}\,\mathrm{d}x\mathrm{d}y.$$

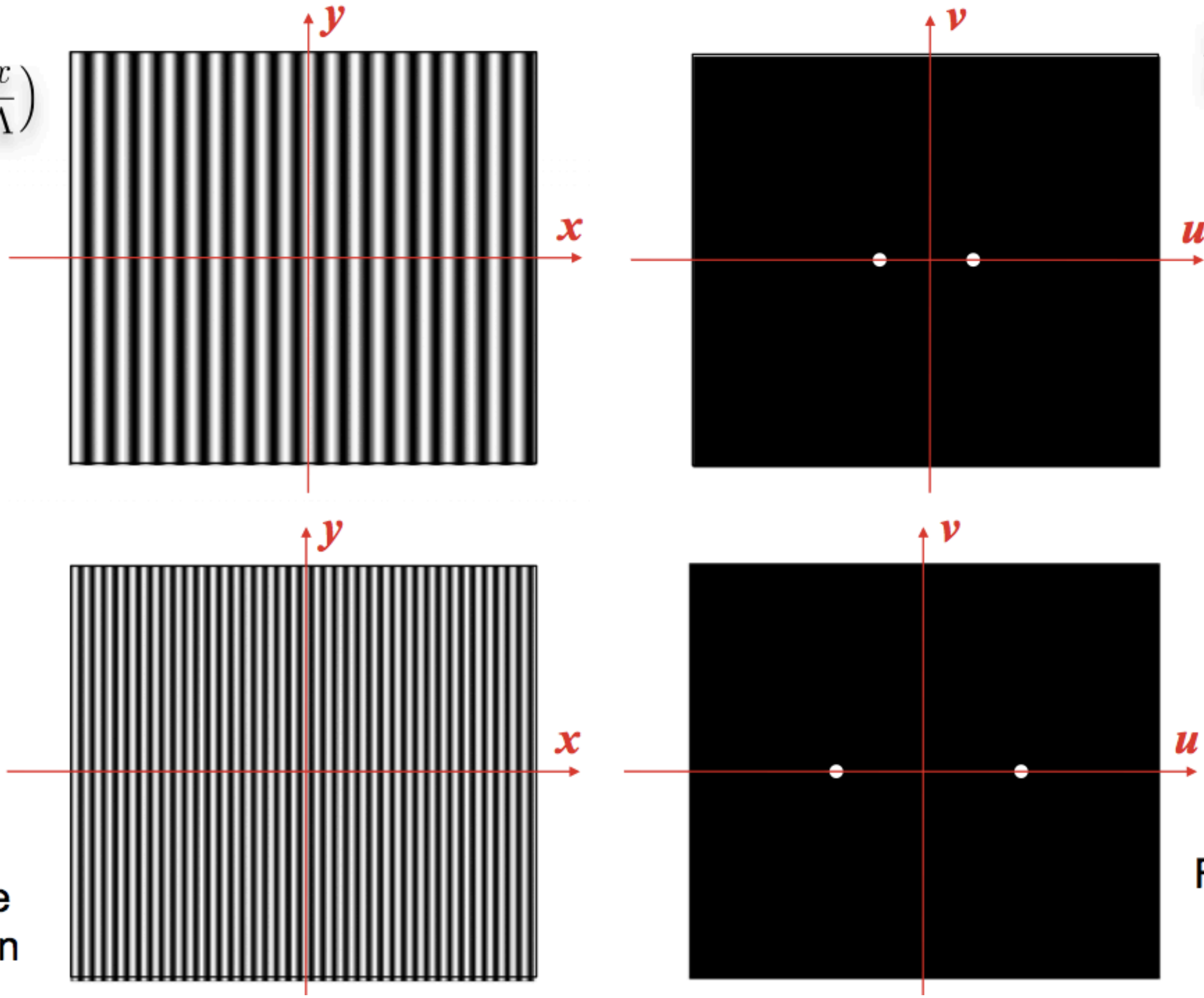
2D

$$g(x,y) = \iint_{-\infty}^{+\infty} G(u,v) \exp\{i2\pi(ux+vy)\}\,\mathrm{d}u\mathrm{d}v.$$

