Error Analysis

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Outline

- **Why quote errors?**
- **Semantics & types of errors**
- **Counting statistics**
- **Error propagation**
- **Curve fitting**

Why do we quote errors?

It provides information about the **precision** of the measurement.

For example the gravitational constant is measured to be

$G_N=6.90 \cdot 10^{-11}\ m^3kg^{-1}s^{-2}$

The "accepted" value is

$G_N=6.6742(10) \cdot 10^{-11}\ m^3kg^{-1}s^{-2}$

Without quoting any errors we don't know if this was just a less precise measurement or a Nobel prize worthy discovery.

Example:

$G_N=(6.90 \pm 0.25) \cdot 10^{-11}\ m^3kg^{-1}s^{-2}$ would be in good agreement

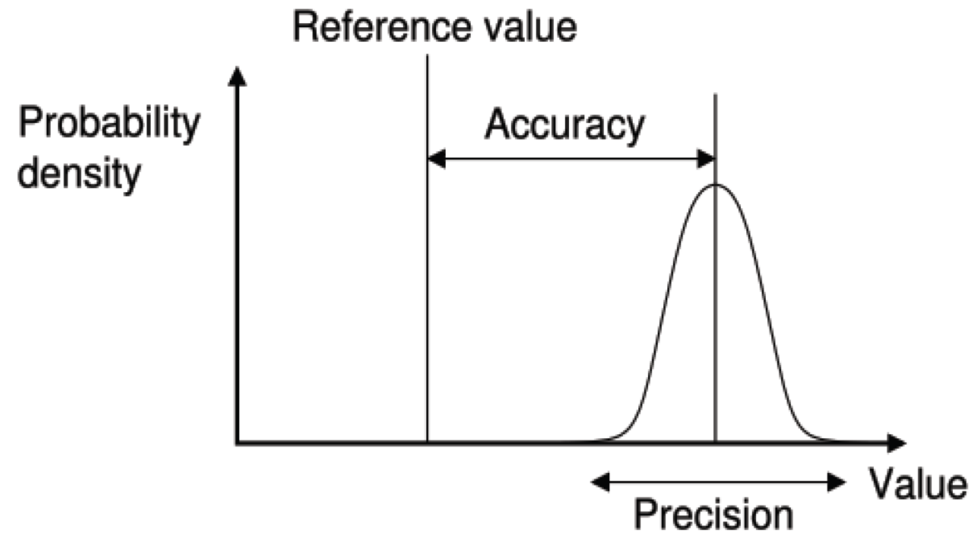
$G_N=(6.90 \pm 0.01) \cdot 10^{-11}\ m^3kg^{-1}s^{-2}$ would be an interesting result …

Semantics

Accuracy vs. Precision

Accuracy is the degree to which a measurement agrees with the true value.

Precision is the repeatability of the measurement.



Error vs. Uncertainty

Error is the degree to which a Measurement agrees with the true value.

Accurate but not precise

Precise but not accurate

Uncertainty is an interval around the measurement in which repeated measurements will fall.

Sloppy and inconsistent language

Error is almost never what we are interested in. In science we typically *do not know the 'true' value*.

Rather we are interested in the **uncertainty**. This is what we need to quantify in any measurement.

We are often very sloppy and inconsistent in our language and call what is actually an uncertainty an error, **e.g. in the title of this lecture**.

When we talk about **measurement error**, by it we will generally understand **measurement uncertainty** and not any of the following:

- It is not **a blunder**
- It is not **an accident**
- It is not due to **incorrectly handled equipment**
- It is not the **difference to an accepted value found in the literature**

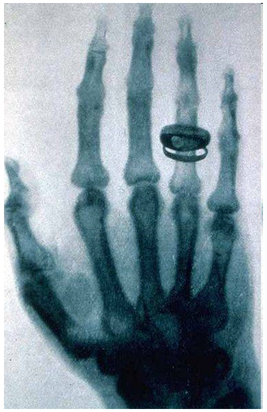
Importance of uncertainty

Example 1
High fiber diets: A study in 1970 claimed that a high fiber diet reduces polyps forming in the colon, being precursors of cancer. A study in 2000 with more analyzed individuals showed no such effect. The **uncertainty** in the first study was too large an not properly accounted for. This left people eating lots of fibers for 30 years.

