Rays in phase space & Gaussian beams

Ivan Bazarov

Cornell Physics Department / CLASSE

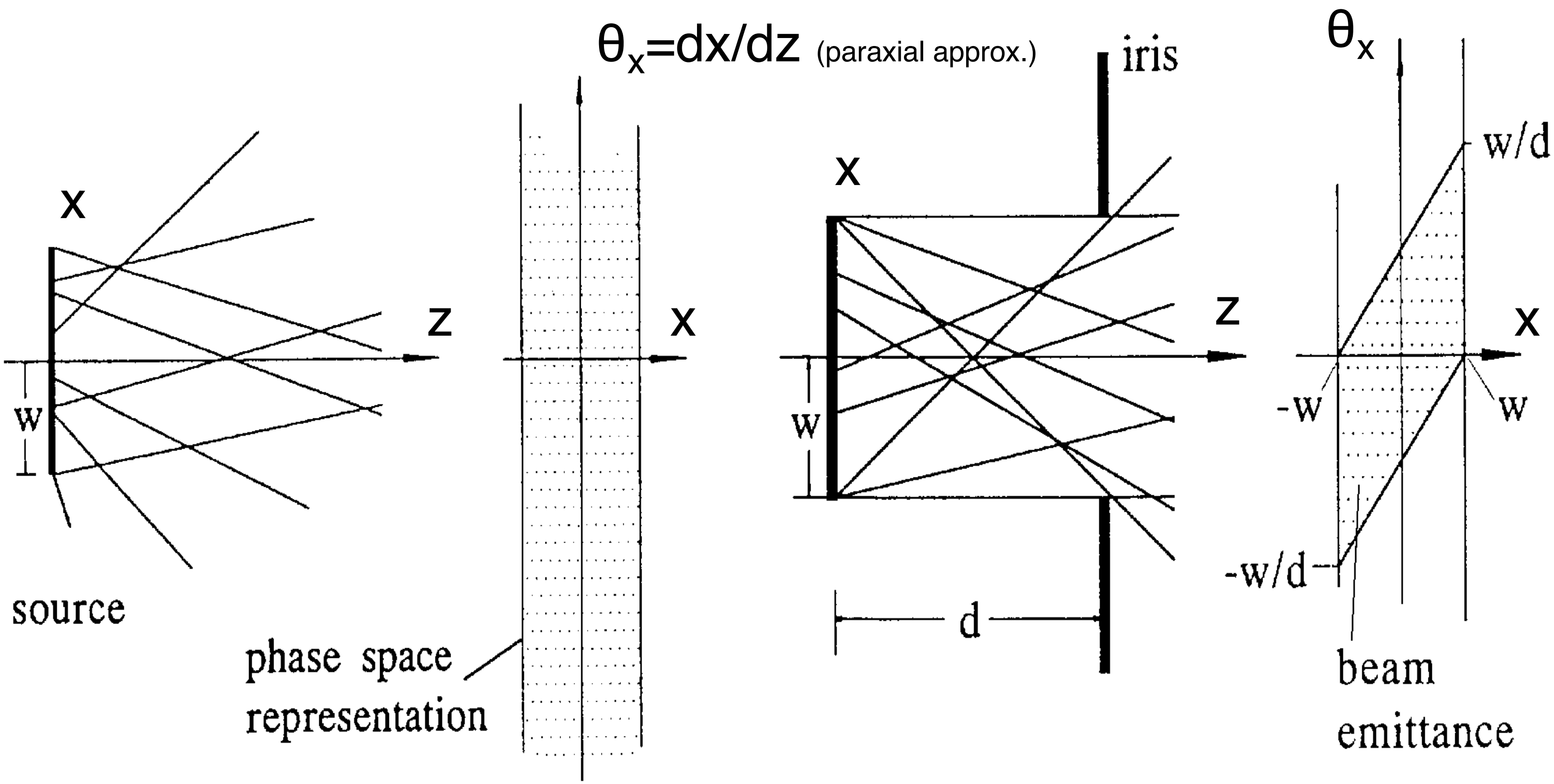
<u>Outline</u>

- **Rays in phase space**
- **Phase space ellipse transformation**
- **Rayleigh range**
- **Diffraction limited phase space area**
- **Gaussian laser beam and its properties**

Phase space definition

(Transverse) phase space – every ray is represented on a 2D plane with (x, θ_x) coordinates.

$\theta_x = dx/dz$ (paraxial approx.)



Why think in terms of phase space?