Spatial domain ↔ (angular) frequency domain

$$\cos\left(2\pi \frac{x}{\Lambda}\right)$$

$$\frac{1}{2}\delta\left(u+\frac{1}{\Lambda}\right)\delta\left(v\right)+\frac{1}{2}\delta\left(u-\frac{1}{\Lambda}\right)\delta\left(v\right)$$



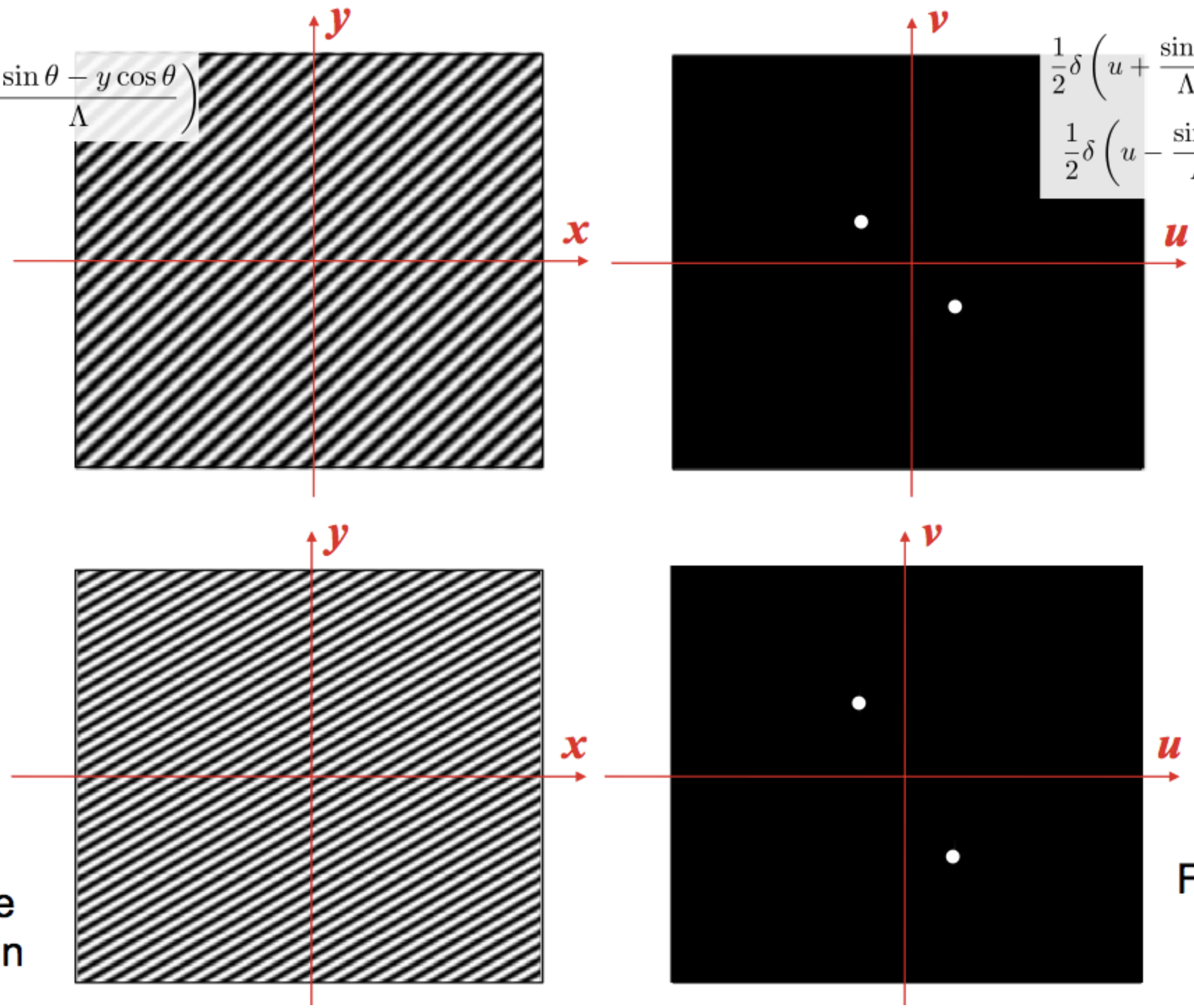
Space domain

Frequency (Fourier) domain

Tilted grating

$$\cos\left(2\pi\frac{x\sin\theta - y\cos\theta}{\Lambda}\right)$$

$$\frac{1}{2}\delta\left(u+\frac{\sin\theta}{\Lambda}\right)\delta\left(v-\frac{\cos\theta}{\Lambda}\right)+$$
$$\frac{1}{2}\delta\left(u-\frac{\sin\theta}{\Lambda}\right)\delta\left(v+\frac{\cos\theta}{\Lambda}\right)$$