Example 2
A study in the late 60s found large levels of iron, which is required for red blood cell production, in spinach. Popular comics tried to promote spinach consumption. A study in the 90s showed that the original measurement had a reading error in the decimal point. The iron levels are a factor of 10 lower than claimed. The incorrect reading of the decimal was a **blunder**, not due to an uncertainty in the measurement. This left children eating lots of spinach for 30 years.

Error analysis helps to limit bias

Fact of scientific life:
Scientists subconsciously bias data to their desired outcome, even when they know about this tendency of their psyche.



Example: N-rays
X-rays discovered in 1895 by Roentgen with huge and fast success.
Another new type of radiation was reported in 1903:
Rene Blondlot (physicist, U Nancy / France) discovered N-rays (with N for Nancy)
100's of papers published within about one year, 26 from Blondlot.
They **go through wood** and **metal** but are **blocked by water**.
They could be **stored in a brick**.
They are **emitted by rabbits, frogs and the human brain** (medical imaging).
Jean Becquerel (son of Henri who discovered radioactivity) found N-rays **transmitted over a wire** (brain scan over telephone…)

Robert Wood (John Hopkins U) visited Blondlot's lab and secretly removed an essential prism from his setup. Blondlot insisted he was still measuring N-rays. Within months no one believed in N-rays any more.

Meaning of an Error

If we measure a voltage $\mathbf{V_{meas} = 10.2 \pm 0.3\ V}$, what does this mean?

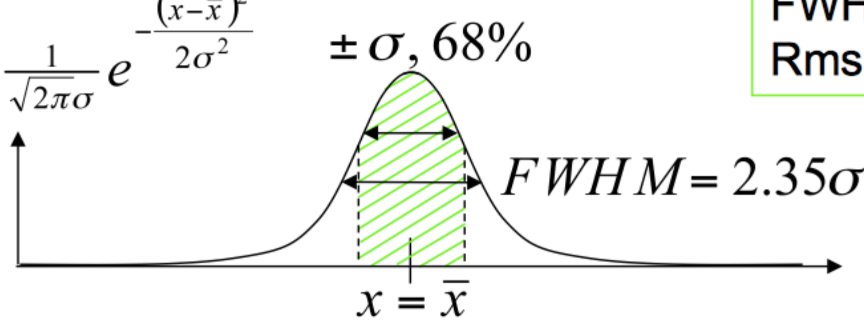
In general there are differences in different science disciplines.

In physics, a 1-sigma error is generally used. If the measurements are normally distributed (Gaussian), this corresponds to a 68% confidence level (CL) interval. Or that 32% of the time the true value would be outside the quoted error range.

For statistical errors, this can be given a precise meaning. Many other errors are harder to estimate.

Gaussian distribution:

$$\rho(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\bar{x})^2}{2\sigma^2}}$$



FWHM	= full width at half maximum
Rms	= root of the mean square

$$rms = \sqrt{\left\langle (x - \bar{x})^2 \right\rangle} = \sigma$$

Different types of errors

- **Statistical**: from finite statistics, originates in the Poisson distribution.
- **Systematic**: e.g. how well can you measure a voltage, length, etc.
- **Theoretical**: for example, if the muon lifetime is measured by capturing muons in matter, there are corrections to the capture rate that comes from theory.

- Sometimes (e.g. High Energy Phys.), these uncertainties are quoted separately:

  $t_\mu = (2.19 \pm 0.05_{stat.} \pm 0.02_{syst.} \pm 0.02_{th.})\mu s$
- Different notations are used for uncertainties, e.g.

  $t_\mu = (2.19(5)_{stat.} \pm (2)_{syst.} \pm (2)_{th.})\mu s$

- In P3330, we'll combine errors (by adding them in quadrature) into a single one

  $t_\mu = (2.19 \pm 0.06)\mu s$ (proceed with caution!)