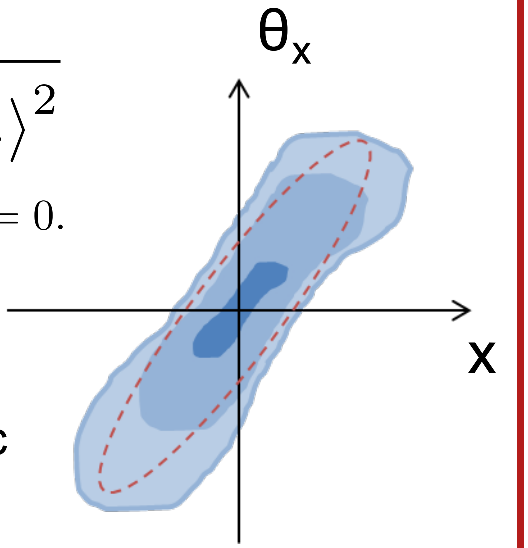
- The main motivation is ***"ray ensemble" description***
- ***Liouville's theorem***: phase space volume is incompressible fluid

- Phase space volume (area) **= *emittance*** (~wavelength for laser beams)
- Linear optics does not change the emittance (ABCD matrix has det = 1)

$$\epsilon = \sigma_{x,\mathrm{waist}} \cdot \sigma_{\theta_x,\mathrm{waist}} = \sqrt{\langle x^2\rangle \langle \theta_x^2\rangle - \langle x\,\theta_x\rangle^2}$$

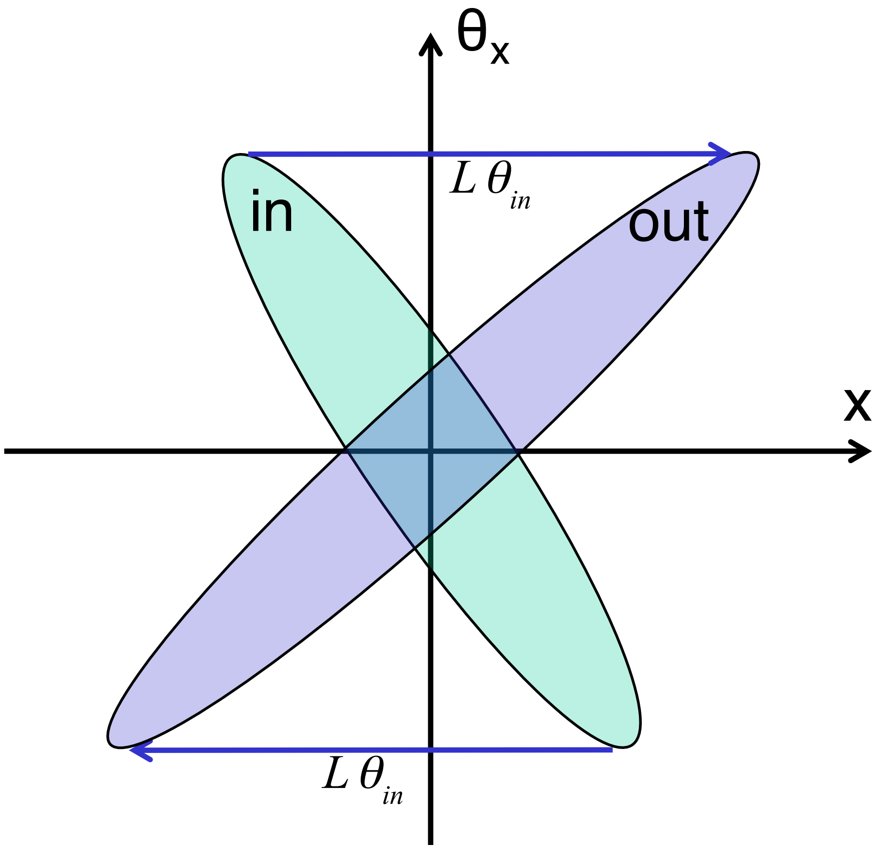
(units of meters!)