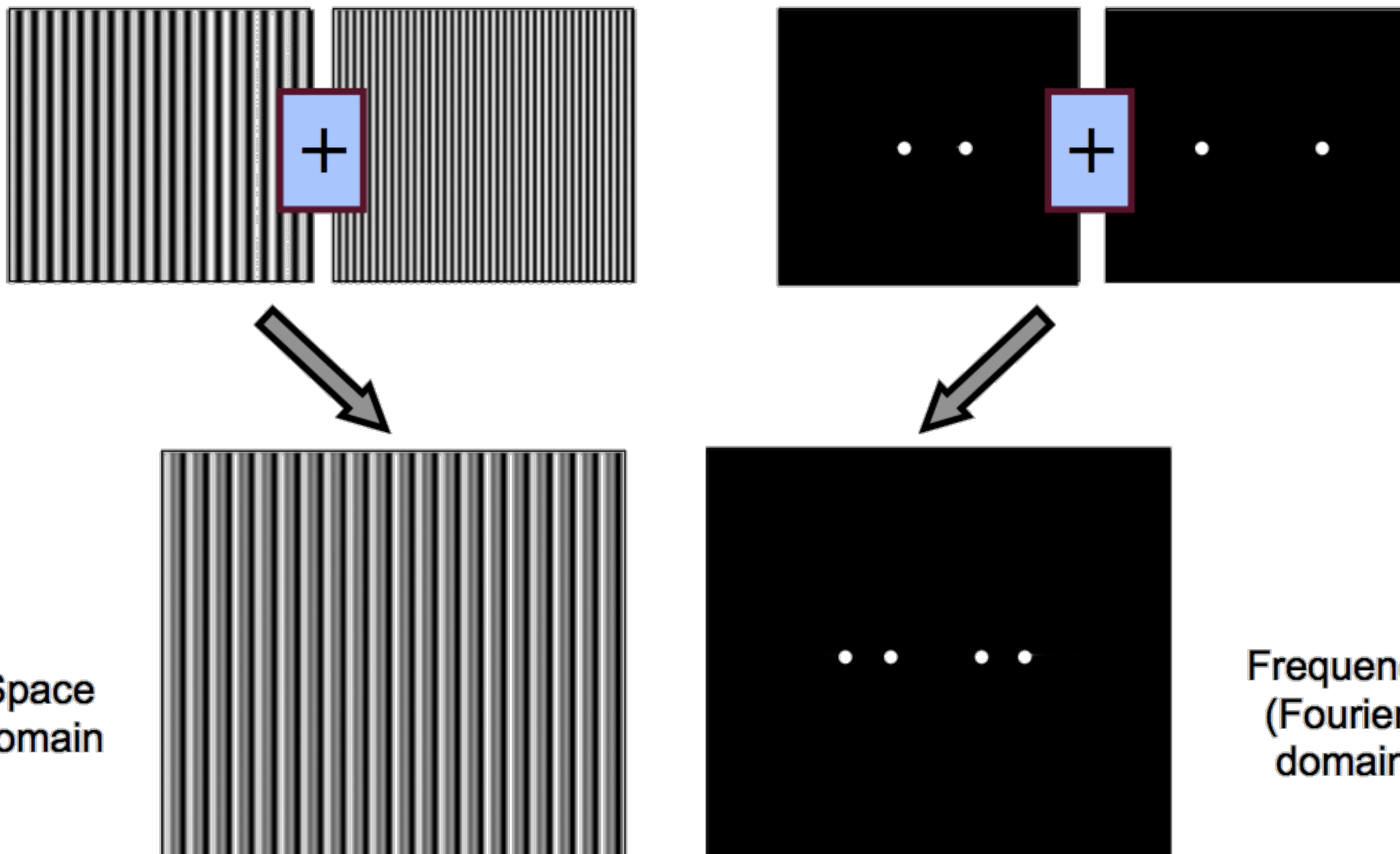
Space domain

Frequency (Fourier) domain

Linear superposition

$$a_1 \cos\left(2\pi \frac{x}{\Lambda_1}\right) + a_2 \cos\left(2\pi \frac{x}{\Lambda_2}\right)$$

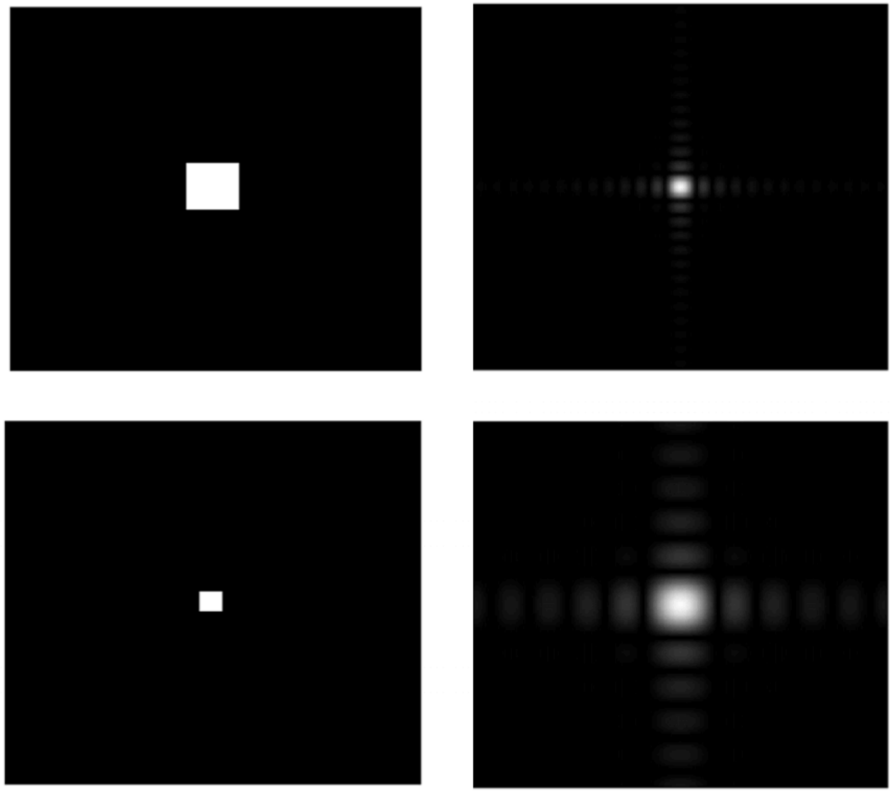
$$\frac{a_1}{2}\delta\left(u + \frac{1}{\Lambda_1}\right)\delta(v) + \frac{a_2}{2}\delta\left(u + \frac{1}{\Lambda_2}\right)\delta(v) +$$
$$\frac{a_1}{2}\delta\left(u - \frac{1}{\Lambda_1}\right)\delta(v) + \frac{a_2}{2}\delta\left(u - \frac{1}{\Lambda_2}\right)\delta(v)$$