- We usually quote absolute (not relative) errors.

- **Note the number of significant digits quoted!**

Accurate arrival:
From 08:45 (Granville) to 15:55 (Montparnasse) drove only 3s/25800s = 10^{-4} too long!

Counting Statistics

Imagine a situation where a number of events occur in a fixed period of time, where these events occur with a known average rate and independently of the time since the last event.

- E.g. a counting experiment is repeated 10 times, which of the 3 outcomes below would you expect?

		mean	rms
a)	99, 100, 98, 101, 101, 99, 100, 101, 100, 99	99.8	0.98
b)	87, 105, 93, 108, 110, 90, 115, 82, 105, 97	99.2	10.4
c)	47, 115, 67, 97, 133, 103, 157, 78, 127, 94	101.8	31.1

Poisson Distribution

$$P(N,\mu) = \frac{\mu^N}{N!} e^{-\mu} \qquad \sum_{N=0}^{\infty} P(N,\mu) = 1$$

$$\langle N \rangle = \sum_{N=0}^{\infty} N \, P(N,\mu) = \sum_{N=1}^{\infty} N \frac{\mu^N}{N!} e^{-\mu} = \mu \sum_{N=1}^{\infty} \frac{\mu^{N-1}}{(N-1)!} e^{-\mu} = \mu$$

$$rms^2 = \left\langle \left(N - \langle N \rangle\right)^2 \right\rangle = \left\langle N^2 - 2N\langle N \rangle + \langle N \rangle^2 \right\rangle = \left\langle N^2 \right\rangle - \langle N \rangle^2$$

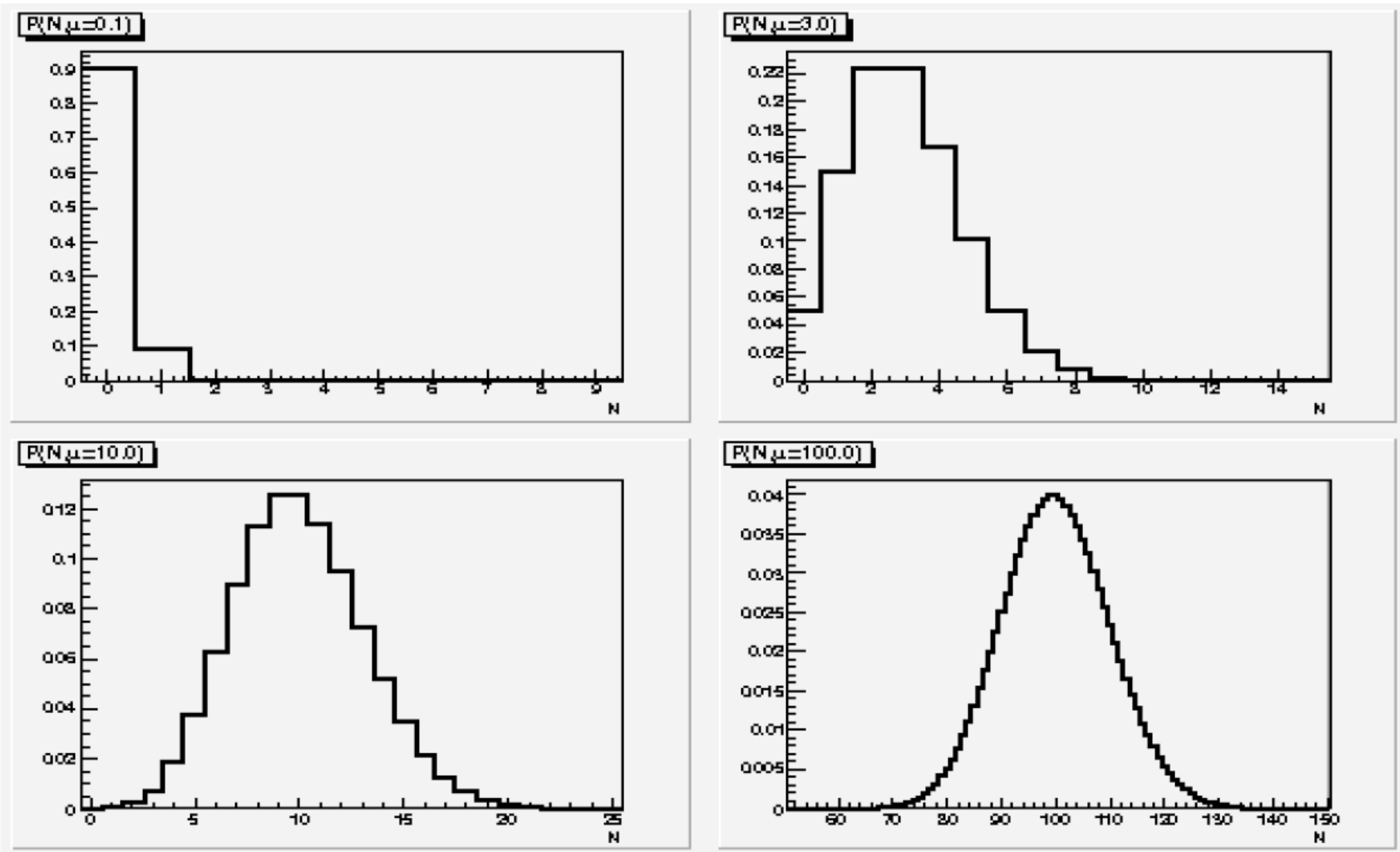
$$\left\langle N^2 \right\rangle = \sum_{N=0}^{\infty} N^2 \, P(N,\mu) = \sum_{N=1}^{\infty} N^2 \frac{\mu^N}{N!} e^{-\mu} = \mu \sum_{N=1}^{\infty} N \frac{\mu^{N-1}}{(N-1)!} e^{-\mu}$$