assuming $\langle x\rangle = 0,\ \langle \theta_x\rangle = 0.$



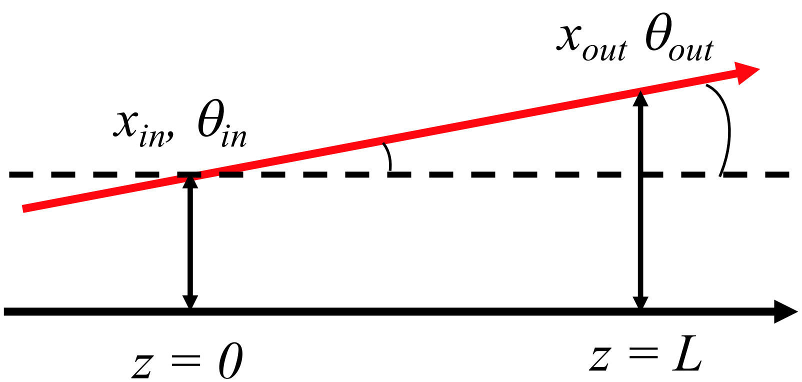
- Limitation of the phase space: only works for geometric optics (no wave phenomena).
- <u>*Important exception to this rule*</u>: ***Gaussian beams*** perfectly account for the diffraction limit and still can be described in terms of classical phase space and its transport (with ABCD matrices!).

Phase space transformation: drift



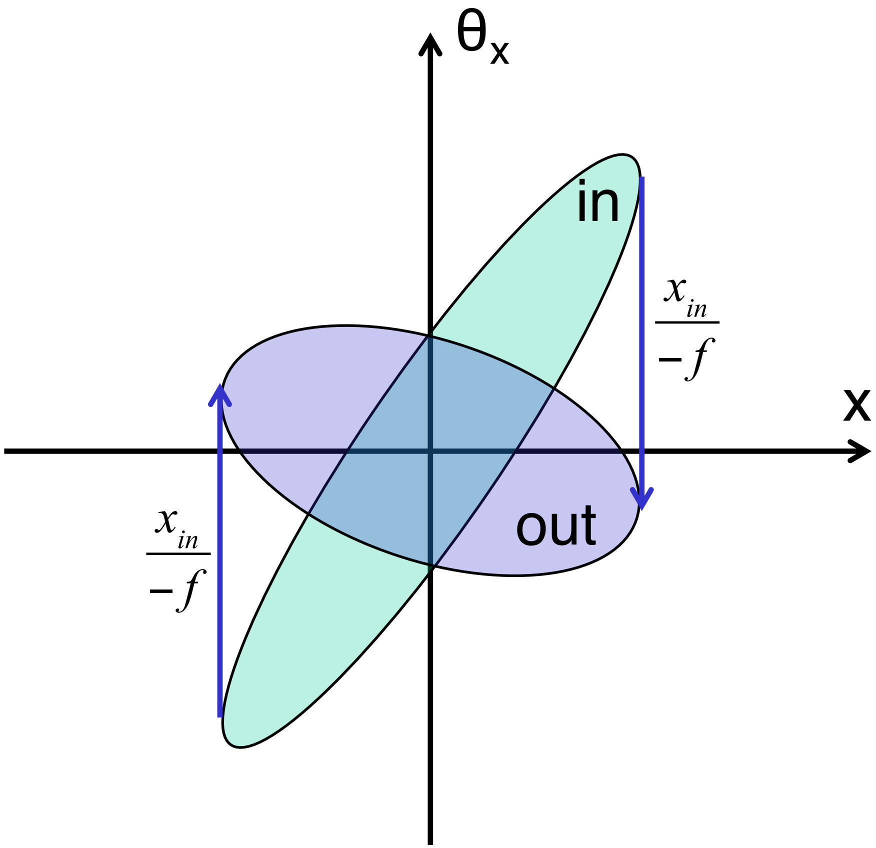
$$x_{out} = x_{in} + L\,\theta_{in}$$

$$\theta_{out} = \theta_{in}$$



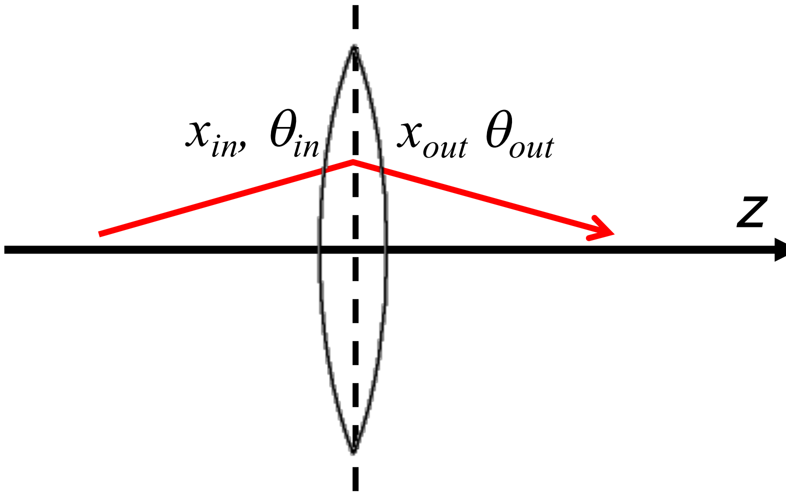
- Phase space is sheered along the x-direction (phase space area is conserved!)