Space domain

Frequency (Fourier) domain

Scaling

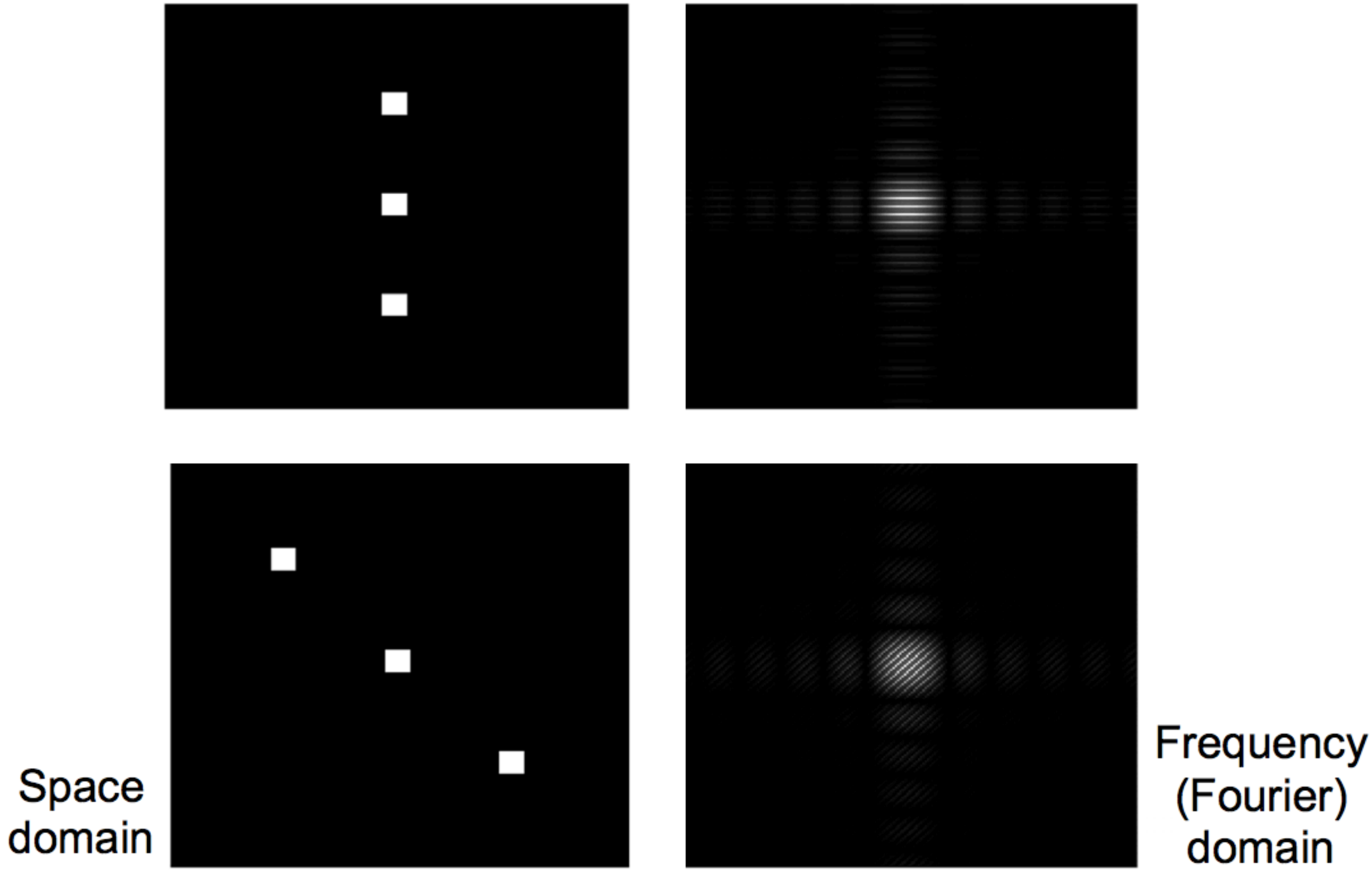


Space domain

Frequency (Fourier) domain

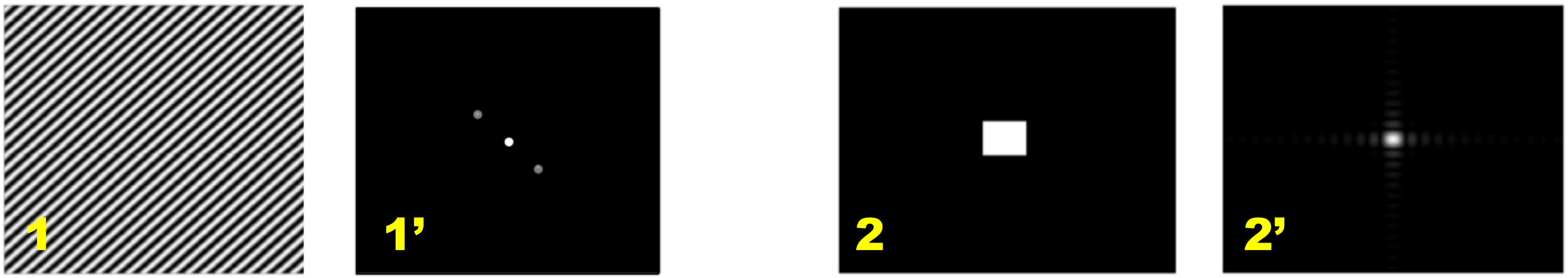
$$\mathcal{F}\left\{g\left(\frac{x}{a},\frac{y}{b}\right)\right\} = |ab|\, G(au, bv)$$