$$= \mu \sum_{N=1}^{\infty} \left[(N-1) \frac{\mu^{N-1}}{(N-1)!} e^{-\mu} + \frac{\mu^{N-1}}{(N-1)!} e^{-\mu} \right] = \mu(\mu + 1)$$

$$rms^2 = \mu \quad \Rightarrow \quad rms = \sqrt{\mu} \quad \Rightarrow \quad \frac{rms}{\langle N \rangle} = \frac{1}{\sqrt{\langle N \rangle}}$$

Limit of the Poisson Distribution

For large <N> (μ>10), the Poisson distribution approaches a normal distribution.



Use of the Poisson Distribution

- The Poisson distribution tells you how probable it is to obtain a given count if the mean is known.
- Typically we don't know the true mean, but our measured count serves as an estimate of the mean.
- We can now use this information to estimate the uncertainty.

- E.g. in a counting experiment we obtain 99 counts. We then assign the uncertainty of $99^{1/2} = 9.9$ to say that the measurement leads to 99 ± 10 counts.

Balancing statistical and systematic errors

- No point in reducing the statistical error much beyond the systematic one.
- E.g. ½ (m)V division on a voltmeter is likely the precision of the device; cannot decrease the uncertainty by simply repeating the measurement.
- When dominated by the statistical error, repeat the measurement *N* times to refine the average and to bring down the error.

$$\bar{X} \pm \sigma_X$$

$$X_{best} = \bar{X} \equiv \frac{\sum X_i}{N}$$

$$\sigma_X \equiv \sqrt{\frac{1}{N}\sum (X_i - \bar{X})^2}$$

Error Propagation

$$c = f(a,b)$$

$$\bar{a} = \langle a \rangle, \quad \delta a = a - \bar{a}, \quad \sigma_a^2 = \langle \delta a^2 \rangle$$

$$\bar{b} = \langle b \rangle, \quad \delta b = b - \bar{b}, \quad \sigma_b^2 = \langle \delta b^2 \rangle$$

$$c = f(a,b) \approx f(\bar{a},\bar{b}) + \partial_a f(\bar{a},\bar{b}) \cdot \delta a + \partial_b f(\bar{a},\bar{b}) \cdot \delta b$$