Phase space transformation: thin lens



$$x_{out} = x_{in}$$

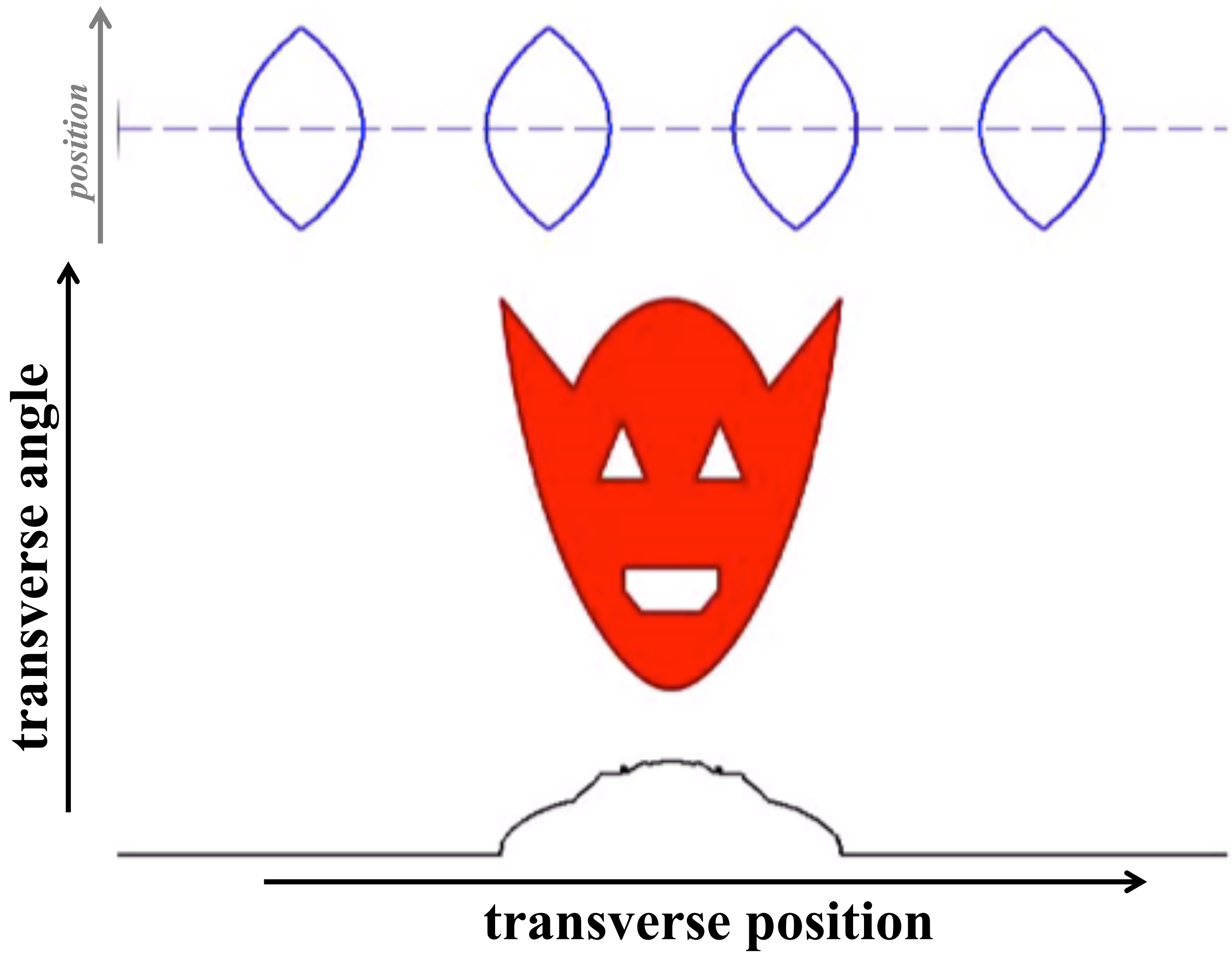
$$\theta_{out} = \theta_{in} + \frac{x_{in}}{-f}$$