Shift theorem



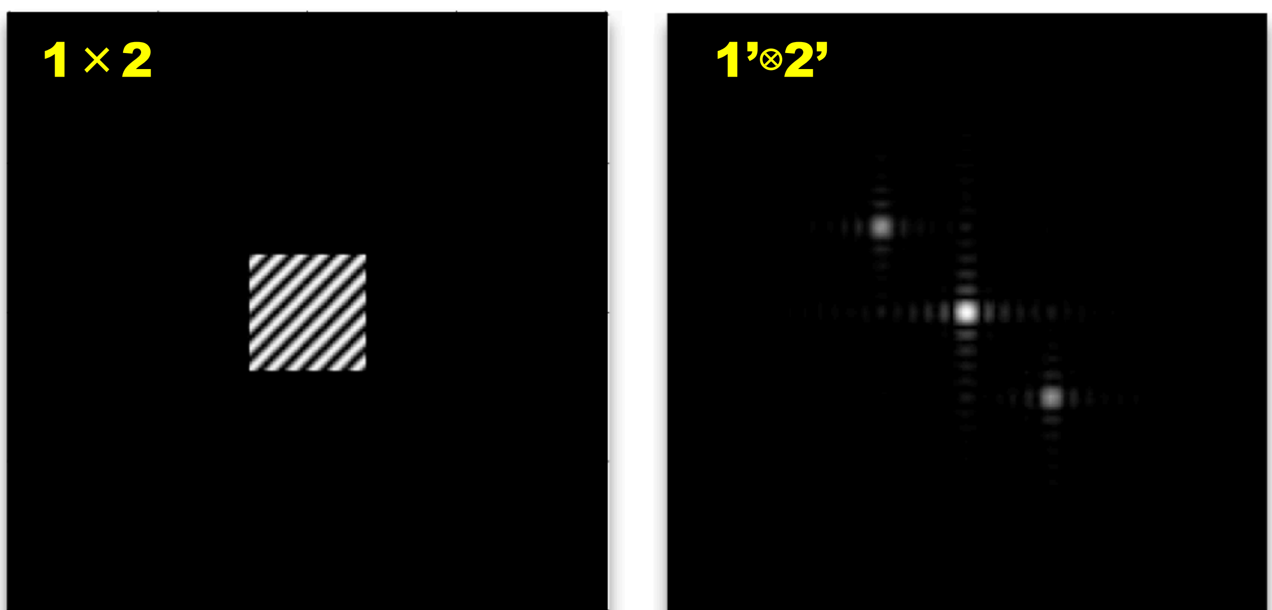
$$\mathcal{F}\left\{g\left(x-a,y-b\right)\right\} = \exp\left\{i2\pi(au+bv)\right\}G(u,v)$$

The convolution theorem



multiplication

convolution



$$\mathcal{F}\{f * g\} = \mathcal{F}\{f\} \cdot \mathcal{F}\{g\} \quad \text{or} \quad \mathcal{F}\{f \cdot g\} = \mathcal{F}\{f\} * \mathcal{F}\{g\}$$

Links/references

http://edu.tnw.utwente.nl/inlopt/overhead_sheets/Herek2010/week7/13.Fresnel%20diffraction.ppt

http://ocw.mit.edu/courses/mechanical-engineering/2-71-optics-spring-2009/video-lectures/lecture-17-fraunhofer-diffraction-fourier-transforms-and-theorems/MIT2_71S09_lec17.pdf