$$\bar{c} = \left\langle f(\bar{a},\bar{b}) + \partial_a f(\bar{a},\bar{b}) \cdot \delta a + \partial_b f(\bar{a},\bar{b}) \cdot \delta b \right\rangle = f(\bar{a},\bar{b})$$

$$\sigma_c^2 = \langle \delta c^2 \rangle \approx \left\langle \left(\partial_a f(\bar{a},\bar{b}) \cdot \delta a + \partial_b f(\bar{a},\bar{b}) \cdot \delta b \right)^2 \right\rangle$$

$$= \left[\partial_a f(\bar{a},\bar{b}) \sigma_a \right]^2 + \left[\partial_b f(\bar{a},\bar{b}) \sigma_b \right]^2 + \partial_a f(\bar{a},\bar{b}) \partial_b f(\bar{a},\bar{b}) \langle \delta a \delta b \rangle$$

Assumption of uncorrelated errors:
Errors in variable *a* vary independently of those in variable *b*.

$$\langle \delta a \delta b \rangle = \langle \delta b \rangle \langle \delta a \rangle = 0$$

$$\sigma_c^2 = \left[\partial_a f(\bar{a},\bar{b}) \sigma_a \right]^2 + \left[\partial_b f(\bar{a},\bar{b}) \sigma_b \right]^2$$

Examples

Example 1: d = a – b + c

$$\sigma_d^2 = \sigma_a^2 + \sigma_b^2 + \sigma_c^2$$

$$\sigma_d = \sqrt{\sigma_a^2 + \sigma_b^2 + \sigma_c^2}$$

Example 2: for velocity measurements
displacement: x = 5.1 ± 0.4 m
during time: t = 0.4 ± 0.1 s

v = ?

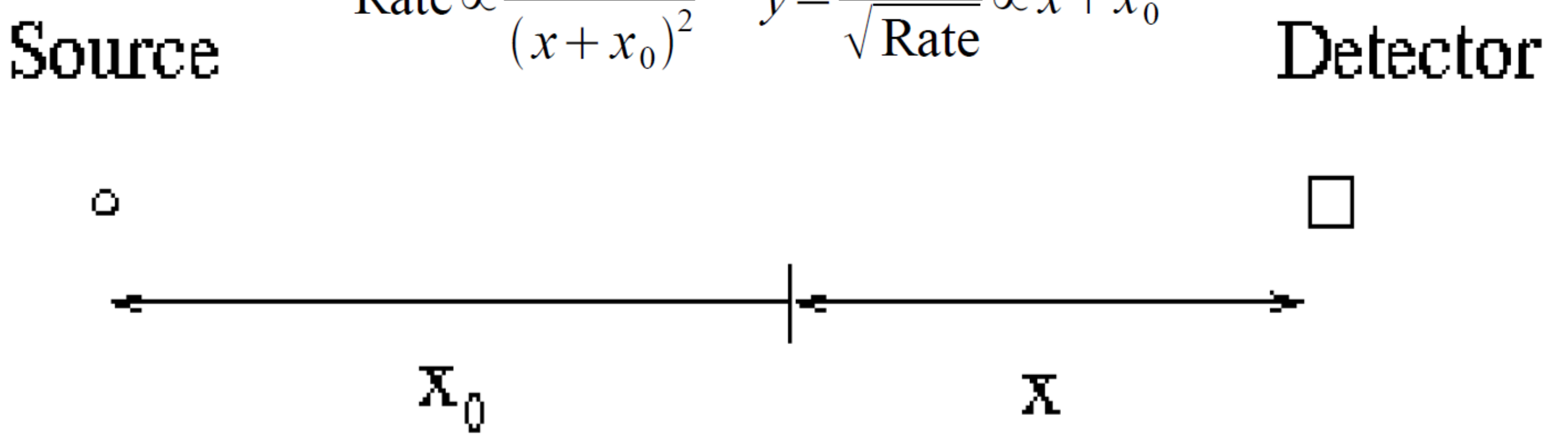
v = x / t = 5.1 m / 0.4 s = 12.75 m/s
$dv = |v| \cdot [(dx/x)^2 + (dt/t)^2]^{1/2}$ = 12.75 m/s ·
$[(0.4/5.1)^2 + (0.1/0.4)^2]^{1/2}$ = 3.34 m/s,
=> ***v = 12.8 ± 3.3 m/s***

Data fitting

E.g. you have a source in a box and a detector, and you need to determine the distance $x+x_0$. You can change x, but do not have access to x_0.