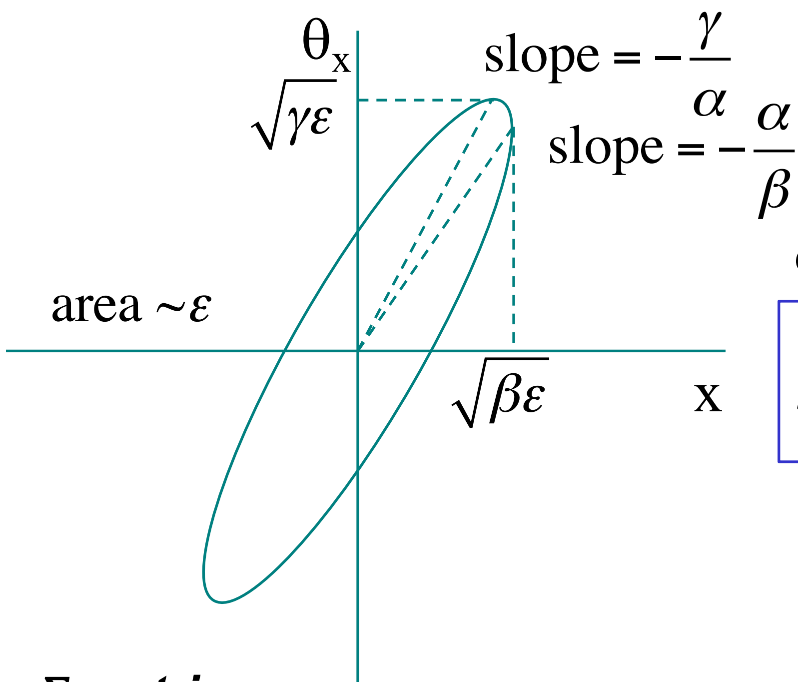
- Phase space is sheered along the θ-direction (phase space area is conserved)

Example: linear optics with four thick lenses

https://www.youtube.com/watch?v=fM4GYnMgGcQ



Phase space ellipse & Σ-matrix



e.g. 2D Gaussian distribution:

$$\rho(x,\theta_x) \propto \exp\left(-\frac{\gamma x^2 + 2\alpha x\theta_x + \beta\theta_x^2}{2\epsilon}\right)$$

$\rho(x,\theta_x)$ = const gives ellipse equations

Σ-matrix:

$$\Sigma = \left\langle \vec{x}\,\vec{x}^T \right\rangle = \begin{bmatrix} \langle xx \rangle & \langle x\theta_x \rangle \\ \langle \theta_x x \rangle & \langle \theta_x\theta_x \rangle \end{bmatrix} = \varepsilon \begin{bmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{bmatrix}$$

$\det[\ldots] = 1$

Twiss parameters
(ellipse orientation)

$$\epsilon = \sqrt{\det \boldsymbol{\Sigma}} = \sqrt{\langle x^2\rangle \langle \theta_x^2\rangle - \langle x\,\theta_x\rangle^2}$$

Sigma-matrix propagation

- Given an ABCD matrix **M:**

$$\mathbf{x}_{\text{out}} = \mathbf{M}\mathbf{x}_{\text{in}} \qquad \begin{bmatrix} x \\ \theta_x \end{bmatrix}_{\text{out}} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} x \\ \theta_x \end{bmatrix}_{\text{in}}$$

how to transform the sigma matrix?

$$\mathbf{\Sigma}_{\text{out}} = \left\langle \mathbf{x}\mathbf{x}^\top \right\rangle\big|_{\text{out}} = \mathbf{M} \left\langle \mathbf{x}\mathbf{x}^\top \right\rangle\big|_{\text{in}} \mathbf{M}^\top$$

$$\mathbf{\Sigma}_{\text{out}} = \mathbf{M}\,\mathbf{\Sigma}_{\text{in}}\,\mathbf{M}^\top$$

- Can rewrite as:

$$\begin{bmatrix} \beta \\ \alpha \\ \gamma \end{bmatrix}_{\text{out}} = \begin{bmatrix} m_{11}^2 & -2m_{11}m_{12} & m_{12}^2 \\ -m_{11}m_{21} & m_{12}m_{21} + m_{22}m_{11} & -m_{12}m_{22} \\ m_{12}^2 & -2m_{22}m_{21} & m_{22}^2 \end{bmatrix} \begin{bmatrix} \beta \\ \alpha \\ \gamma \end{bmatrix}_{\text{in}}$$

Twiss-parameter transformation for a drift

Given the ray transformation for a drift of length z:

$$\begin{bmatrix} x \\ \theta_x \end{bmatrix}_z = \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ \theta_x \end{bmatrix}_0$$

Twiss parameter transformation matrix becomes:

$$\begin{bmatrix} \beta \\ \alpha \\ \gamma \end{bmatrix}_z = \begin{bmatrix} 1 & -2z & z^2 \\ 0 & 1 & -z \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \beta \\ \alpha \\ \gamma \end{bmatrix}_0$$

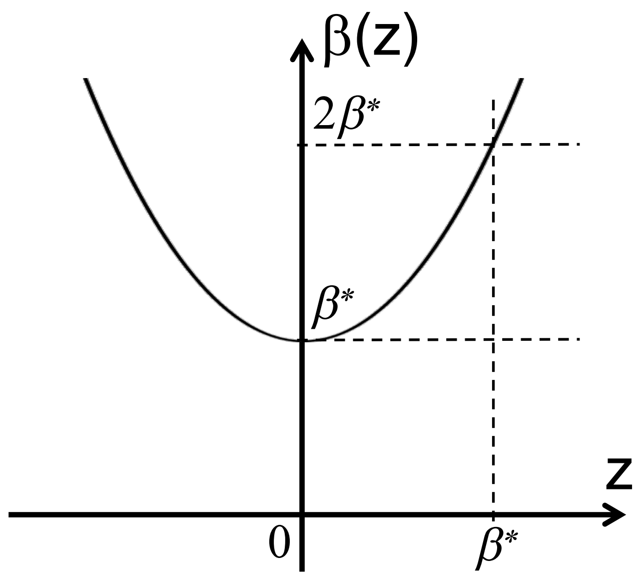
If $z = 0$ is location of beam waist, then $\alpha = 0$ and since $\beta\gamma - \alpha^2 = 1$ we have:

$$\beta(z) = \beta^* + \frac{z^2}{\beta^*}$$

beam waist value

Rayleigh range

- This β-function is also known as the **Rayleigh range** (units of meters)
- Recall that $\sigma_x = \sqrt{\langle x^2 \rangle} = \sqrt{\beta\epsilon}$



- Physical meaning of the Rayleigh range
  - *how tight a focus spot one can get;*
  - *distance over which the cross-section area of the beam doubles.*

Twiss-parameter transformation for a lens

Given the ray transformation for a lens with focal length *f* **:**

$$\begin{bmatrix} x \\ \theta_x \end{bmatrix}_{\text{after}} = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} x \\ \theta_x \end{bmatrix}_{\text{before}}$$

Twiss parameter transformation matrix becomes:

$$\begin{bmatrix} \beta \\ \alpha \\ \gamma \end{bmatrix}_{\text{after}} = \begin{bmatrix} 1 & 0 & 0 \\ 1/f & 1 & 0 \\ 1/f^2 & 2/f & 1 \end{bmatrix} \begin{bmatrix} \beta \\ \alpha \\ \gamma \end{bmatrix}_{\text{before}}$$

Smallest epsilon (phase space area) of light?

- ε can be arbitrarily large depending on the source.

- What is the smallest phase space area?

$$\sigma_x \sigma_{p_x} \geq \hbar/2$$

Uncertainty Principle

photon's full momentum

$$\sigma_x (p\sigma_{\theta_x}) \geq \hbar/2$$

$$\sigma_x \sigma_{\theta_x} \geq \hbar/2p$$

$$\epsilon \geq \hbar/2p$$

wavenumber

$$\epsilon \geq \hbar/(2\hbar k)$$

$$\epsilon \geq \lambda/4\pi$$

wavelength

- ***Absolute smallest ε for light is $\varepsilon = \lambda/4\pi$ which is realized only for a Gaussian mode laser beam.***

Example1: shoot laser to the Moon

- Suppose you are trying to send a focused laser beam to the Moon to establish a communication link while minimizing the overall beam diameter. What is the smallest lens/mirror diameter you would require on Earth for that (ignore any lensing due to the Earth's atmosphere)?

Distance to the Moon d = 384,400 km = 3.844×10^8 m

Laser wavelength λ = 0.4 μm (violet, perfectly Gaussian laser mode)

Choose the Rayleigh range to be equal to the distance d: $\beta^* = d$

Then the rms laser spot at the "lens" on Earth is: $\sigma_x = \sqrt{2\beta^*(\lambda/4\pi)} \approx 4.9\,\mathrm{m}$

So the "lens" at the full width half max (FWHM) must be: $2.35 \times \sigma_x \approx \varnothing 11.5$ m

The rms spot on the Moon would be: $\sigma_{x,\mathrm{Moon}} = \sqrt{\beta^*(\lambda/4\pi)} \approx 3.5\,\mathrm{m}$

Remark1: the actual spot will be larger because the "lens" is clipping the Gaussian beam, which will lead to the diffraction effects blowing up the size…

Remark2: what sort of accuracy to its shape must such a lens/mirror have? Is it even feasible?