Task: determine x_0

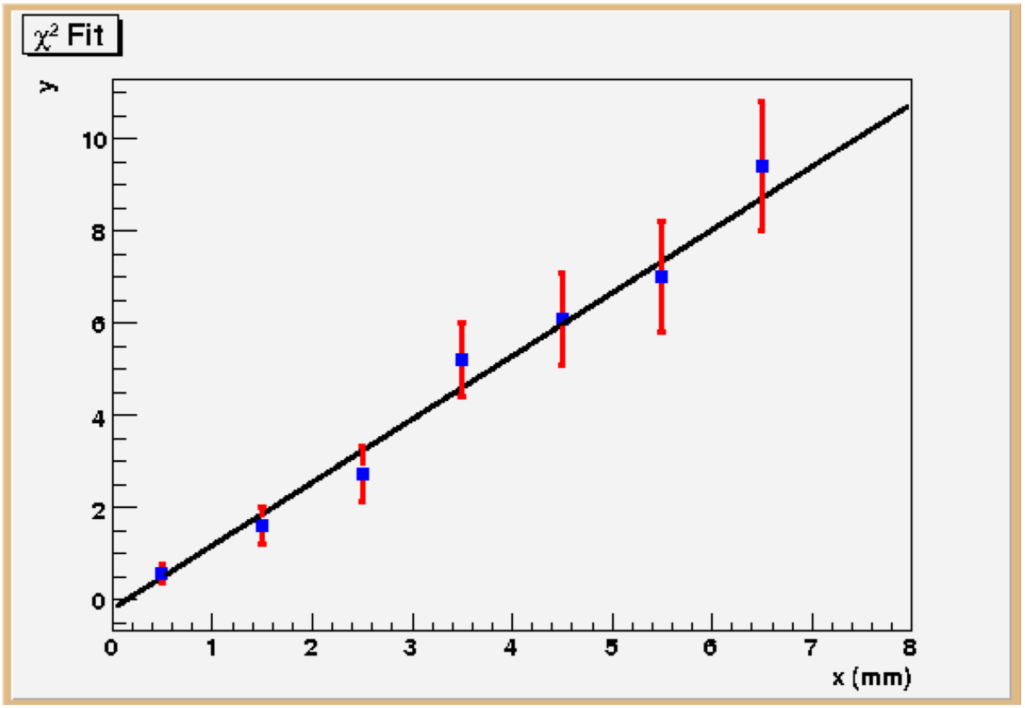
$$\text{Rate} \propto \frac{1}{(x+x_0)^2} \qquad y = \frac{1}{\sqrt{\text{Rate}}} \propto x + x_0$$



χ^2 or LS fit (line)

$$\chi^2(a,b) = \sum_{i=1}^{N} \frac{\left(y_i^{\text{meas}} - (a + b x_i)\right)^2}{\sigma_i^2}$$

$$\text{Minimize the } \chi^2 \text{ i.e. solve } \frac{\partial \chi^2}{\partial a} = \frac{\partial \chi^2}{\partial b} = 0$$



χ^2 fit (any function)

More generally: N measurement points x_i of the variable y_i with an error σ_i (i = 1,2,…,N). We desire to fit a function f(x; a_j) that depends on M parameters a_j (j = 1,2,…,M). Least squared method (or χ^2) prescribes to minimize the LS sum

$$S = \sum_{i=1}^{N} \left[\frac{y_i - f(x_i; a_j)}{\sigma_i} \right]^2 = \chi^2$$

To find parameters a_j, must solve M equations: $$\frac{\partial S}{\partial a_j} = 0$$

Depending on f(x; a_j) form, there may or may not exist analytical solutions. In practice, use MATLAB, pylab, Octave, Mathematica for all curve fitting.

χ^2-fit: error of parameters

The second derivatives of χ^2 at best values of a_j form the inverse of the covariance of the error matrix **V**:

$$(\mathbf{V}^{-1})_{jk} = \frac{1}{2}\frac{\partial^2 S}{\partial a_j \partial a_k}$$

The diagonal elements of **V** are the variances of a_j:

$$\mathbf{V} = \begin{pmatrix} \sigma_{a_1}^2 & \mathrm{cov}(a_1, a_2) & \cdots & \mathrm{cov}(a_1, a_M) \\ \mathrm{cov}(a_2, a_1) & \sigma_{a_2}^2 & \cdots & \mathrm{cov}(a_2, a_M) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{cov}(a_M, a_1) & \mathrm{cov}(a_M, a_2) & \cdots & \sigma_{a_M}^2 \end{pmatrix}$$

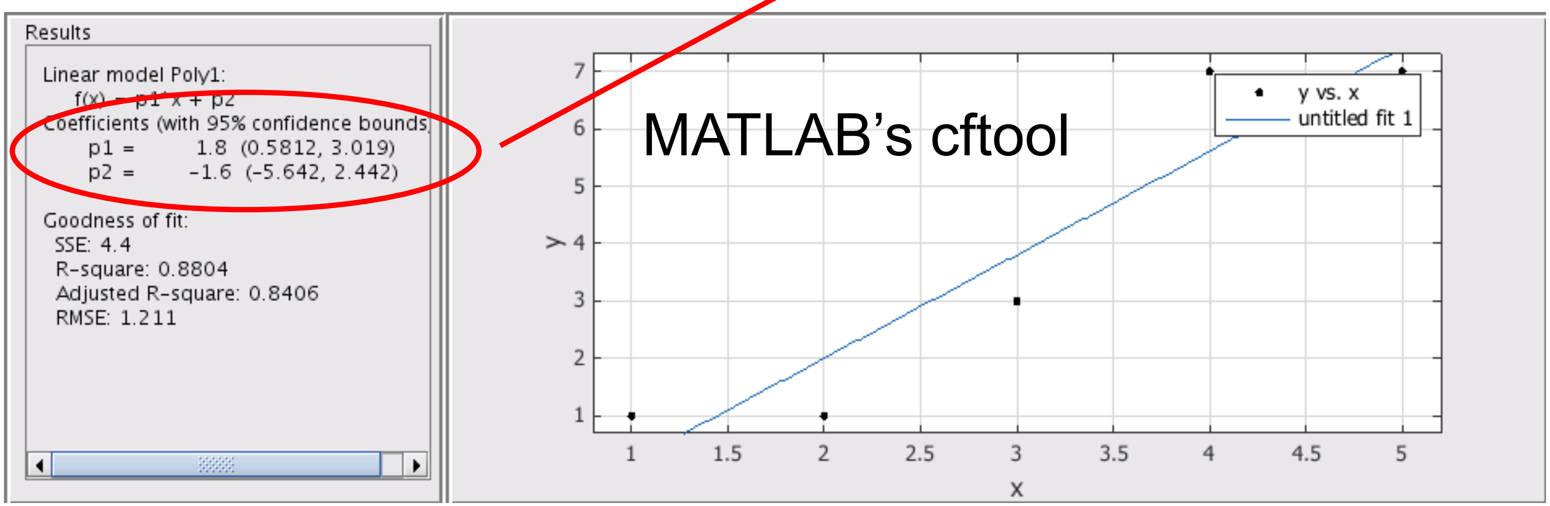
Example 1

x	y
1	1
2	1
3	3
4	7
5	7

Fit a line: $y = mx + c$

convert 95% confidence interval to 1-sigma: divide by 4 (3.92)