Example2: laser cutter design choices

- You have designed a laser welder that has a 1 micron rms spot size on the target using a 0.4 μm laser Gaussian laser source. How precisely should the distance to the target be controlled so that the intensity fluctuations remain within a factor of 2?

This is exactly the definition of the Rayleigh range. So, we find:

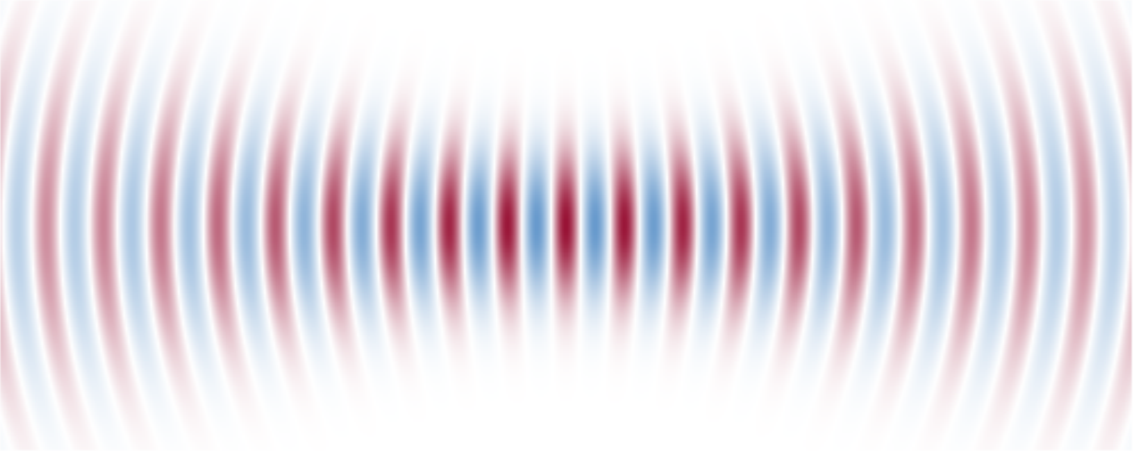
$$\beta = \sigma_x^2/(\lambda/4\pi) \approx 31.4\mu\text{m}$$

In other words, such a laser welder would have a very small depth-of-focus and wouldn't be able to cut very deep into metal.

Relaxing the focused spot requirement to 10 microns would result in 3.1 mm depth-of-focus, which is way more reasonable for cutting thin metal sheets.

Gaussian beam

Gaussian beam – an exact solution to Maxwell equations within the paraxial approximation



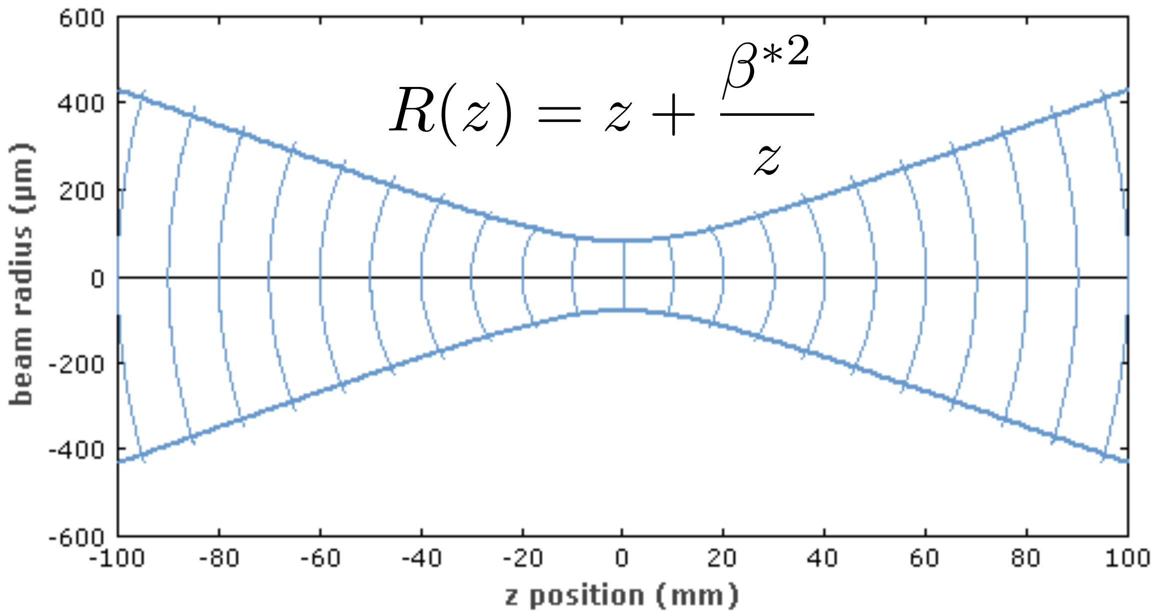
Its propagation can be treated classically (using ABCD matrix) assuming a pure Gaussian distribution in phase space with a ***given β-function (=Rayleigh range)*** and ***emittance $\varepsilon = \lambda/4\pi$***

This will always work provided that the optics is linear (lenses are ideal) and no clipping of light happens anywhere!

Wavefront curvature of Gaussian beams

Even though the classical propagation works for Gaussian beam, it's a more rich object = mode with electric field defined everywhere and perfectly coherent.

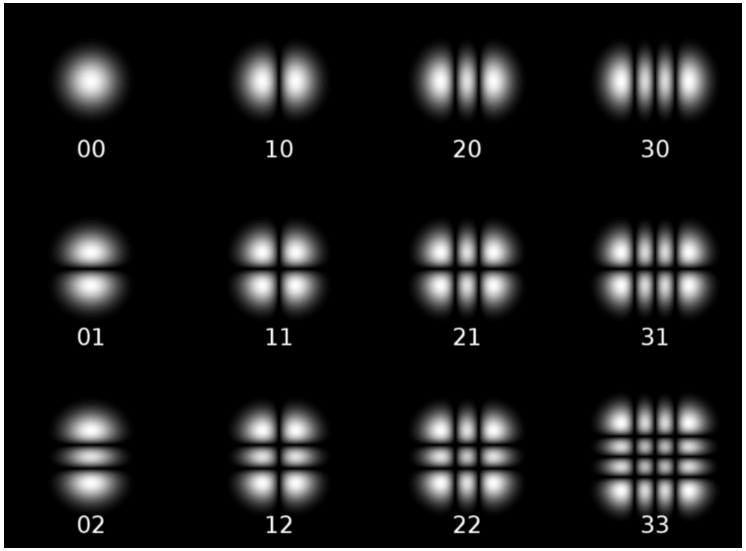
E.g. the field wavefront has a perfectly defined phase, with its curvature given by



Note: when $z \gg \beta^*$, the beam behaves as a spherical wave
when $z \ll \beta^*$, the beam behaves as a planar wave

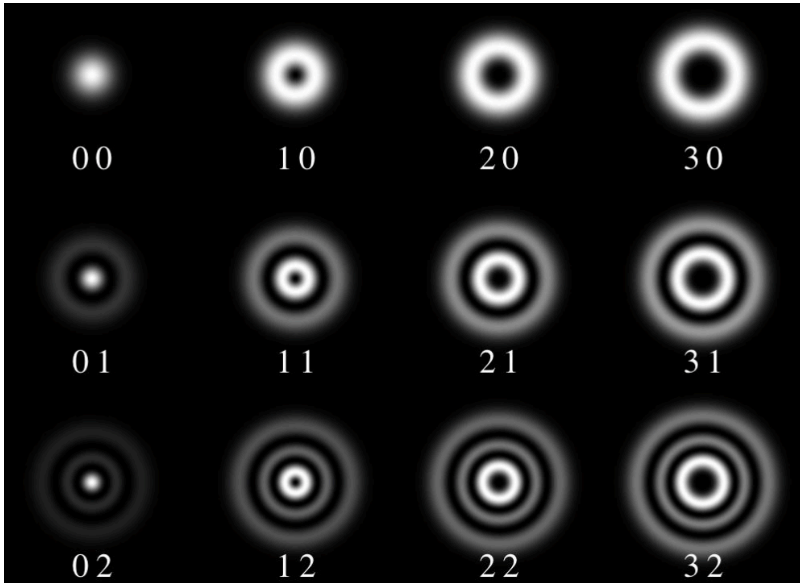
Other solutions to Maxwell equations

Hermite-Gaussian



TEM$_{mn}$

Laguerre-Gaussian



Emittance for Hermite-Gaussian beam is given by

$$\epsilon_x = \frac{\lambda}{4\pi}(2m+1)$$

$$\epsilon_y = \frac{\lambda}{4\pi}(2n+1)$$

M^2 beam quality factor

$$M^2 = \frac{\epsilon}{\lambda/4\pi} \geq 1$$

$M^2 = 1$ only for a pure Gaussian beam!

Links/References

Gaussian beam plots taken from Encyclopedia of Laser Physics and Technology

Hermite-, Laguerre-Gauss beam pics taken from Wikipedia