$m = 1.8 \pm 0.6$
$c = -1.6 \pm 2.0$



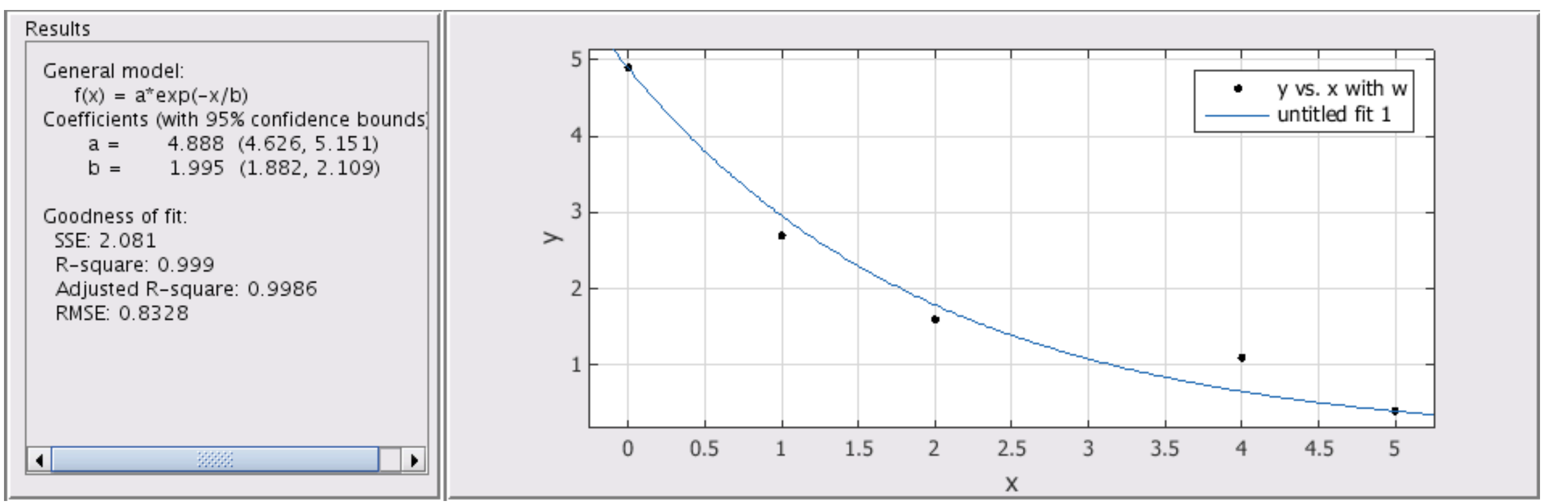
Example 2

x	y
0	4.9 ± 0.1
1	2.7 ± 0.4
2	1.6 ± 0.3
4	1.1 ± 0.4
5	0.40 ± 0.02

Fit a curve: y = a*exp(-x/b)

Hint: use 1/error^2 as weights in fitting

a = 4.9 ± 0.1
b = 2.00 ± 0.06



Accumulating knowledge from measurements

If you measured a quantity by N independent procedures and obtained the values c_i with uncertainty σ_i, what is the best combined measurement and uncertainty ?

$$c = \frac{\sum_{i=1}^{N} c_i / \sigma_i^2}{\sum_{i=1}^{N} 1 / \sigma_i^2}$$

The error propagation formula $\sigma_c^2 = \left[\partial_a f(\bar{a},\bar{b})\sigma_a\right]^2 + \left[\partial_b f(\bar{a},\bar{b})\sigma_b\right]^2$ leads to

$$\sigma_c^2 = \sum_{i=1}^{N}\left(\frac{1/\sigma_i^2}{\sum_{i=1}^{N} 1/\sigma_i^2}\sigma_i\right)^2 = \sum_{i=1}^{N}\left(\frac{1/\sigma_i}{\sum_{i=1}^{N} 1/\sigma_i^2}\right)^2, \quad \sigma_c^2 = \frac{1}{\sum_{i=1}^{N} 1/\sigma_i^2}$$

For N identical uncertainties $\sigma_i = \sigma$: $\sigma_c = \frac{\sigma}{\sqrt{N}}$